

RANDOM GRAPHS

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Graph Theory Preliminaries A graph G , formally speaking, is a pair $(V(G), E(G))$ where the elements $v \in V(G)$ are called vertices and the elements of $E(G)$, called edges, are two element subsets $\{v, w\}$ of $V(G)$. When $\{v, w\} \in E(G)$ we say v, w are adjacent. (In standard graph theory terminology our graphs are undirected and have no loops and no multiple edges.) Pictorially, we often display the $v \in V(G)$ as points and draw an arc between v and w when they are adjacent. We call $V(G)$ the vertex set of G and $E(G)$ the edge set of G . (When G is understood we shall write simply V and E respectively. We also often write $v \in G$ or $\{v, w\} \in G$ instead of the formally correct $v \in V(G)$ and $\{v, w\} \in E(G)$ respectively.) A set $S \subseteq V$ is called a *clique* if all pairs $x, y \in S$ are adjacent. The clique number, denoted by $\omega(G)$, is the largest size of a clique in G . The complete graph on k vertices, denoted by K_k , consists of a vertex set of size k with all pairs x, y adjacent.

1 Lecture 1: Random Graphs

1.1 What is a Random Graph

Let n be a positive integer, $0 \leq p \leq 1$. The random graph $G(n, p)$ is a probability space over the set of graphs on the vertex set $\{1, \dots, n\}$ determined by

$$\Pr[\{i, j\} \in G] = p \tag{1}$$

with these events mutually independent.

Random Graphs is an active area of research which combines probability theory and graph theory. The subject began in 1960 with the monumental paper *On the Evolution of Random Graphs* by Paul Erdős and Alfred Rényi. The book *Random Graphs* by Béla Bollobás is the standard source for the field. The book *The Probabilistic Method* by Noga Alon and this author contains much of the material in these notes, and more.

There is a compelling dynamic model for random graphs. For all pairs i, j let $x_{i,j}$ be selected uniformly from $[0, 1]$, the choices mutually independent. Imagine p going from 0 to 1. Originally, all potential edges are “off”. The edge from i to j (which we may imagine as a neon light) is turned on when p reaches $x_{i,j}$ and then stays on. At $p = 1$ all edges

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are “on”. At time p the graph of all “on” edges has distribution $G(n, p)$. As p increases $G(n, p)$ evolves from empty to full.

In their original paper Erdős and Rényi let $G(n, e)$ be the random graph with n vertices and precisely e edges. Again there is a dynamic model: Begin with no edges and add edges randomly one by one until the graph becomes full. Generally $G(n, e)$ will have very similar properties as $G(n, p)$ with $p \sim \frac{e}{\binom{n}{2}}$. We will work on the probability model exclusively.

1.2 Threshold Functions

The term “the random graph” is, strictly speaking, a misnomer. $G(n, p)$ is a probability space over graphs. Given any graph theoretic property A there will be a probability that $G(n, p)$ satisfies A , which we write $\Pr[G(n, p) \models A]$. When A is monotone $\Pr[G(n, p) \models A]$ is a monotone function of p . As an instructive example, let A be the event “ G is triangle free”. Let X be the number of triangles contained in $G(n, p)$. Linearity of expectation gives

$$E[X] = \binom{n}{3} p^3 \tag{2}$$

This suggests the parametrization $p = c/n$. Then

$$\lim_{n \rightarrow \infty} E[X] = \lim_{n \rightarrow \infty} \binom{n}{3} p^3 = c^3/6 \tag{3}$$

We shall see that the distribution of X is asymptotically Poisson. In particular

$$\lim_{n \rightarrow \infty} \Pr[G(n, p) \models A] = \lim_{n \rightarrow \infty} \Pr[X = 0] = e^{-c^3/6} \tag{4}$$

Note that

$$\lim_{c \rightarrow 0} e^{-c^3/6} = 1 \tag{5}$$

$$\lim_{c \rightarrow \infty} e^{-c^3/6} = 0 \tag{6}$$

When $p = 10^{-6}/n$, $G(n, p)$ is very unlikely to have triangles and when $p = 10^6/n$, $G(n, p)$ is very likely to have triangles. In the dynamic view the first triangles almost always appear at $p = \Theta(1/n)$. If we take a function such as $p(n) = n^{-.9}$ with $p(n) \gg n^{-1}$ then $G(n, p)$ will almost always have triangles. Occasionally we will abuse notation and say, for example, that $G(n, n^{-.9})$ contains a triangle - this meaning that the probability that it contains a triangle approaches 1 as n approaches infinity. Similarly, when $p(n) \ll n^{-1}$, for example, $p(n) = 1/(n \ln n)$, then $G(n, p)$ will almost always not contain a triangle and we abuse notation and say that $G(n, 1/(n \ln n))$ is trianglefree. It was a central observation of Erdős and Rényi that many natural graph theoretic properties become true in a very narrow range of p . They made the following key definition.

Definition 1. $r(n)$ is called a threshold function for a graph theoretic property A if

1. When $p(n) \ll r(n)$, $\lim_{n \rightarrow \infty} \Pr[G(n, p) \models A] = 0$
2. When $p(n) \gg r(n)$, $\lim_{n \rightarrow \infty} \Pr[G(n, p) \models A] = 1$

or *visa versa*.

In our example, $1/n$ is a threshold function for A . Note that the threshold function, when one exists, is not unique. We could equally have said that $10/n$ is a threshold function for A .

Lets approach the problem of $G(n, c/n)$ being trianglefree once more. For every set S of three vertices let B_S be the event that S is a triangle. Then $\Pr[B_S] = p^3$. Then “trianglefreeness” is precisely the conjunction $\bigwedge \overline{B_S}$ over all S . If the B_S were mutually independent then we *would* have

$$\Pr[\bigwedge \overline{B_S}] = \prod [\overline{B_S}] = (1 - p^3)^{\binom{n}{3}} \sim e^{-\binom{n}{3}p^3} \rightarrow e^{-c^3/6} \quad (7)$$

The reality is that the B_S are not mutually independent though when $|S \cap T| \leq 1$, B_S and B_T are mutually independent. This is quite a typical situation in the study of random graphs in which we must deal with events that are “almost”, but not precisely, mutual independent.

1.3 Variance

Here we introduce the Variance in a form that is particularly suited to the study of random graphs. The expressions Δ and Δ^* defined in this section will appear often in these notes.

Let X be a nonnegative integral valued random variable and suppose we want to bound $\Pr[X = 0]$ given the value $\mu = E[X]$. If $\mu < 1$ we may use the inequality

$$\Pr[X > 0] \leq E[X] \quad (8)$$

so that if $E[X] \rightarrow 0$ then $X = 0$ almost always. (Here we are imagining an infinite sequence of X dependent on some parameter n going to infinity. This is the standard situation with the random graph $G(n, p(n))$.) But now suppose $E[X] \rightarrow \infty$. It does *not* necessarily follow that $X > 0$ almost always. For example, let X be the number of deaths due to nuclear war in the twelve months after reading this paragraph. Calculation of $E[X]$ can make for lively debate but few would deny that it is quite large. Yet we may believe - or hope - that $\Pr[X \neq 0]$ is very close to zero. We can sometimes deduce $X > 0$ almost always if we have further information about $Var[X]$.

Theorem 1.1.

$$\Pr[X = 0] \leq \frac{Var[X]}{E[X]^2} \quad (9)$$

Proof. Set $\lambda = \mu/\sigma$ in Chebyshev’s Inequality. Then

$$\Pr[X = 0] \leq \Pr[|X - \mu| \geq \lambda\sigma] \leq \frac{1}{\lambda^2} = \frac{\sigma^2}{\mu^2} \quad (10)$$

□

We generally apply this result in asymptotic terms.

