Counting with Permanents

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Permanents

The *permanent* of an $n \times n$ matrix $A = [a_{i,j}]$ is defined by

$$\text{per}(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where the sum is over all permutations σ of $\{1, 2, \ldots, n\}$. In other words we take the product of the entries along each diagonal, and add up these products. This is the same definition as the *determinant*, except we leave out the ± 1's.

Example:

$$\text{per}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1.5.9 + 1.6.8 + 2.4.9 + 2.6.7 + 3.5.7 + 3.4.8 = 450$$

Basic properties

$\text{per}(A) = \text{per}(A^T)$ for any A.
If k is any scalar and A has order n then $\text{per}(kA) = k^n \,\text{per}(A)$.

The permanent of a permutation matrix is 1.
The permanent is unaffected by permuting rows and/or columns.
So

$$\text{per}(AB) = \text{per}(A)\,\text{per}(B) \qquad (\dagger)$$

if A or B is a permutation matrix.
However $(\dagger)$ does not hold for arbitrary A and B.

This is a devastating blow, since most of the nice properties of determinants follow from $\det(AB) = \det(A)\det(B)$.

In particular, we cannot use Gaussian elimination to calculate permanents. (There is strong evidence that there is no fast algorithm to calculate permanents.)

"Determinants are angels and permanents are devils"

–Doron Zeilberger

Counting with permanents

For a square $(0,1)$-matrix A the permanent *counts* the positive diagonals of A. In other words, it counts the permutation matrices P such that $P \leqslant A$ (entrywise).

Example: In how many ways can n married couples dance so that nobody dances with their spouse?
In other words, how many permutations are there with no fixed point? Such a permutation is called a *derangement*.

$$D_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

$$D_6 = \text{per} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

More spouse avoidance

In how many ways can n married couples sit around a circular table so that nobody sits next to their spouse? (Protocol insists that sexes alternate around the table, and that ladies are seated first).
Once the ladies are seated, the gents can be seated according to a permutation. The number of such permutations is called the n-th Menagé number, M_n.
The Menagé numbers can be counted by permanents:
Example:

$$\text{per}\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} = M_6.$$

$$M_n = \sum_k (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

Biadjacency matrices

Any $(0, 1)$-matrix can be interpreted as the *biadjacency matrix* of a bipartite graph B. The two colour classes of the graph are the rows and columns respectively, and the 1's record where the edges occur between a row vertex and a column vertex.

A case of special importance in this talk will be when the bipartite graph is regular, say each vertex has degree k. Then each row and column of the biadjacency matrix sums to k.

We let Λ_n^k denote the set of $(0, 1)$-matrices of order n, where each row and column sums to k.

Example: Our derangement example came from Λ_6^5 and the Menagé example came from Λ_6^4.

Counting matchings with permanents

Theorem: For a bipartite graph G, the number of perfect matchings in G is equal to the permanent of the biadjacency matrix of G.

Matchings are vital in many pure and applied problems. eg. All the problems discussed so far; assigning lecturers to classes, or drivers to vehicles. We don't always want *perfect* matchings.

Good news... Permanents can count k-matchings too!
Define the *k-th subpermanent sum* $\sigma_k(A)$ to be the sum of the permanents of all $k \times k$ submatrices of A.

Theorem: For a bipartite graph G, the number of k-matchings in G is equal to σ_k of the biadjacency matrix of G.

Hall's marriage theorem

Hall's theorem (as introduced by Richard yesterday) is sometimes known as the *marriage theorem*, because it can be phrased as the answer to the following question:

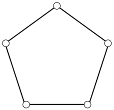
Suppose that there is a group of eligible bachelors for a group of single women to choose a husband from. Further, suppose that each woman has a list of men she is willing to marry. Under what conditions is it possible for each women to marry a man on her list?

We want an SDR from the women's sets of potential husbands. The number of these is $\sigma_w(N)$ if there are w women and N is the incidence matrix for their lists.

An application of Hall's theorem

Theorem: Every regular bipartite graph has a perfect matching.

Proof: Suppose G is a k-regular bipartite graph with bipartition (U, V). Suppose that $W \subseteq U$. There are $k|W|$ edges from W to V and no vertex in V lies on more than k of them. Hence W has at least $k|W|/k = |W|$ neighbours, so by Hall's theorem G has a perfect matching. $\square$

NB. We need the condition that G is bipartite. eg. 

The above theorem says $\text{per}(A) \geqslant 1$ for all $A \in \Lambda_n^k$ with $k \geqslant 1$. As a corollary, A can be written as the sum of k permutation matrices.

Latin rectangles

A $k \times n$ *Latin rectangle* is a matrix containing n different symbols arranged so that each symbol occurs exactly once per row and at most once per column. An $n \times n$ Latin rectangle is a *Latin square*.

Example: A 3×6 Latin rectangle and a 5×5 Latin square:

$$\begin{pmatrix} 3 & 6 & 1 & 2 & 5 & 4 \\ 5 & 2 & 3 & 4 & 1 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

A natural way to build Latin squares is one row at a time. Let L be a $k \times n$ Latin rectangle. Define $R \in \Lambda_n^{n-k}$ to be the $(0,1)$-matrix that records a 1 in cell (i,j) if symbol i has not yet been used in column j of L. Then $\mathrm{per}(R)$ counts the number of extensions of L to a $(k+1) \times n$ Latin rectangle.

