CMT I. Applications of the Rado-Hall Theorem

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ICTP/IPM Workshop, 3-7 Sept 2012

September 4, 2012

P. Hall's Theorem

$\mathcal{A} = (A_1, A_2, \ldots, A_n)$, a family of n subsets of a finite set X.

A **System of Representatives (SR)** of $\mathcal{A}$ is a family of elements $(x_1, x_2, \ldots, x_n)$ of X with

$$x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n.$$

If $x_1, x_2, \ldots, x_n$ are all different, then $(x_1, x_2, \ldots, x_n)$ is a **System of Distinct Representatives (SDR)** of $\mathcal{A}$.

Example: $X = \{1, 2, 3, 4, 5, 6, 7\}$

$\mathcal{A} = (\{1, 3, 6, 7\}, \{2, 3, 5\}, \{1, 4, 6, 7\}, \{2, 5, 7\}, \{1, 3, 7\})$

Question: When does a family $\mathcal{A}$ have an SDR?

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P. Hall's Theorem continued

Obvious necessary condition for $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ to have an SDR:

$$(*) \quad |\cup_{i \in K} A_i| \geq |K| \quad K \subseteq \{1, 2, \ldots, n\}.$$

For instance, if 5 sets collectively contained only 4 elements then surely there cannot be an SDR.

Hall's Theorem: Condition $(*)$ is both necessary and sufficient.

Induction can be used to prove Hall's Theorem. My favorite proof is due to Richard Rado. It is based on the observation that if each of the sets $A_1, A_2, \ldots, A_n$ contain exactly one element: $A = \{x_1\}, A_2 = \{x_2\}, \ldots, A_n = \{x_n\}$, then (*) implies that $x_1, x_2, \ldots, x_n$ are all different and so $(x_1, x_2, \ldots, x_n)$ is an SDR. So we try to reduce our problem to this situation.

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Rado's proof of Hall's Theorem

(Keep in mind that for sets, $|X_1 \cup X_2| + |X_1 \cap X_2| = |X_1| + |X_2|$.)

So suppose that one of the sets, say A_1 contains (at least) two elements, a and b where $a \neq b$.

Suppose that neither $(A_1 \setminus \{a\}, A_2, \ldots, A_n)$ nor $(A_1 \setminus \{b\}, A_2, \ldots, A_n)$ satisfy the (*) condition. Then there exists $K_1, K_2 \subseteq \{2, \ldots, n\}$ such that

$$|(A_1 \setminus \{a\}) \cup \cup_{i \in K_1} A_i| \leq |K_1|, \quad |(A_1 \setminus \{b\}) \cup \cup_{i \in K_2} A_i| \leq |K_2|.$$

Then,

$$\begin{aligned} |K_1| + |K_2| &\geq |(A_1 \setminus \{a\}) \cup \cup_{i \in K_1} A_i| + |(A_1 \setminus \{b\}) \cup \cup_{i \in K_2} A_i| \\ &\geq |A_1 \cup \cup_{i \in K_1 \cup K_2} A_i| + |\cup_{i \in K_1 \cap K_2} A_i| \\ &\geq |K_1 \cup K_2| + 1 + |K_1 \cap K_2| \\ &= |K_1| + |K_2| + 1, \text{ a contradiction.} \end{aligned}$$

Linear Independence of Vectors in a Vector Space

Let X be a finite set of vectors in a vector space V , and let $\mathcal{I}$ be the collection of subsets of X that are **linearly independent**. Then $\mathcal{I}$ satisfies the following familiar properties:

- $\emptyset \in \mathcal{I}$,
- $A \in \mathcal{I}, A' \subseteq A$ implies $A' \in \mathcal{I}$,
- $A_1, A_2 \in \mathcal{I}, |A_1| < |A_2|$ implies $\exists x \in A_2 \setminus A_1$ such that $A_1 \cup \{x\} \in \mathcal{I}$.

The **rank** $\rho(Y)$ of any subset Y of X is the dimension of the subspace they span: the maximum number of linearly independent vectors in Y. Recall that for subspaces U and W of V: $\dim(U+W)+\dim(U\cap W)=\dim(U)+\dim(W)$ gives, for subsets Y, Z of X: $\text{rank}(Y\cup Z)+\text{rank}(Y\cap Z)\leq \text{rank}(Y)+\text{rank}(Z)$.

These properties when viewed axiomatically are one way to define a **matroid**.