Corollary 1.2. *If $\text{Var}[X] = o(E[X]^2)$ then $X > 0$ a.a.*

The proof of Theorem 1.3 actually gives that for any $\epsilon > 0$

$$\Pr[|X - E[X]| \geq \epsilon E[X]] \leq \frac{\text{Var}[X]}{\epsilon^2 E[X]^2} \quad (11)$$

and thus in asymptotic terms we actually have the following stronger assertion:

Corollary 1.3. *If $\text{Var}[X] = o(E[X]^2)$ then $X \sim E[X]$ a.a.*

Suppose again $X = X_1 + \dots + X_m$ where X_i is the indicator random variable for event A_i . For indices i, j write $i \sim j$ if $i \neq j$ and the events A_i, A_j are not independent. We set (the sum over ordered pairs)

$$\Delta = \sum_{i \sim j} \Pr[A_i \wedge A_j] \quad (12)$$

Note that when $i \sim j$

$$\text{Cov}[X_i, X_j] = E[X_i X_j] - E[X_i]E[X_j] \leq E[X_i X_j] = \Pr[A_i \wedge A_j] \quad (13)$$

and that when $i \neq j$ and not $i \sim j$ then $\text{Cov}[X_i, X_j] = 0$. Thus

$$\text{Var}[X] \leq E[X] + \Delta \quad (14)$$

Corollary 1.4. *If $E[X] \rightarrow \infty$ and $\Delta = o(E[X]^2)$ then $X > 0$ almost always. Furthermore $X \sim E[X]$ almost always.*

Let us say X_1, \dots, X_m are *symmetric* if for every $i \neq j$ there is an automorphism of the underlying probability space that sends event A_i to event A_j . Examples will appear in the next section. In this instance we write

$$\Delta = \sum_{i \sim j} \Pr[A_i \wedge A_j] = \sum_i \Pr[A_i] \sum_{j \sim i} \Pr[A_j | A_i] \quad (15)$$

and note that the inner summation is independent of i . We set

$$\Delta^* = \sum_{j \sim i} \Pr[A_j | A_i] \quad (16)$$

where i is any fixed index. Then

$$\Delta = \sum_i \Pr[A_i] \Delta^* = \Delta^* \sum_i \Pr[A_i] = \Delta^* E[X] \quad (17)$$

Corollary 1.5. *If $E[X] \rightarrow \infty$ and $\Delta^* = o(E[X])$ then $X > 0$ almost always. Furthermore $X \sim E[X]$ almost always.*

The condition of Corollary 1.4 has the intuitive sense that conditioning on any specific A_i holding does not substantially increase the expected number $E[X]$ of events holding.

1.4 Appearance of Small Subgraphs

What is the threshold function for the appearance of a given graph H . This problem was solved in the original papers of Erdős and Rényi. We begin with an instructive special case.

Theorem 1.6. *The property $\omega(G) \geq 4$ has threshold function $n^{-2/3}$.*

Proof. For every 4-set S of vertices in $G(n, p)$ let A_S be the event “ S is a clique” and X_S its indicator random variable. Then

$$E[X_S] = \Pr[A_S] = p^6 \tag{18}$$

as six different edges must all lie in $G(n, p)$. Set

$$X = \sum_{|S|=4} X_S \tag{19}$$

so that X is the number of 4-cliques in G and $\omega(G) \geq 4$ if and only if $X > 0$. Linearity of Expectation gives

$$E[X] = \sum_{|S|=4} E[X_S] = \binom{n}{4} p^6 \sim \frac{n^4 p^6}{24} \tag{20}$$

When $p(n) \ll n^{-2/3}$, $E[X] = o(1)$ and so $X = 0$ almost surely.

Now suppose $p(n) \gg n^{-2/3}$ so that $E[X] \rightarrow \infty$ and consider the Δ^* of Corollary 1.5. (All 4-sets “look the same” so that the X_S are symmetric.) Here $S \sim T$ if and only if $S \neq T$ and S, T have common edges - i.e., if and only if $|S \cap T| = 2$ or 3 . Fix S . There are $O(n^2)$ sets T with $|S \cap T| = 2$ and for each of these $\Pr[A_T | A_S] = p^5$. There are $O(n)$ sets T with $|S \cap T| = 3$ and for each of these $\Pr[A_T | A_S] = p^3$. Thus

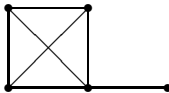
$$\Delta^* = O(n^2 p^5) + O(np^3) = o(n^4 p^6) = o(E[X])$$

since $p \gg n^{-2/3}$. Corollary 1.5 therefore applies and $X > 0$, i.e., there *does* exist a clique of size 4, almost always. \square

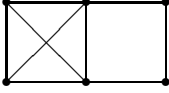
The proof of Theorem 1.6 appears to require a fortuitous calculation of Δ^* . The following definitions will allow for a description of when these calculations work out.

Definition 2. *Let H be a graph with v vertices and e edges. We call $\rho(H) = e/v$ the density of H . We call H balanced if every subgraph H' has $\rho(H') \leq \rho(H)$. We call H strictly balanced if every proper subgraph H' has $\rho(H') < \rho(H)$.*

Examples. K_4 and, in general, K_k are strictly balanced. The graph



is not balanced as it has density $7/5$ while the subgraph K_4 has density $3/2$. The graph



is balanced but not strictly balanced as it and its subgraph K_4 have density $3/2$.

Theorem 1.7. *Let H be a balanced graph with v vertices and e edges. Let $A(G)$ be the event that H is a subgraph (not necessarily induced) of G . Then $p = n^{-v/e}$ is the threshold function for A .*

Proof. We follow the argument of Theorem 1.6. For each v -set S let A_S be the event that $G|_S$ contains H as a subgraph. Then

$$p^e \leq \Pr[A_S] \leq v!p^e \quad (21)$$

(Any particular placement of H has probability p^e of occurring and there are at most $v!$ possible placements. The precise calculation of $\Pr[A_S]$ is, in general, complicated due to the overlapping of potential copies of H .) Let X_S be the indicator random variable for A_S and

$$X = \sum_{|S|=v} X_S \quad (22)$$

so that A holds if and only if $X > 0$. Linearity of Expectation gives

$$E[X] = \sum_{|S|=v} E[X_S] = \binom{n}{v} \Pr[A_S] = \Theta(n^v p^e) \quad (23)$$

If $p \ll n^{-v/e}$ then $E[X] = o(1)$ so $X = 0$ almost always.

Now assume $p \gg n^{-v/e}$ so that $E[X] \rightarrow \infty$ and consider the Δ^* of Corollary 1.5. (All v -sets look the same so the X_S are symmetric.) Here $S \sim T$ if and only if $S \neq T$ and S, T have common edges - i.e., if and only if $|S \cap T| = i$ with $2 \leq i \leq v-1$. Let S be fixed. We split

$$\Delta^* = \sum_{T \sim S} \Pr[A_T | A_S] = \sum_{i=2}^{v-1} \sum_{|T \cap S|=i} \Pr[A_T | A_S] \quad (24)$$

For each i there are $O(n^{v-i})$ choices of T . Fix S, T and consider $\Pr[A_T | A_S]$. There are $O(1)$ possible copies of H on T . Each has - since, critically, H is balanced - at most $\frac{ie}{v}$ edges with both vertices in S and thus at least $e - \frac{ie}{v}$ other edges. Hence

$$\Pr[A_T | A_S] = O(p^{e - \frac{ie}{v}}) \quad (25)$$

and

$$\Delta^* = \sum_{i=2}^{v-1} O(n^{v-i} p^{e - \frac{ie}{v}}) = \sum_{i=2}^{v-1} O((n^v p^e)^{1 - \frac{i}{v}}) \quad (26)$$

so that

$$\Delta^* = o(n^v p^e) = o(E[X]) \quad (27)$$

as $n^v p^e \rightarrow \infty$. Hence Corollary 1.5 applies. \square

Theorem 1.8. *In the notation of Theorem 1.7 if H is not balanced then $p = n^{-v/e}$ is not the threshold function for A .*

Proof. Let H_1 be a subgraph of H with v_1 vertices, e_1 edges and $e_1/v_1 > e/v$. Let α satisfy $v/e < \alpha < v_1/e_1$ and set $p = n^{-\alpha}$. The expected number of copies of H_1 is then $o(1)$ so almost always $G(n, p)$ contains no copy of H_1 . But if it contains no copy of H_1 then it surely can contain no copy of H . \square

The threshold function for the property of containing a copy of H , for general H , was examined in the original papers of Erdős and Rényi. Let H_1 be that subgraph with maximal density $\rho(H_1) = e_1/v_1$. (When H is balanced we may take $H_1 = H$.) They showed that $p = n^{-v_1/e_1}$ is the threshold function. This will follow fairly quickly from the methods of theorem 1.7.