Extending Latin rectangles

Since every regular bipartite graph contains a perfect matching, it follows that every Latin rectangle can be extended to a Latin square.

In fact we can say a little more:

Theorem: [Frobenius-Kőnig] A non-negative $n \times n$ matrix has zero permanent if and only if it contains an $r \times s$ submatrix of zeroes, for some r, s satisfying $r + s = n + 1$.

Using this, Brualdi and Csima ['86] showed that if $k < n/2$ we can fix any $n - 2k$ entries in the next row, and still be sure of extending a $k \times n$ Latin rectangle to a $(k + 1) \times n$ Latin rectangle.

Brègman's theorem

What is the maximum permanent over Λ_n^k?

Theorem: [Brègman] If A is a $(0,1)$ matrix with row sums $r_1, r_2, \ldots, r_n$ then $\operatorname{per} A \leqslant \prod_i (r_i!)^{1/r_i}$.

Example: The maximum permanent in Λ_9^3 is achieved by

$$\begin{pmatrix} 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&1&1 \end{pmatrix}$$

The complement of Brègman's theorem

[McKay/W'98] It turns out the complement of $J_k \oplus J_k \oplus \cdots \oplus J_k$

$$\begin{pmatrix} 0_k & J_k & J_k & J_k & \cdots & J_k \\ J_k & 0_k & J_k & J_k & \cdots & J_k \\ J_k & J_k & 0_k & J_k & \cdots & J_k \\ J_k & J_k & J_k & 0_k & \cdots & J_k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ J_k & J_k & J_k & J_k & \cdots & 0_k \end{pmatrix}$$

maximises the permanent in Λ_n^{n-k} when $n = mk$ for integer m

. . . at least it does when $m = 2$ or $m \geqslant 5$.
It doesn't when $m = 3$.

Minimum permanent in Λ_n^k

Much less is known about the structure of matrices that minimise the permanent in Λ_n^k.

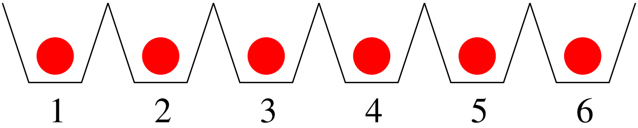
Thanks to Schrijver ['98] and Gurvits ['08], we do know

$$\operatorname{per} A \geqslant \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n$$

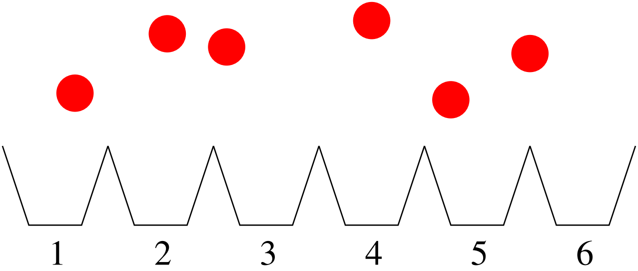
for all $A \in \Lambda_n^k$.

The base constant is best possible in the sense that

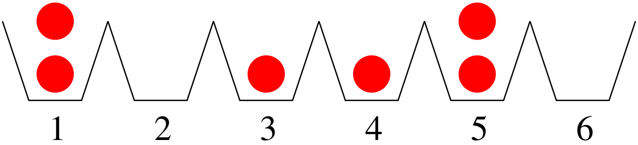
$$\lim_{n\to\infty} \big(\min_{A\in\Lambda_n^k} \operatorname{per} A\big)^{1/n} = \frac{(k-1)^{k-1}}{k^{k-2}}.$$



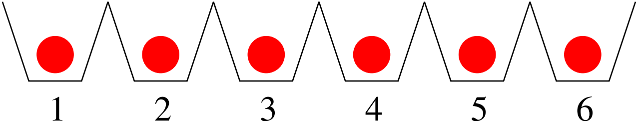
n identical balls in n labelled buckets on the back of a truck.
One ball per bucket.



The truck goes over a bump. . .



. . . then the balls fall back into the buckets.



(Or they may fall back neatly, one per bucket).

Suppose that a ball in bucket i has a probability $p_{i,j}$ of being jolted into bucket j.

$$\sum_j p_{i,j} = 1$$

The matrix $P = [p_{i,j}]$ is row stochastic.

What is the probability that the state is preserved (ie. we end up with one ball per bucket)?

The balls must have been permuted by some permutation τ.
The probability of τ occurring is $\prod_{i=1}^{n} p_{i,\tau(i)}$.
The total probability is the sum over all possible τ, (since these events are mutually exclusive).

Hence per(P) is the probability that the state is preserved.
ie. it is the "permanence" of the state.

Doubly stochastic matrices

A matrix is *row stochastic* if its entries are non-negative real numbers and each row has a total of 1. A matrix D is *doubly stochastic* if both D and D^T are row stochastic.

The set of doubly stochastic matrices of order n is traditionally denoted Ω_n.