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Matroids

X a finite set, $\mathcal{I}$ a collection of subsets of X **called independent sets**, satisfying:

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These properties can be used to show the **submodular inequality**: For subsets Y and Z of X:

$$\rho(Y \cup Z) + \rho(Y \cap Z) \leq \rho(Y) + \rho(Z).$$

A matroid can also be defined in terms of **circuits** (minimal dependent subsets of X), or **bases** (maximal independent subsets of X), ...

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Rado's Theorem - the Hall-Rado Theorem

$\mathcal{A} = (A_1, A_2, \ldots, A_n)$: a family of n subsets of a finite set X.
$\mathcal{I}$: the collection of independent sets of a matroid on X.

Then $\mathcal{A}$ has an SDR $(x_1, x_2, \ldots, x_n)$, with

$$x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n,$$

such that its set of elements $\{x_1, x_2, \ldots, x_n\}$ is an independent set if and only if

$$(*) \quad \rho(\cup_{i \in K} A_i) \geq |K|, \quad K \subseteq \{1, 2, \ldots, n\}.$$

Hall's theorem is the special case where every subset of X is independent (a trivial matroid) and the rank function is $\rho(Y) = |Y|$. The proof of Hall's theorem can be adapted to give a proof of Rado's theorem using the submodularity property of the rank function.

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$(0,1)$-matrices

Let $A = [a_{ij}]$ be an $m \times n$ $(0,1)$-matrix. Let

$$r_i = \text{ the number of 1s in row } i \text{ and}$$

$$s_j = \text{ the number of 1s in column } j.$$

Then $R = (r_1, r_2, \ldots, r_m)$ is the **row sum vector** of A and $S = (s_1, s_2, \ldots, s_n)$ is the **column sum vector** of A.

Example: $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $R = (3,3,2,2), S = (3,3,3,1)$

WLOG (by permuting rows and columns) we may assume that R and S are **monotone**: $r_1 \geq r_2 \geq \cdots \geq r_m$ and $s_1 \geq s_2 \geq \cdots \geq s_n$.

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Vector Majorization/Dominance

If $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ are two real monotone vectors, then $X \preceq Y$ (X is **majorized** by Y) provided

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \quad (1 \leq i \leq k)$$

with equality when $k = n$. If Y is a nonnegative integral vector, then the **conjugate** of Y is the vector $Y^* = (y_1^*, y_2^*, y_3^*, \ldots)$ where

$$y_k^* = |\{i : y_i \geq k\}|.$$

Example: $Y = (4, 3, 1, 1)$, $Y^* = (4, 2, 2, 1)$:

1	1	1	1	4
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The Inverse Problem: Gale-Ryser Theorem

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be monotone nonnegative integral vectors. Then there exists an $m \times n$ $(0, 1)$-matrix A with row sum vector R and column sum vector S if and only if

$$S \preceq R^*,$$

that is, **S is majorized by the conjugate of R**.

The condition is satisfied if such a matrix A exists: Slide all the 1s of A to the left obtaining a matrix with column sum vector R^*.
E.G.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ \hline 4 & 3 & 4 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 5 & 5 & 4 & 2 & 0 \end{bmatrix}$$

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Gale-Ryser Theorem: Outline of Sufficiency

Let $X = \{(i,j) : 1 \le i \le m, 1 \le j \le n\}$, the set of positions of an $m \times n$ matrix and let $(X_1, X_2, \dots, X_m)$ where X_k is the set of positions (k,j) of the kth row. Define a matroid on X by: A set P of positions is **independent** if it has $\le r_k$ positions from row k:

$$|P \cap X_k| \le r_k \quad (1 \le k \le m).$$

(**Check this defines a matroid**) Its rank function is:

$$\rho(F) = \sum_{k=1}^{m} \min\{|F \cap X_k|, r_k\} \quad (F \subseteq X).$$

Let

$$\mathcal{A} = (A_1, A_2, \dots, A_p) = (\underbrace{E_1, \dots, E_1}_{s_1}, \underbrace{E_2, \dots, E_2}_{s_2}, \dots, \underbrace{E_n, \dots, E_n}_{s_n})$$

where $p = s_1 + s_2 + \cdots + s_n$ and E_j is the set of positions in column j. **An independent SDR of $\mathcal{A}$ 'is' a $(0,1)$-matrix with row sum vector R and column sum vector S.**

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Gale-Ryser Theorem: Outline of Sufficiency cont.