We finish this section with two strengthenings of Theorem 1.7.

Theorem 1.9. *Let H be strictly balanced with v vertices, e edges and a automorphisms. Let X be the number of copies of H in $G(n, p)$. Assume $p \gg n^{-v/e}$. Then almost always*

$$X \sim \frac{n^v p^e}{a} \tag{28}$$

Proof. Label the vertices of H by $1, \dots, v$. For each ordered x_1, \dots, x_v let A_{x_1, \dots, x_v} be the event that x_1, \dots, x_v provides a copy of H in that order. Specifically we define

$$A_{x_1, \dots, x_v} : \{i, j\} \in E(H) \Rightarrow \{x_i, x_j\} \in E(G) \tag{29}$$

We let I_{x_1, \dots, x_v} be the corresponding indicator random variable. We define an equivalence class on v -tuples by setting $(x_1, \dots, x_v) \equiv (y_1, \dots, y_v)$ if there is an automorphism σ of $V(H)$ so that $y_{\sigma(i)} = x_i$ for $1 \leq i \leq v$. Then

$$X = \sum I_{x_1, \dots, x_v} \tag{30}$$

gives the number of copies of H in G where the sum is taken over one entry from each equivalence class. As there are $(n)_v/a$ terms

$$E[X] = \frac{(n)_v}{a} E[I_{x_1, \dots, x_v}] = \frac{(n)_v p^e}{a} \sim \frac{n^v p^e}{a} \tag{31}$$

Our assumption $p \gg n^{-v/e}$ implies $E[X] \rightarrow \infty$. It suffices therefore to show $\Delta^* = o(E[X])$. Fixing x_1, \dots, x_v ,

$$\Delta^* = \sum_{(y_1, \dots, y_v) \sim (x_1, \dots, x_v)} \Pr[A_{(y_1, \dots, y_v)} | A_{(x_1, \dots, x_v)}] \tag{32}$$

There are $v!/a = O(1)$ terms with $\{y_1, \dots, y_v\} = \{x_1, \dots, x_v\}$ and for each the conditional probability is at most one (actually, at most p), thus contributing $O(1) = o(E[X])$ to Δ^* . When $\{y_1, \dots, y_v\} \cap \{x_1, \dots, x_v\}$ has i elements, $2 \leq i \leq v - 1$ the argument of Theorem 1.7 gives that the contribution to Δ^* is $o(E[X])$. Altogether $\Delta^* = o(E[X])$ and we apply Corollary 1.5 \square

Theorem 1.10. *Let H be any fixed graph. For every subgraph H' of H (including H itself) let $X_{H'}$ denote the number of copies of H' in $G(n, p)$. Assume p is such that $E[X_{H'}] \rightarrow \infty$ for every H' . Then*

$$X_H \sim E[X_H] \tag{33}$$

almost always.

Proof. Let H have v vertices and e edges. As in Theorem 4.4 it suffices to show $\Delta^* = o(E[X])$. We split Δ^* into a finite number of terms. For each H' with w vertices and f edges we have those (y_1, \dots, y_w) that overlap with the fixed (x_1, \dots, x_v) in a copy of H' . These terms contribute, up to constants,

$$n^{v-w} p^{e-f} = \Theta\left(\frac{E[X_H]}{E[X_{H'}]}\right) = o(E[X_H])$$

to Δ^* . Hence Corollary 1.5 does apply. \square

1.5 Connectivity

In this section we give a relatively simple example of what we call the Poisson Paradigm: the rough notion that if there are many rare and nearly independent events then the number of events that hold has approximately a Poisson distribution. This will yield one of the most beautiful of the Erdős-Rényi results, a quite precise description of the threshold behavior for connectivity. A vertex $v \in G$ is *isolated* if it is adjacent to no $w \in V$. In $G(n, p)$ let X be the number of isolated vertices.

Theorem 1.11. *Let $p = p(n)$ satisfy $n(1-p)^{n-1} = \mu$. Then*

$$\lim_{n \rightarrow \infty} \Pr[X = 0] = e^{-\mu} \tag{34}$$

Proof. We let X_i be the indicator random variable for vertex i being isolated so that $X = X_1 + \dots + X_n$. Then $E[X_i] = (1-p)^{n-1}$ so by linearity of expectation $E[X] = \mu$. Now consider the r -th factorial moment $E[(X)_r]$ ($(X)_r := \prod_{i=0}^{r-1} (X-i)$) for any fixed r . By the symmetry $E[(X)_r] = (n)_r E[X_1 \cdots X_r]$. For vertices $1, \dots, r$ to all be isolated the $r(n-1) - \binom{r}{2}$ pairs $\{i, x\}$ overlapping $1, \dots, r$ must all not be edges. Thus

$$E[(X)_r] = (n)_r (1-p)^{r(n-1) - \binom{r}{2}} \sim n^r (1-p)^{r(n-1)} \sim \mu^r \tag{35}$$

(That is, the dependence among the X_i was asymptotically negligible.) All the moments of X approach those of $P(\mu)$. This implies (a nonobvious result in probability theory) that X approaches $P(\mu)$ in distribution. \square

Now we give the Erdős-Rényi famous “double exponential” result.

Theorem 1.12. *Let*

$$p = p(n) = \frac{\log n}{n} + \frac{c}{n} + o\left(\frac{1}{n}\right) \tag{36}$$

Then

$$\lim_{n \rightarrow \infty} \Pr[G(n, p) \text{ is connected}] = e^{-e^{-c}} \tag{37}$$

Proof. For such p , $n(1-p)^{n-1} \sim \mu = e^{-c}$ and by the above argument the probability that X has no isolated vertices approaches $e^{-\mu}$. If G has no isolated vertices but is not connected there is a component of k vertices for some $2 \leq k \leq \frac{n}{2}$. Letting B be this event

$$\Pr[B] \leq \sum_{k=2}^{n/2} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-1)-\binom{k}{2}} \quad (38)$$

The first factor is the choice of a component set $S \subset V(G)$. The second factor is a choice of tree on S . The third factor is the probability that those tree pairs are in $E(G)$. The final factor is that there be no edge from S to $V(G) - S$. Some calculation (which we omit but note that $k = 2$ provides the main term) gives that $\Pr[B] = o(1)$ so that $X \neq 0$ and connectivity have the same limiting probability. \square

1.6 The Janson Inequalities

In many instances we would like to bound the probability that none of a set of bad events $B_i, i \in I$ occur. If the events are mutually independent then

$$\Pr[\wedge_{i \in I} \overline{B}_i] = \prod_{i \in I} \Pr[\overline{B}_i] \quad (39)$$

When the B_i are “mostly” independent the Janson Inequalities allow us, sometimes, to say that these two quantities are “nearly” equal.

Let Ω be a finite universal set and let R be a random subset of Ω given by

$$\Pr[r \in R] = p_r, \quad (40)$$

these events mutually independent over $r \in \Omega$. (In application to $G(n, p)$, Ω is the set of pairs $\{i, j\}$, $i, j \in V(G)$ and all $p_r = p$ so that R is the edge set of $G(n, p)$.) Let $A_i, i \in I$, be subsets of Ω , I a finite index set. Let B_i be the event $A_i \subseteq R$. (That is, each point $r \in \Omega$ “flips a coin” to determine if it is in R . B_i is the event that the coins for all $r \in A_i$ came up “heads”.) Let X_i be the indicator random variable for B_i and $X = \sum_{i \in I} X_i$ the number of $A_i \subseteq R$. The event $\wedge_{i \in I} \overline{B}_i$ and $X = 0$ are then identical. For $i, j \in I$ we write $i \sim j$ if $i \neq j$ and $A_i \cap A_j \neq \emptyset$. Note that when $i \neq j$ and not $i \sim j$ then B_i, B_j are independent events since they involve separate coin flips. Furthermore, and this plays a crucial role in the proofs, if $i \notin J \subset I$ and not $i \sim j$ for all $j \in J$ then B_i is mutually independent of $\{B_j | j \in J\}$, i.e., independent of any Boolean function of those B_j . This is because the coin flips on A_i and on $\cup_{j \in J} A_j$ are independent. We define

$$\Delta = \sum_{i \sim j} \Pr[B_i \wedge B_j] \quad (41)$$

Here the sum is over ordered pairs so that $\Delta/2$ gives the same sum over unordered pairs. (This will be the same Δ as defined earlier.) We set

$$M = \prod_{i \in I} \Pr[\overline{B}_i], \quad (42)$$

the value of $\Pr[\wedge_{i \in I} \overline{B}_i]$ if the B_i were independent.