The minimum permanent of a row stochastic matrix of order $n > 1$ is, trivially, zero.

However, finding the minimum permanent on Ω_n was a famous unsolved problem for more than 50 years.

van der Waerden's conjecture

In 1926 B. L. van der Waerden conjectured that the minimum permanent in Ω_n is achieved (uniquely) by the matrix

$$\frac{1}{n}J_n = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$$

which has permanent $n!/n^n$. This conjecture was finally solved (independently) by Egorychev and Falikman around 1980. Then in 1982 Friedland showed that $\frac{1}{n}J_n$ also minimises σ_k on Ω_n (for any k), proving a conjecture of Tverberg.

Any matrix $A \in \Lambda_n^k$ satisfies $\frac{1}{k}A \in \Omega_n$ and hence $\mathrm{per}(A) \geqslant n!(k/n)^n$.

The number of Latin squares

The number of Latin squares is only known exactly for order $n \leqslant 11$. For $n = 11$ there are 776966836171770144107444346734230682311065600000.

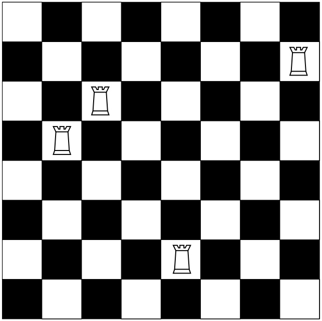
Say L_n is the number of Latin squares of order n. We can get bounds on L_n given that we know the number of extensions of a $k \times n$ Latin rectangle to a $(k+1) \times n$ Latin rectangle is the permanent of some matrix in $A \in \Lambda_n^{n-k}$. Combining the Egorychev/Falikman and Brègman bounds gives:

Theorem:

$$\frac{n!^{2n}}{n^{n^2}} \leqslant L_n \leqslant \Big(\prod_{k=1}^{n} k!^{1/k} \Big)^n$$

Time to play chess!

In how many ways can 4 non-attacking rooks be placed on the white squares of a chessboard?



Represent the chessboard as a $(0,1)$-matrix $C \in \Lambda_8^4$, with the 1's corresponding to the permissible positions (white squares).

Then $\sigma_i(C)$ counts the number of ways that i non-attacking rooks can be placed on the permissible positions. In particular $\sigma_4(C) = 8304$.

Rook polynomials

The problem can obviously be generalised in a number of ways. Any $(0,1)$-matrix can be interpreted as a 'board' with certain allowed and other prohibited positions. In such a case σ_i will always count the number of arrangements of i non-attacking rooks. It is in this context that we define the *rook polynomial* of a matrix A by

$$\rho(A,x) = \rho(A) = \sum_{i=0}^{n} (-1)^i \sigma_i(A) x^{n-i}.$$

Example: If $A \in \Lambda_n^1$ then $\sigma_i(A) = \binom{n}{i}$ so

$$\rho(A) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{n-i} = (x-1)^n.$$

Take care! The mathematical literature contains several polynomials closely related to our rook polynomial, and unfortunately some of them are also known as "rook polynomials".

Roots of rook polynomials

Suppose $A \in \Lambda_n^k$ and consider the roots $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $\rho(A)$.

If $k = 1$ then $\rho(A) = (x-1)^n$ so $\lambda_i = 1$ for each i.

For $k \geqslant 2$, Heilmann and Lieb showed that λ_i is real for each i, and

$$0 < \lambda_i < 4(k-1).$$

In particular, since the roots of the rook polynomial are real, its coefficients form a log-concave sequence. In fact

$$\frac{\sigma_i(A)}{\sigma_{i-1}(A)} \geqslant \frac{(i+1)(m-i+1)}{i(m-i)} \frac{\sigma_{i+1}(A)}{\sigma_i(A)} > \frac{\sigma_{i+1}(A)}{\sigma_i(A)}.$$

A remarkable integral

Theorem: [Joni&Rota,Godsil] For any $A \in \Lambda_n^k$,

$$\mathrm{per}(\overline{A}) = \int_0^\infty e^{-x} \rho(A)\, dx$$

where $\overline{A}$ is the (bipartite) complement of A.

Proof: This can be viewed as an inclusion-exclusion result as follows. The number of i-matchings in A is counted by $\sigma_i(A)$, and there are $(n-i)!$ diagonals of A which include any given i-matching. Now, $\mathrm{per}(\overline{A})$ is the number of diagonals of A which contain no edges at all, so

$$\mathrm{per}(\overline{A}) = n! - (n-1)!\, \sigma_1(A) + (n-2)!\, \sigma_2(A) - \ldots (-1)^n 0!\, \sigma_n(A).$$

Since $\int_0^\infty e^{-x} x^i \, dx = i!$, the right hand side turns out to be precisely $\int_0^\infty e^{-x} \rho(A)\, dx$. $\square$

Counting with permanents (summary)

There are a number of important combinatorial problems that involve counting permutations.
The permanent counts

- permutations with restricted positions
- perfect matchings in bipartite graphs
- systems of distinct representatives
- extensions to Latin rectangles
- placements of rooks
- and so on ...