On the first eigenvalue/eigenvector in sparse random symmetric matrices

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The first (largest) eigenvalue and its eigenvector (first eigenvector) play key roles in many problems in information science. In multivariate data analysis, the first eigenvector of the variance-covariance matrix represents the most significant component that underlies a set of data under the assumption that the data are generated from a multivariate Gaussian distribution, and the first eigenvalue indicates its relevance. The well-known Google PageRank$^{\mathrm{TM}}$ ranks World Wide Web pages based on the first eigenvector of a transition matrix over a huge network of pages. The first eigenvectors of certain matrices expressing a given graph can also be utilized as a practical solution to combinatorial problems such as graph three coloring- and graph bisection problems.

In those applications, it is expected that the first eigenvector contains useful information being *ordered* to a certain preferential direction that is embedded in a matrix. On the other hand, it is also known that the first eigenvector tends to be *localized* making almost all entries vanishingly small for some ensembles of sparse matrices. In order to gain insights about how these two different features can appear in the first eigenvector, we examined an ensemble of $N \times N$ sparse random symmetric matrices which is characterized by a bimodal degree distribution $p(k) = p_1\delta_{k,c_1} + p_2\delta_{k,c_2}$ and a binary distribution of entries $P_J(J_{ij}) = \frac{1+\Delta}{2}\delta(J_{ij}-1) + \frac{1-\Delta}{2}\delta(J_{ij}+1)$, where $c_2 \geq c_1$ and $0 \leq \Delta \leq 1$ [1]. Our results include the followings:

- When $c_2 = c_1$, the first eigenvalue is given as $\Lambda = (c_1-1)\Delta + \Delta^{-1}$ for $\Delta > \Delta_{\mathrm{c}} \equiv (c_1-1)^{-1/2}$ and $2\sqrt{c_1-1}$ for $\Delta < \Delta_{\mathrm{c}}$. The localization of the first eigenvector does not occur.

- Let us suppose a situation that only a single site has a larger degree c_2, which corresponds to the case of $p_2 = 1 - p_1 = N^{-1}$. In such a situation, the first eigenvector is localized for sufficiently small Δ if $c_2 > 2(c_1-1)$ and for $\forall\Delta$ if $c_2 \geq c_1(c_1-1)$. The first eigenvalue for the localized case is given as $\Lambda = c_2(c_2-c_1+1)^{-1/2}$.

- In more general cases that $p_1, p_2 \sim O(1)$ and c_2 is sufficiently larger than c_1, $\Lambda = 2\sqrt{c_2-1}$ and the first eigenvector is not localized in the limit of $N \to \infty$. However, experimentally observing this is difficult as the finite size effect is very large.

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[1] Y. Kabashima and H. Takahashi, J. Phys. A: Math. Theor. **45**, 325001 (2012)