Theorem 1.13 (The Janson Inequality). *Let $B_i, i \in I, \Delta, M$ be as above and assume all $\Pr[B_i] \leq \epsilon$. Then*

$$M \leq \Pr[\wedge_{i \in I} \overline{B}_i] \leq M e^{\frac{1-\Delta}{1-\epsilon} \frac{\Delta}{2}} \quad (43)$$

Now set

$$\mu = E[X] = \sum_{i \in I} \Pr[B_i] \quad (44)$$

For each $i \in I$

$$\Pr[\overline{B}_i] = 1 - \Pr[B_i] \leq e^{-\Pr[B_i]} \quad (45)$$

so, multiplying over $i \in I$,

$$M \leq e^{-\mu} \quad (46)$$

It is often more convenient to replace the upper bound of Theorem 1.13 with

$$\Pr[\wedge_{i \in I} \overline{B}_i] \leq e^{-\mu + \frac{1-\Delta}{1-\epsilon} \frac{\Delta}{2}} \quad (47)$$

As an example, set $p = cn^{-2/3}$ and consider the probability that $G(n, p)$ contains no K_4 . The B_i then range over the $\binom{n}{4}$ potential K_4 - each being a 6-element subset of Ω . Here, as is often the case, $\epsilon = o(1)$, $\Delta = o(1)$ (as calculated previously) and μ approaches a constant, here $k = c^6/24$. In these instances $\Pr[\wedge_{i \in I} \overline{B}_i] \rightarrow e^{-k}$. Thus we have the fine structure of the threshold function of $\omega(G) = 4$.

As Δ becomes large the Janson Inequality becomes less precise. Indeed, when $\Delta \geq 2\mu(1 - \epsilon)$ it gives an upper bound for the probability which is larger than one. At that point (and even somewhat before) the following result kicks in.

Theorem 1.14. (*Generalized Janson Inequality*) *Under the assumptions of Theorem 1.13 and the further assumption that $\Delta \geq \mu(1 - \epsilon)$*

$$\Pr[\wedge_{i \in I} \overline{B}_i] \leq e^{-\frac{\mu^2(1-\epsilon)}{2\Delta}} \quad (48)$$

The Generalized Janson Inequality (when it applies) often gives a much stronger result than Chebyshev's Inequality as used earlier. We can bound $\text{Var}[X] \leq \mu + \Delta$ so that

$$\Pr[\wedge_{i \in I} \overline{B}_i] = \Pr[X = 0] \leq \frac{\text{Var}[X]}{E[X]^2} \leq \frac{\mu + \Delta}{\mu^2} \quad (49)$$

Suppose $\epsilon = o(1)$, $\mu \rightarrow \infty$, $\mu \ll \Delta$, and $\gamma = \frac{\mu^2}{\Delta} \rightarrow \infty$. Chebyshev's upper bound on $\Pr[X = 0]$ is then roughly γ^{-1} while Janson's upper bound is roughly $e^{-\gamma}$.

1.7 The Proofs

The original proofs of Janson are based on estimates of the Laplace transform of an appropriate random variable. The proof presented here follows that of Boppana and Spencer [1989]. We shall use the inequalities

$$\Pr[B_i | \wedge_{j \in J} \overline{B}_j] \leq \Pr[B_i] \quad (50)$$

valid for all index sets $J \subset I, i \notin J$ and

$$\Pr[B_i|B_k \wedge \bigwedge_{j \in J} \overline{B_j}] \leq \Pr[B_i|B_k] \quad (51)$$

valid for all index sets $J \subset I, i, k \notin J$. The first follows from general Correlation Inequalities. The second is equivalent to the first since conditioning on B_k is the same as assuming $p_r = \Pr[r \in R] = 1$ for all $r \in A_k$. We note that Janson's Inequality actually applies to any set of events B_i and relation \sim for which (50,51) apply.

Proof. (Thm. 1.13) The lower bound follows immediately. Order the index set $I = \{1, \dots, m\}$ for convenience. For $1 \leq i \leq m$

$$\Pr[B_i | \bigwedge_{1 \leq j < i} \overline{B_j}] \leq \Pr[B_i] \quad (52)$$

so

$$\Pr[\overline{B_i} | \bigwedge_{1 \leq j < i} \overline{B_j}] \geq \Pr[\overline{B_i}] \quad (53)$$

and

$$\Pr[\bigwedge_{i \in I} \overline{B_i}] = \prod_{i=1}^m \Pr[\overline{B_i} | \bigwedge_{1 \leq j < i} \overline{B_j}] \geq \prod_{i=1}^m \Pr[\overline{B_i}] \quad (54)$$

Now the upper bound. For a given i renumber, for convenience, so that $i \sim j$ for $1 \leq j \leq d$ and not for $d+1 \leq j < i$. We use the inequality $\Pr[A|B \wedge C] \geq \Pr[A \wedge B|C]$, valid for any A, B, C . With $A = B_i, B = \overline{B_1} \wedge \dots \wedge \overline{B_d}, C = \overline{B_{d+1}} \wedge \dots \wedge \overline{B_{i-1}}$

$$\Pr[B_i | \bigwedge_{1 \leq j < i} \overline{B_j}] = \Pr[A|B \wedge C] \geq \Pr[A \wedge B|C] = \Pr[A|C] \Pr[B|A \wedge C] \quad (55)$$

From the mutual independence $\Pr[A|C] = \Pr[A]$. We bound

$$\Pr[B|A \wedge C] \geq 1 - \sum_{j=1}^d \Pr[B_j|B_i \wedge C] \geq 1 - \sum_{j=1}^d \Pr[B_j|B_i] \quad (56)$$

from the Correlation Inequality. Thus

$$\Pr[B_i | \bigwedge_{1 \leq j < i} \overline{B_j}] \geq \Pr[B_i] - \sum_{j=1}^d \Pr[B_j \wedge B_i] \quad (57)$$

Reversing

$$\Pr[\overline{B_i} | \bigwedge_{1 \leq j < i} \overline{B_j}] \leq \Pr[\overline{B_i}] + \sum_{j=1}^d \Pr[B_j \wedge B_i] \quad (58)$$

$$\leq \Pr[\overline{B_i}] \left(1 + \frac{1}{1-\epsilon} \sum_{j=1}^d \Pr[B_j \wedge B_i] \right) \quad (59)$$

since $\Pr[\overline{B_i}] \geq 1 - \epsilon$. Employing the inequality $1 + x \leq e^x$,

$$\Pr[\overline{B_i} | \bigwedge_{1 \leq j < i} \overline{B_j}] \leq \Pr[\overline{B_i}] e^{\frac{1}{1-\epsilon} \sum_{j=1}^d \Pr[B_j \wedge B_i]} \quad (60)$$

For each $1 \leq i \leq m$ we plug this inequality into

$$\Pr[\wedge_{i \in I} \overline{B}_i] = \prod_{i=1}^m \Pr[\overline{B}_i \wedge \wedge_{1 \leq j < i} \overline{B}_j] \quad (61)$$

The terms $\Pr[\overline{B}_i]$ multiply to M . The exponents add: for each $i, j \in I$ with $j < i$ and $j \sim i$ the term $\Pr[B_j \wedge B_i]$ appears once so they add to $\Delta/2$. \square

Proof. (Theorem 1.14) As discussed earlier, the proof of Theorem 1.13 gives

$$\Pr[\wedge_{i \in I} \overline{B}_i] \leq e^{-\mu + \frac{1}{1-\epsilon} \frac{\Delta}{2}} \quad (62)$$

which we rewrite as

$$-\ln[\Pr[\wedge_{i \in I} \overline{B}_i]] \geq \sum_{i \in I} \Pr[B_i] - \frac{1}{2(1-\epsilon)} \sum_{i \sim j} \Pr[B_i \wedge B_j] \quad (63)$$