Now we apply Rado's Theorem. In checking Rado's condition $\rho(\cup_{i\in K} A_i) \geq |K|$, the left side depends only on whether at least one copy of E_j appears while the right side depends on how many copies appear. So it is equivalent to:
$\rho(\cup_{j\in K} E_j) \geq \sum_{j\in K} s_j \quad (K \subseteq \{1, 2, \ldots, n\})$. $\cup_{j\in K} E_j$ is the set of positions in the columns indexed by K, and so

$$\rho(\cup_{j\in K} E_j) = \sum_{i=1}^{m} \min\{|K|, r_i\},$$

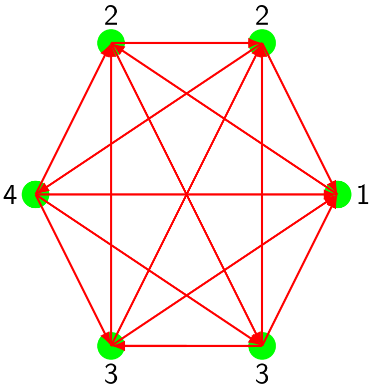
and so depends only on $k = |K|$ and not K itself. By monotonicity, $\sum_{j\in K} s_j$ is largest at $\sum_{j=1}^{k} s_j$. Thus Rado's condition reduces to

$$\left(\sum_{i=1}^{k} r_i^* =\right) \sum_{i=1}^{m} \min\{k, r_i\} \geq \sum_{j=1}^{k} s_j \quad (1 \leq k \leq n).$$

Tournaments (Tournament matrices)

A **tournament** T_n of order n is an orientation of the complete graph K_n.

Example: K_6 and an orientation of it giving a T_6:



$4, 3, 3, 2, 2, 1$ is the **score sequence**

Adjacency Matrix of T_6

Often it is easier to visualize a tournament by constructing its **adjacency matrix**. The adjacency matrix of T_6 with a particular ordering of its vertices is

$$A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

All 0s on the main diagonal; of each pair of diagonally opposite entries one is a 1 and the other is a 0: $a_{ij} + a_{ji} = 1$ for $i \neq j$. The **scores** of T_6 are the **row sums** of A_6:

$$4, 3, 3, 1, 2, 2.$$

Adjacency Matrix of T_6

Often it is easier to visualize a tournament by constructing its **adjacency matrix**. The adjacency matrix of T_6 with a particular ordering of its vertices is

$$A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

All 0s on the main diagonal; of each pair of diagonally opposite entries one is a 1 and the other is a 0: $a_{ij} + a_{ji} = 1$ for $i \neq j$. The **scores** of T_6 are the **row sums** of A_6:

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$$4, 3, 3, 1, 2, 2.$$

Transitive Tournaments

Vertices can be ordered so that all arrows go down:

$$\begin{array}{cc} 1 & \\ 2 & \downarrow \\ 3 & \\ & \downarrow \\ \vdots & \\ n & \end{array}$$

Scores are $\mathbf{n-1, n-2, \ldots, 2, 1, 0}$. In this case, it is absolutely clear who the best and worst players are!

Adjacency Matrix of a Transitive Tournament of order 6

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Landau's Theorem

If $R = (r_1, r_2, \ldots, r_n)$ is a sequence of nonnegative integers, then R is the score sequence of a tournament of order n iff

$$(*) \quad \sum_{i \in I} r_i \geq \binom{|I|}{2}, \quad (I \subseteq \{1, 2, \ldots, n\})$$

with equality when $I = \{1, 2, \ldots, n\}$.

Necessity of $(*)$ is clear: Any set I of vertices with $|I| = k$ induces a tournament T_I of order k; T_I has $\binom{k}{2}$ edges; all of these edges, and more in general, are accounted for in the sum of their scores $\sum_{i \in I} r_i$. Thus $\sum_{i \in I} r_i \geq \binom{k}{2}$.

WLOG we assume monotonicity of the scores in the form:$r_1 \leq r_2 \leq \cdots \leq r_n$. Then $(*)$ is equivalent to

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2}, \quad (k = 1, 2, \ldots, n)$$

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Proof of Landau's Theorem

Several proofs exist:

- A constructive proof of Ryser: Inductively construct row and column i for $i = n, \cdots, 2, 1$
- Minimal counterexample proofs of Mahmoodian and Thomassen: Choose a counterexample with n minimum and then with minimum r_1.
- As a corollary of Rado's theorem; see next slide.