For any set of indices $S \subset I$ the same inequality applied only to the $B_i, i \in S$ gives

$$-\ln[\Pr[\wedge_{i \in S} \overline{B}_i]] \geq \sum_{i \in S} \Pr[B_i] - \frac{1}{2(1-\epsilon)} \sum_{i, j \in S, i \sim j} \Pr[B_i \wedge B_j] \quad (64)$$

Let now S be a random subset of I given by

$$\Pr[i \in S] = p \quad (65)$$

with p a constant to be determined, the events mutually independent. (Here we are using probabilistic methods to prove a probability theorem!)

$$E[-\ln[\Pr[\wedge_{i \in S} \overline{B}_i]]] \geq E\left[\sum_{i \in S} \Pr[B_i]\right] - \frac{1}{2(1-\epsilon)} E\left[\sum_{i, j \in S, i \sim j} \Pr[B_i \wedge B_j]\right] \quad (66)$$

Each term $\Pr[B_i]$ then appears with probability p and each term $\Pr[B_i \wedge B_j]$ with probability p^2 so that

$$E[-\ln[\Pr[\wedge_{i \in S} \overline{B}_i]]] \geq p\mu - \frac{1}{1-\epsilon} p^2 \frac{\Delta}{2} \quad (67)$$

We set

$$p = \frac{\mu(1-\epsilon)}{\Delta} \quad (68)$$

so as to maximize this quantity. The added assumption of Theorem 1.14 assures us that the probability p is at most one. Then

$$E[-\ln[\Pr[\wedge_{i \in S} \overline{B}_i]]] \geq \frac{\mu^2(1-\epsilon)}{2\Delta} \quad (69)$$

Therefore there is a specific $S \subset I$ for which

$$-\ln[\Pr[\wedge_{i \in S} \overline{B}_i]] \geq \frac{\mu^2(1-\epsilon)}{2\Delta} \quad (70)$$

That is,

$$\Pr[\wedge_{i \in S} \overline{B_i}] \leq e^{-\frac{\mu^2(1-\epsilon)}{2\Delta}} \quad (71)$$

But

$$\Pr[\wedge_{i \in I} \overline{B_i}] \leq \Pr[\wedge_{i \in S} \overline{B_i}] \quad (72)$$

completing the proof. \square

1.8 Appearance of Small Subgraphs Revisited

Generalizing the fine threshold behavior for the appearance of K_4 we find the fine threshold behavior for the appearance of any strictly balanced graph H .

Theorem 1.15. *Let H be a strictly balanced graph with v vertices, e edges and a automorphisms. Let $c > 0$ be arbitrary. Let A be the property that G contains no copy of H . Then with $p = cn^{-v/e}$,*

$$\lim_{n \rightarrow \infty} \Pr[G(n, p) \models A] = \exp[-c^e/a] \quad (73)$$

Proof. Let $A_\alpha, 1 \leq \alpha \leq \binom{n}{v}v!/a$, range over the edge sets of possible copies of H and let B_α be the event $G(n, p) \supseteq A_\alpha$. We apply Janson's Inequality. As

$$\lim_{n \rightarrow \infty} \mu = \lim_{n \rightarrow \infty} \binom{n}{v}v!p^e/a = c^e/a \quad (74)$$

we find

$$\lim_{n \rightarrow \infty} M = \exp[-c^e/a] \quad (75)$$

Now we examine (similar to Theorem 1.7)

$$\Delta = \sum_{\alpha \sim \beta} \Pr[B_\alpha \wedge B_\beta] \quad (76)$$

We split the sum according to the number of *vertices* in the intersection of copies α and β . Suppose they intersect in j vertices. If $j = 0$ or $j = 1$ then $A_\alpha \cap A_\beta = \emptyset$ so that $\alpha \sim \beta$ cannot occur. For $2 \leq j \leq v$ let f_j be the maximal $|A_\alpha \cap A_\beta|$ where $\alpha \sim \beta$ and α, β intersect in j vertices. As $\alpha \neq \beta$, $f_v < e$. When $2 \leq j \leq v - 1$ the critical observation is that $A_\alpha \cap A_\beta$ is a subgraph of H and hence, as H is strictly balanced,

$$\frac{f_j}{j} < \frac{e}{v} \quad (77)$$

There are $O(n^{2v-j})$ choices of α, β intersecting in j points since α, β are determined, except for order, by $2v - j$ points. For each such α, β

$$\Pr[B_\alpha \wedge B_\beta] = p^{|A_\alpha \cup A_\beta|} = p^{2e - |A_\alpha \cap A_\beta|} \leq p^{2e - f_j} \quad (78)$$

Thus

$$\Delta = \sum_{j=2}^v O(n^{2v-j})O(n^{-\frac{v}{e}(2e-f_j)}) \quad (79)$$

But

$$2v - j - \frac{v}{e}(2e - f_j) = \frac{vf_j}{e} - j < 0 \quad (80)$$

so each term is $o(1)$ and hence $\Delta = o(1)$. By Janson's Inequality

$$\lim_{n \rightarrow \infty} \Pr[\wedge \overline{B_\alpha}] = \lim_{n \rightarrow \infty} M = \exp[-c^\epsilon/a] \quad (81)$$

completing the proof. \square

The fine threshold behavior for the appearance of an arbitrary graph H has been worked out but it can get quite complicated.

1.9 Some Very Low Probabilities

Let A be the property that G does not contain K_4 and consider $\Pr[G(n, p) \models A]$ as p varies. (Results with K_4 replaced by an arbitrary H are discussed at the end of this section.) We know that $p = n^{-2/3}$ is a threshold function so that for $p \gg n^{-2/3}$ this probability is $o(1)$. Here we want to estimate that probability. Our estimates here will be quite rough, only up to a $o(1)$ additive factor in the hyperexponent, though with more care the bounds differ by “only” a constant factor in the exponent. If we were to consider all potential K_4 as giving mutually independent events then we would be led to the estimate $(1 - p^6)^{\binom{n}{4}} = e^{-n^{4+o(1)}p^6}$. For p appropriately small this turns out to be correct. But for, say, $p = \frac{1}{2}$ it would give the estimate $e^{-n^{4+o(1)}}$. This must, however, be way off the mark since with probability $2^{-\binom{n}{2}} = e^{-n^{2+o(1)}}$ the graph G could be empty and hence trivially satisfy A .

Rather than giving the full generality we assume $p = n^{-\alpha}$ with $\frac{2}{3} > \alpha > 0$. The result is:

$$\Pr[G(n, p) \models A] = e^{-n^{4-6\alpha+o(1)}} \quad (82)$$

for $\frac{2}{3} > \alpha \geq \frac{2}{5}$ and

$$\Pr[G(n, p) \models A] = e^{-n^{2-\alpha+o(1)}} \quad (83)$$

for $\frac{2}{5} \geq \alpha > 0$.

The upper bound follows from the inequality

$$\Pr[G(n, p) \models A] \geq \max \left[(1 - p^6)^{\binom{n}{4}}, (1 - p)^{\binom{n}{2}} \right] \quad (84)$$

This is actually two inequalities. The first comes from the probability of G not containing a K_4 being at most the probability as if all the potential K_4 were independent. The second is the same bound on the probability that G doesn't contain a K_2 - i.e., that G has no edges. Calculation shows that the “turnover” point for the two inequalities occurs when $p = n^{-2/5+o(1)}$.

The upper bound follows from the Janson inequalities. For each four set α of vertices B_α is that that 4-set gives a K_4 and we want $\Pr[\wedge \overline{B_\alpha}]$. We have $\mu = \Theta(n^4 p^6)$ and $-\ln M \sim \mu$ and (as shown earlier) $\Delta = \Theta(\mu \Delta^*)$ with $\Delta^* = \Theta(n^2 p^5 + np^3)$. With $p = n^{-\alpha}$ and $\frac{2}{3} > \alpha > \frac{2}{5}$ we have $\Delta^* = o(1)$ so that

$$\Pr[\wedge \overline{B_\alpha}] \leq e^{-\mu(1+o(1))} = e^{-n^{4-6\alpha+o(1)}}$$

When $\frac{2}{5} > \alpha > 0$ then $\Delta^* = \Theta(n^2 p^5)$ (somewhat surprisingly the np^3 never is significant in these calculations) and the extended Janson inequality gives

$$\Pr[\wedge \overline{B_\alpha}] \leq e^{-\Theta(\mu^2/\Delta)} = e^{-\Theta(\mu/\Delta^*)} = e^{-n^{2-\alpha}}$$

The general result has been found by T. Łuczak, A. Ruciński and S. Janson (1990). Let H be any fixed graph and let A be the property of not containing a copy of H . For any subgraph H' of H the correlation inequality gives

$$\Pr[G(n, p) \models A] \leq e^{-E[X_{H'}]} \tag{85}$$

where $X_{H'}$ is the number of copies of H' in G . Now let $p = n^{-\alpha}$ where we restrict to those α for which p is past the threshold function for the appearance of H . Then

$$\Pr[G(n, p) \models A] = e^{n^{o(1)}} \min_{H'} e^{-E[X_{H'}]} \tag{86}$$

2 Lecture II: The Phase Transition

2.1 Branching Processes

Paul Erdős and Alfred Rényi, in their original 1960 paper, discovered that the random graph $G(n, p)$ undergoes a remarkable change at $p = 1/n$. Speaking roughly, let first $p = c/n$ with $c < 1$. Then $G(n, p)$ will consist of small components, the largest of which is of size $\Theta(\ln n)$. But now suppose $p = c/n$ with $c > 1$. In that short amount of “time” many of the components will have joined together to form a “giant component” of size $\Theta(n)$. The remaining vertices are still in small components, the largest of which has size $\Theta(\ln n)$. They dubbed this phenomenon the *Double Jump*. We prefer the descriptive term Phase Transition because of the connections to percolation.

To better understand the Phase Transition we make a lengthy detour into the subject of Branching Processes. Imagine that we are in a unisexual universe and we start with a single organism. Imagine that this organism has a number of children given by a given random variable Z . (For us, Z will be Poisson with mean c .) These children then themselves have children, the number again being determined by Z . These grandchildren then have children, etc. As $Z = 0$ will have nonzero probability there will be some chance that the line dies out entirely. We want to study the total number of organisms in this process, with particular eye to whether or not the process continues forever.

Now let's be more precise. Let Z_1, Z_2, \dots be independent random variables, each with distribution Z . Define Y_0, Y_1, \dots by initial value $Y_0 = 1$ and the recursion

$$Y_i = Y_{i-1} + Z_i - 1 \tag{87}$$

and let T be the least t for which $Y_t = 0$. If no such t exists (the line continuing forever) we say $T = +\infty$. The Y_i and Z_i mirror the Branching Process as follows. We view all organisms as living or dead. Initially there is one live organism and no dead ones. At each time unit we select one of the live organisms, it has Z_i children, and then it dies. The number Y_i of live organisms at time i is then given by the recursion. The process

stops when $Y_t = 0$ (extinction) but it is a convenient fiction to define the recursion for all t . Note that T is not affected by this fiction since once $Y_t = 0$, T has been defined. T (whether finite or infinite) is the total number of organisms, including the original, in this process. (A natural approach, found in many probability texts, is to have all organisms of a given generation have their children at once and study the number of children of each generation. While we may think of the organisms giving birth by generation it will not affect our model.)

We shall use the major result of Branching Processes that when $E[Z] = c < 1$ with probability one the process dies out ($T < \infty$) but when $E[Z] = c > 1$ then there is a nonzero probability that the process goes on forever ($T = \infty$).

When a branching process dies we call $H = (Z_1, \dots, Z_T)$ the *history* of the process. A sequence (z_1, \dots, z_t) is a possible history if and only if the sequence y_i given by $y_0 = 1$, $y_i = y_{i-1} + z_i - 1$ has $y_i > 0$ for $0 \leq i < t$ and $y_t = 0$. When Z is Poisson with mean λ

$$\Pr[H = (z_1, \dots, z_t)] = \prod_{i=1}^t \frac{e^{-\lambda} \lambda^{z_i}}{z_i!} = \frac{e^{-\lambda} (\lambda e^{-\lambda})^{t-1}}{\prod_{i=1}^t z_i!} \quad (88)$$

since $z_1 + \dots + z_t = t - 1$.

Definition 3. We call $d < 1 < c$ a conjugate pair if

$$de^{-d} = ce^{-c} \quad (89)$$

The function $f(x) = xe^{-x}$ increases from 0 to e^{-1} in $[0, 1)$ and decreases back to 0 in $(1, \infty)$ so that all $c \neq 1$ have a unique conjugate. Let $c > 1$ and $y = \Pr[T < \infty]$ so that $y = e^{c(y-1)}$. Then $(cy)e^{-cy} = ce^{-c}$ so

$$d = cy \quad (90)$$

For future use we note that if $c = 1 + \epsilon$ then its dual d satisfies

$$d = 1 - \epsilon + O(\epsilon^2) \quad (91)$$

as $\epsilon \rightarrow 0^+$.

Theorem 2.1. (*Duality Principle*) Let $d < 1 < c$ be conjugates. The Branching Process with mean c , conditional on extinction, has the same distribution as the Branching Process with mean d .

Proof. It suffices to show that for every history $H = (z_1, \dots, z_t)$

$$\frac{e^{-c}(ce^{-c})^{t-1}}{y \prod_{i=1}^t z_i!} = \frac{e^{-d}(de^{-d})^{t-1}}{\prod_{i=1}^t z_i!} \quad (92)$$

This is immediate as $ce^{-c} = de^{-d}$ and $ye^{-d} = ye^{-cy} = e^{-c}$. □

2.2 The Giant Component

Now let's return to random graphs. We define a procedure to find the component $C(v)$ containing a given vertex v in a given graph G . We are motivated by Karp [1990] in which this approach is applied to random digraphs. In this procedure vertices will be live, dead or neutral. Originally v is live and all other vertices are neutral, time $t = 0$ and $Y_0 = 1$. Each time unit t we take a live vertex w and check all pairs $\{w, w'\}$, w' neutral, for membership in G . If $\{w, w'\} \in G$ we make w' live, otherwise it stays neutral. After searching all neutral w' we set w dead and let Y_t equal the new number of live vertices. When there are no live vertices the process terminates and $C(v)$ is the set of dead vertices. Let Z_t be the number of w' with $\{w, w'\} \in G$ so that $Y_0 = 1$ and

$$Y_t = Y_{t-1} + Z_t - 1 \quad (93)$$

With $G = G(n, p)$ each neutral w' has independent probability p of becoming live. Here, critically, no pair $\{w, w'\}$ is ever examined twice so that the conditional probability for $\{w, w'\} \in G$ is always p . As $t - 1$ vertices are dead and Y_{t-1} are live

$$Z_t \sim B[n - (t - 1) - Y_{t-1}, p] \quad (94)$$

Let T be the least t for which $Y_t = 0$. Then $T = |C(v)|$. As in Section 2.1 we continue the recursive definition of Y_t , this time for $0 \leq t \leq n$.

Theorem 2.2. *For all t*

$$Y_t \sim B[n - 1, 1 - (1 - p)^t] + 1 - t \quad (95)$$

Proof. It is more convenient to deal with

$$N_t = n - t - Y_t \quad (96)$$

the number of neutral vertices at time t and show, equivalently,

$$N_t \sim B[n - 1, (1 - p)^t] \quad (97)$$

This is reasonable since each $w \neq v$ has independent probability $(1 - p)^t$ of staying neutral t times. Formally, as $N_0 = n - 1$ and

$$\begin{aligned} N_t &= n - t - Y_t &= n - t - B[n - (t - 1) - Y_{t-1}, p] - Y_{t-1} + 1 \\ &= N_{t-1} - B[N_{t-1}, p] \\ &= B[N_{t-1}, 1 - p] \end{aligned}$$

the result follows by induction. □

We set $p = c/n$. When t and Y_{t-1} are small we may approximate Z_t by $B[n, c/n]$ which is approximately Poisson with mean c . Basically small components will have size distribution as in the Branching Process. The analogy must break down for $c > 1$ as the Branching Process may have an infinite population whereas $|C(v)|$ is surely at most n .

Essentially, those v for which the Branching Process for $C(v)$ does not “die early” all join together to form the giant component.