Landau's Theorem from Rado's Theorem

Proof Outline: RAB and K. Kiernan 2009

- Define a matroid $\mathbf{M}$ on $X = \{(i,j); 1 \leq i,j \leq n, i \neq j\}$ whose **independent sets** are those subsets of X that do not contain a symmetric pair $(i,j),(j,i)$ with $i \neq j$.
- Thus the minimal dependent sets - the **circuits** - are the $\binom{n}{2}$ disjoint sets $\{(i,j),(j,i)\}$ of two pairs in X with $i \neq j$.
- We have $\rho(X) = \binom{n}{2}$.
- Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ where $A_i = \{(i,j) : 1 \leq j \leq n, j \neq i\} \quad (i = 1, 2, \ldots, n)$. Thus A_i consists of all the non-diagonal positions of row i.
- There exists a tournament with score sequence $r_1, r_2, \ldots, r_n$ if and only if there exists $P_1, P_2, \ldots, P_n$, with $P_i \subseteq A_i$ and $|P_i| = r_i$ $(1 \leq i \leq n)$, such that $P = P_1 \cup P_2 \cup \cdots \cup P_n$ is an independent set of $\mathbf{M}$.

Landau's Theorem from Rado's Theorem continued

- There exists a tournament with score sequence $r_1, r_2, \ldots, r_n$ if and only if there exists $P_1, P_2, \ldots, P_n$, with $P_i \subseteq A_i$ and $|P_i| = r_i$ $(1 \le i \le n)$, such that $P = P_1 \cup P_2 \cup \cdots \cup P_n$ is an independent set of $\mathbf{M}$,
- Equivalently, if and only if the family

$$\mathcal{A}' = (\underbrace{A_1, \ldots, A_1}_{r_1}, \underbrace{A_2, \ldots, A_2}_{r_2}, \ldots, \underbrace{A_n, \ldots, A_n}_{r_n})$$

  has an SDR whose elements form an independent set.
- The desired tournament has vertices $1, 2, \ldots, n$ and an edge from i to j if and only (i, j) is in P_i. The independence of P then implies that there is no edge from j to i.
- Now one checks that Rado's condition holds if the inequalities of Landau hold.

Stronger Landau Inequalities

RAB and J. Shen 2001: If $R = (r_1, r_2, \ldots, r_n)$ is a sequence of nonnegative integers, then R is the score sequence of a tournament of order n if and only if

$$(**) \quad \sum_{i \in I} r_i \geq \frac{1}{2} \sum_{i \in I} (i-1) + \frac{1}{2} \binom{|I|}{2} \quad (I \subseteq \{1, 2, \ldots, n\})$$

with equality when $I = \{1, 2, \ldots, n\}$.

Since

$$\sum_{i \in I} (i-1) \geq \sum_{i=1}^{|I|} (i-1) = \binom{|I|}{2},$$

the inequalities $(**)$ are individually, but not collectively, stronger than Landau's inequalities

$$(*) \quad \sum_{i \in I} r_i \geq \binom{|I|}{2}.$$

What good are these "stronger" inequalities?

Certainly not to show the existence of a tournament with specified scores — they only make it more difficult.

Answer: These stronger inequalities allow an especially simple proof of the wonderful theorem first proved by Ao and Hanson 1998 and, independently, Guiduli, Gyárás, and Weidl 1998: that shows the existence of a very special tournament for each possible score vector:

If there is a tournament with score sequence $r_1 \leq r_2 \leq \cdots \leq r_n$, then there is one such that the induced tournaments on the vertex sets

$$\{1, 3, 5, \ldots\} \textbf{ and } \{2, 4, 6, \ldots, \}$$

are transitive tournaments, that is, there is the arc $i \longrightarrow j$ whenever $i > j$ and $i \equiv j$ mod 2.

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Example

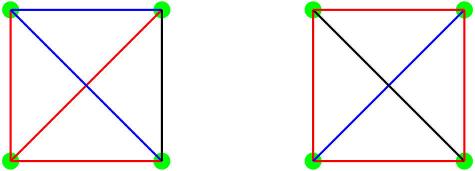
Thus to construct a tournament T_{10} with score sequence $R = (1, 3, 4, 4, 4, 5, 5, 5, 7, 7)$, we have only to construct A below:

$$
\left|\begin{array}{ccccc|ccccc||c}
0 & 0 & 0 & 0 & 0 & & & & & & 1 \\
1 & 0 & 0 & 0 & 0 & & & & & & 4 \\
1 & 1 & 0 & 0 & 0 & & & A & & & 4 \\
1 & 1 & 1 & 0 & 0 & & & & & & 5 \\
1 & 1 & 1 & 1 & 0 & & & & & & 7 \\
\hline
 & & & & & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & & & 1 & 0 & 0 & 0 & 0 & 4 \\
 & & A' & & & 1 & 1 & 0 & 0 & 0 & 5 \\
 & & & & & 1 & 1 & 1 & 0 & 0 & 5 \\
 & & & & & 1 & 1 & 1 & 1 & 0 & 7 \\
\hline
8 & 5 & 5 & 4 & 2 & 6 & 5 & 4 & 4 & 2 &
\end{array}\right| \qquad (A' = J - A^t)
$$

with row and column sum vectors $(1, 3, 2, 2, 3)$ and $(2, 2, 2, 3, 2)$.

Biclique Partitions

A **biclique** of a graph G of order n is a complete bipartite subgraph $K_{r,s}$ with $r, s \geq 1$ (when $r = s = 1$ we just get one edge). The **biclique partition number** $bp(G)$ of G is the smallest number of bicliques into which its edges can be partitioned. For example,



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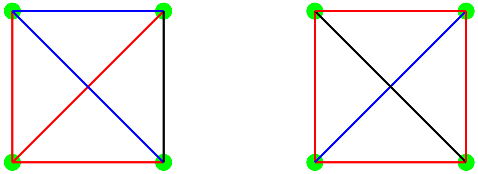

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Graham-Pollak Theorem I

bp(K$_n$) = n − 1: The proof uses elementary linear algebra.

A biclique partition of K_n into bicliques corresponds to a tournament of order n:

Let T_n be the corresponding tournament matrix so that $T_n + T_n^t = J_n - I_n$. Let T_n^* be the $n \times (n+1)$ matrix obtaining by adjoining a column of all 1s to T_n, Let x be a vector in the left nullspace of T_n^*: $x^t T_n^* = 0$. On the one hand,

$$x^t(J_n - I_n)x = x^t J_n x - x^t x = -x^t x.$$

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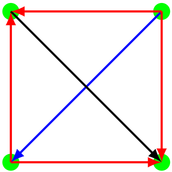

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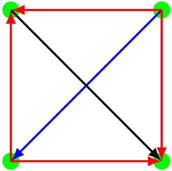

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Graham-Pollak Theorem II

$$x^t(J_n - I_n)x = x^t J_n x - x^t x = -x^t x.$$

On the other hand,

$$x^t(J_n - I_n)x = x^t(T_n + T_n^t)x = x^t T_n x + x^t T_n^t x = (x^t T_n)x + x^t(x^t T_n)^t$$

$$= 0 + 0 = 0$$

Thus $x^t x = 0$ and so $x = 0$. Thus the rank of T_n^* equals n and so the rank of T_n is $\geq n - 1$.

But a decomposition of K_n into r bicliques expresses the matrix $J_n - I_n$ as the sum $J_n - I_n = Q + 2(A_1 + A_2 + \cdots + A_r)$ where the A_i have rank 1 and Q is skew-symmetric. Thus $I_n + Q = J_n - 2(A_1 + A_2 + \cdots + A_r)$ has eigenvalues with real part equal to 1 and is nonsingular. So $r + 1 \geq n$ and $r \geq n - 1$.

Graham-Pollak Theorem II

$$x^t(J_n - I_n)x = x^tJ_nx - x^tx = -x^tx.$$

On the other hand,

$$x^t(J_n - I_n)x = x^t(T_n + T_n^t)x = x^tT_nx + x^tT_n^tx = (x^tT_n)x + x^t(x^tT_n)^t$$

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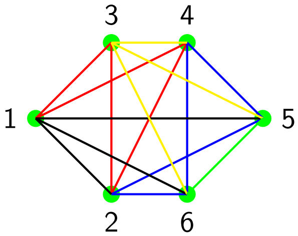
Graham-Pollak Theorem III

Example of a Q:



$$Q = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

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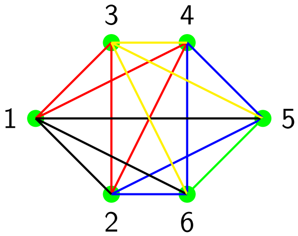

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Conclusion

The Rado-Hall Theorem is a wonderful theorem to have in ones's mathematical toolchest!