Fix c . Let $Y_0^*, Y_1^*, \dots, T^*, Z_1^*, Z_2^*, \dots, H^*$ refer to the Branching Process and let the unstarred $Y_0, Y_1, \dots, T, Z_1, Z_2, \dots, H$ refer to the Random Graph process. For any possible history (z_1, \dots, z_t)

$$\Pr[H^* = (z_1, \dots, z_t)] = \prod_{i=1}^t \Pr[Z^* = z_i] \quad (98)$$

where Z^* is Poisson with mean c while

$$\Pr[H = (z_1, \dots, z_t)] = \prod_{i=1}^t \Pr[Z_i = z_i] \quad (99)$$

where Z_i has Binomial Distribution $B[n-1-z_1-\dots-z_{i-1}, c/n]$. The Poisson distribution is the limiting distribution of Binomials. When $m = m(n) \sim n$ and c, i are fixed

$$\lim_{n \rightarrow \infty} \Pr[B[m, c/n] = i] = \lim_{n \rightarrow \infty} \binom{m}{z} \left(\frac{c}{n}\right)^z \left(1 - \frac{c}{n}\right)^{m-z} = e^{-c} c^z / z! \quad (100)$$

hence

$$\lim_{n \rightarrow \infty} \Pr[H = (z_1, \dots, z_t)] = \Pr[H^* = (z_1, \dots, z_t)] \quad (101)$$

Assume $c < 1$. For any fixed t , $\lim_{n \rightarrow \infty} \Pr[T = t] = \Pr[T^* = t]$. We now bound the size of the largest component. For any t

$$\Pr[T > t] \leq \Pr[Y_t > 0] = \Pr[B[n-1, 1 - (1-p)^t] \geq t] \leq \Pr[B[n, tc/n] \geq t] \quad (102)$$

as $1 - (1-p)^t \leq tp$ and $n-1 < n$. By Large Deviation Results

$$\Pr[T > t] < e^{-\alpha t} \quad (103)$$

where $\alpha = \alpha(c) > 0$. Let $\beta = \beta(c)$ satisfy $\alpha\beta > 1$. Then

$$\Pr[T > \beta \ln n] < n^{-\alpha\beta} = o(n^{-1}) \quad (104)$$

There are n choices for initial vertex v . Thus almost always *all* components have size $O(\ln n)$.

Now assume $c > 1$. For any fixed t , $\lim_{n \rightarrow \infty} \Pr[T = t] = \Pr[T^* = t]$ but what corresponds to $T^* = \infty$? For $t = o(n)$ we may estimate $1 - (1-p)^t \sim pt$ and $n-1 \sim n$ so that

$$\Pr[Y_t \leq 0] = \Pr[B[n-1, 1 - (1-p)^t] \leq t-1] \sim \Pr[B[n, tc/n] \leq t] \quad (105)$$

drops exponentially in t by Large Deviation results. When $t = \alpha n$ we estimate $1 - (1-p)^t$ by $1 - e^{-c\alpha}$. The equation $1 - e^{-c\alpha} = \alpha$ has solution $\alpha = 1 - y$ where y is the extinction probability. For $\alpha < 1 - y$, $1 - e^{-c\alpha} > \alpha$ and

$$\Pr[Y_t \leq 0] \sim \Pr[B[n, 1 - e^{-c\alpha}] \leq \alpha n] \quad (106)$$

is exponentially small while for $\alpha > 1 - y$, $1 - e^{-c\alpha} < \alpha$ and $\Pr[Y_t \leq 0] \sim 1$. Thus almost always $Y_t = 0$ for some $t \sim (1 - y)n$. Basically, $T^* = \infty$ corresponds to $T \sim (1 - y)n$. Let $\epsilon, \delta > 0$ be arbitrarily small. With somewhat more care to the bounds we may show that there exists t_0 so that for n sufficiently large

$$\Pr[t_0 < T < (1 - \delta)n(1 - y) \text{ or } T > (1 + \delta)n(1 - y)] < \epsilon \quad (107)$$

Pick t_0 sufficiently large so that

$$y - \epsilon \leq \Pr[T^* \leq t_0] \leq y \quad (108)$$

Then as $\lim_{n \rightarrow \infty} \Pr[T \leq t_0] = \Pr[T^* \leq 0]$ for n sufficiently large

$$y - 2\epsilon \leq \Pr[T \leq t_0] \leq y + \epsilon \quad (109)$$

$$1 - y - 2\epsilon \leq \Pr[(1 - \delta)n(1 - y) < T < (1 + \delta)n(1 - y)] < 1 - y + 3\epsilon \quad (110)$$

Now we expand our procedure to find graph components. We start with $G \sim G(n, p)$, select $v = v_1 \in G$ and compute $C(v_1)$ as before. Then we delete $C(v_1)$, pick $v_2 \in G - C(v_1)$ and iterate. At each stage the remaining graph has distribution $G(m, p)$ where m is the number of vertices. (Note, critically, that no pairs $\{w, w'\}$ in the remaining graph have been examined and so it retains its distribution.) Call a component $C(v)$ small if $|C(v)| \leq t_0$, giant if $(1 - \delta)n(1 - y) < |C(v)| < (1 + \delta)n(1 - y)$ and otherwise failure. Pick $s = s(\epsilon)$ with $(y + \epsilon)^s < \epsilon$. (For ϵ small $s \sim K \ln \epsilon^{-1}$.) Begin this procedure with the full graph and terminate it when either a giant component or a failure component is found or when s small components are found. At each stage, as only small components have thus far been found, the number of remaining points is $m = n - O(1) \sim n$ so the conditional probabilities of small, giant and failure remain asymptotically the same. The chance of ever hitting a failure component is thus $\leq s\epsilon$ and the chance of hitting all small components is $\leq (y + \epsilon)^s \leq \epsilon$ so that with probability at least $1 - \epsilon'$, where $\epsilon' = (s + 1)\epsilon$ may be made arbitrarily small, we find a series of less than s small components followed by a giant component. The remaining graph has $m \sim yn$ points and $pm \sim cy = d$, the conjugate of c as defined earlier. As $d < 1$ the previous analysis gives the maximal components. In summary: almost always $G(n, c/n)$ has a giant component of size $\sim (1 - y)n$ and all other components of size $O(\ln n)$. Furthermore, the Duality Principle has a discrete analog.

Theorem 2.3. (*Discrete Duality Principle*) *Let $d < 1 < c$ be conjugates. The structure of $G(n, c/n)$ with its giant component removed is basically that of $G(m, d/m)$ where m , the number of vertices not in the giant component, satisfies $m \sim ny$.*

Well, this is not a precisely stated theorem – but the concept is quite compelling.

2.3 A Static View

The small components of $G(n, c/n)$ can also be examined from a static view. For a fixed k let X be the number of tree components of size k . Then

$$E[X] = \binom{n}{k} k^{k-2} p^{k-1} (1 - p)^{k(n-k) + \binom{k}{2} - (k-1)} \quad (111)$$

Here we use the nontrivial fact, due to Cayley, that there are k^{k-2} possible trees on a given k -set. For c, k fixed

$$E[X] \sim n \frac{e^{-ck} k^{k-2} c^{k-1}}{k!} \quad (112)$$

As trees are strictly balanced a second moment method gives $X \sim E[X]$ almost always. Thus $\sim p_k n$ points lie in tree components of size k where

$$p_k = \frac{e^{-ck} (ck)^{k-1}}{k!} \quad (113)$$

It can be shown analytically that $p_k = \Pr[T = k]$ in the Branching Process with mean c . Let Y_k denote the number of cycles of size k and Y the total number of cycles. Then

$$E[Y_k] = \frac{(n)_k}{2k} \left(\frac{c}{n}\right)^k \sim \frac{c^k}{2k} \quad (114)$$

for fixed k . For $c < 1$

$$E[Y] = \sum E[Y_k] \rightarrow \sum_{k=1}^{\infty} \frac{c^k}{2k} \quad (115)$$

has a finite limit whereas for $c > 1$, $E[Y] \rightarrow \infty$. Even for $c > 1$ for any fixed k the number of k -cycles has a limiting expectation and so do not asymptotically affect the number of components of a given size.

3 Lecture III: Inside the Phase Transition

In the evolution of the random graph $G(n, p)$ a crucial change takes place in the vicinity of $p = c/n$ with $c = 1$. The small components at that time are rapidly joining together to form a giant component. This corresponds to the Branching Process when births are Poisson with mean 1. There the number T of organisms will be finite almost always and yet have infinite expectation. No wonder that the situation for random graphs is extremely delicate. In recent years there has been much interest in looking “inside” the phase transition at the growth of the largest components. The appropriate parametrization is, perhaps surprisingly,

$$p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$$

3.1 The Big Picture

Let us assume $p \sim \frac{1}{n}$ to avoid trivialities. Also, a component is called *simple* if it is either a tree or unicyclic, otherwise it is called *complex*.

1. $\lambda \rightarrow -\infty$ PreCritical
 - (a) The largest components are all asymptotically the same size.
 - (b) All components have size $o(n^{2/3})$.
 - (c) All components are simple.

2. $\lambda \rightarrow +\infty$ PostCritical
 - (a) The largest component is bigger than the second largest component by any fixed factor. We call it the *dominant component*.
 - (b) The dominant component is complex.
 - (c) With the dominant component removed the remaining graph is like the Pre-Critical phase, given by replacing λ by $-\lambda$.
 - (d) The dominant component has size $\gg n^{2/3}$, all others have size $o(n^{2/3})$.
3. $-\infty < \lambda < \lambda$ constant. The Critical Window. Inside the Phase Transition.
 - (a) Chaotic Behavior. Most interesting questions have a distribution, rather than an almost sure, answer.
 - (b) The ratio U of largest to second largest component size has a continuous distribution with support the reals $[1, \infty)$.
 - (c) The largest components have size $\Theta(n^{2/3})$. The constant term has a continuous distribution on the positive reals.
 - (d) The value of edges minus vertices plus one (zero for trees, one for unicyclic, etc.) can be any integer for the largest (or t -th largest, t fixed) component with positive limiting probability.

We hasten to add that there is no finer scaling than that given by the critical window. In other words, the value $\lambda = 0$ is not special, the values $\lambda = +3.1987$ or $\lambda = -23.4019284$ have equal validity to being consider the critical point. Indeed, we do not speak of a critical value but rather of a critical window.

3.2 The Dominant Component

Set $p = \frac{1+\epsilon}{n}$ with $\epsilon = \epsilon(n) \gg n^{-1/3}$. (In our parametrization this means $\lambda \rightarrow +\infty$, the postcritical phase.) We assume $\epsilon = o(1)$ to avoid trivialities. We follow the arguments and notation of Section 2.2 on the Giant Component. Our object is to show that almost surely there will be a dominant component of size $\sim 2\epsilon n$. We give it only in outline form. We start with a pure branching process result that seems to encapsulate the core ideas.

Theorem 3.1. *Let $\epsilon > 0$. Let T^1, T^2, \dots be independent trials, each with distribution T_ϵ^* . Let J be the first integer with $T^J = +\infty$ (with probability one there will be one) and set $Y = \sum_{i < J} T^i$. (If $J = 1$ set $Y = 0$.) Then, asymptotically as $\epsilon \rightarrow 0^+$*

$$E[Y] \sim \frac{1}{2}\epsilon^{-2}$$

Proof. As $\Pr[T^i = +\infty] \sim 2\epsilon$, $E[J] \sim 1/(2\epsilon)$. Conditioning on $T^i < \infty$ gives T^i the distribution $T_{-\epsilon}^*$ by duality (using (91)) and so the conditional expectation of T^i is ϵ^{-1} . \square

We generate the component of a vertex v with a sequence Y_t , the number of live vertices at time t , with component size T equal the minimal t with $Y_t = 0$. Call the component small if $T = O(\epsilon^{-2})$ and dominant if $T \sim 2\epsilon n$. Equation (95) still holds. So

$$E[Y_t] = (n-1)(1-(1-p)^t) + 1 - t \sim n(1-(1-p)^t) - t \quad (116)$$

Here $1-(1-p)^t = pt - \frac{1}{2}p^2t^2 + O(p^3t^3)$ so that

$$n(1-(1-p)^t) - t = t\epsilon - \frac{1}{2}t^2n^{-1} + O(t^3n^{-2}) \quad (117)$$

The third term is negligible and this is zero for $t \sim 2n\epsilon$. One gets that with probability $1 - o(\epsilon)$ we have $Y_t > 0$ in the range $\epsilon^{-2} \ll t < 2\epsilon n(1 - o(1))$ and $Y_t < 0$ in the range $t > 2\epsilon n(1 + o(1))$. So with probability $1 - o(\epsilon)$ the component size is either small or dominant. The probability of being small is like the probability that the Poisson process is small, which is like the probability that it is finite, which is $1 - (2 + o(1))\epsilon$. The probability of being dominant is then $\sim 2\epsilon$. That is, T has distribution like T_ϵ^* over the small range, but it also has a “spike” probability of $\sim 2\epsilon$ for $T \sim 2\epsilon n$, and this corresponds to the spike at infinity of T_ϵ^* .

We continue to run the experiment, generating components, until we reach a dominant one. But now (in a somewhat circular argument), have we used up so many vertices that our original arguments are invalid. From Theorem 3.1 the number of vertices used up will be $O(\epsilon^{-2})$. At that point $n \leftarrow n - O(\epsilon^{-2})$ or, equivalently, $\epsilon \leftarrow \epsilon - O(\epsilon^{-2}n^{-1})$. Here we make critical use of $\epsilon \gg n^{-1/3}$ as that implies that $\epsilon^{-2}n^{-1} = o(\epsilon)$ so that ϵ has not substantially changed.

Once we receive a dominant component we have $n \leftarrow n(1 - 2\epsilon)$ so that $\epsilon \leftarrow -\epsilon$ and we are now in the precritical phase. In Section 3.8 we see that even inside the critical window the largest component has size $O(n^{3/2})$, thus that is true earlier in the precritical phase. (The size of the largest component in the precritical phase has been well studied – it is actually of size $o(n^{2/3})$ and asymptotic formulae can be given in terms of ϵ .) As $\epsilon \gg n^{-1/3}$ the dominant component of size $2\epsilon n \gg n^{2/3}$ is indeed much larger than all other components.

3.3 A Delicate Calculation

Let's fix λ, c and let X be the number of tree components of size $k = cn^{2/3}$. Then

$$E[X] = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k) + \binom{k}{2} - (k-1)}$$

Watch the terms cancel!

$$\binom{n}{k} = \frac{(n)_k}{k!} \sim \frac{n^k e^k}{k^k \sqrt{2\pi k}} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right)$$

For $i < k$

$$-\ln\left(1 - \frac{i}{n}\right) = \frac{i}{n} + \frac{i^2}{2n^2} + O\left(\frac{i^3}{n^3}\right)$$

so that

$$\sum_{i=1}^{k-1} -\ln\left(1 - \frac{i}{n}\right) = \frac{k^2}{2n} + \frac{k^3}{6n^2} + o(1) = \frac{k^2}{2n} + \frac{c^3}{6} + o(1)$$

Also

$$p^{k-1} = n^{1-k} \left(1 + \frac{\lambda}{n^{1/3}}\right)^{k-1}$$

$$(k-1) \ln\left(1 + \frac{\lambda}{n^{1/3}}\right) = (k-1) \left(\frac{\lambda}{n^{1/3}} - \frac{\lambda^2}{2n^{2/3}} + O(n^{-1})\right) = \frac{\lambda k}{n^{1/3}} - \frac{\lambda^2 c}{2} + o(1)$$

Also

$$\ln(1-p) = -p + O(n^{-2}) = -\frac{1}{n} - \frac{\lambda}{n^{4/3}} + O(n^{-2})$$

and

$$k(n-k) + \binom{k}{2} - (k-1) = kn - \frac{k^2}{2} + O(n^{2/3})$$

so that

$$[k(n-k) + \binom{k}{2} - (k-1)] \ln(1-p) = -k + \frac{k^2}{2n} - \frac{\lambda k}{n^{1/3}} + \frac{\lambda c^2}{2} + o(1)$$

and

$$E[X] \sim \frac{n^k k^{k-2}}{k^k \sqrt{2\pi k n^{k-1}}} e^A$$

where

$$\begin{aligned} A &= k - \frac{k^2}{2n} - \frac{c^3}{6} + \frac{\lambda k}{n^{1/3}} - \frac{\lambda^2 c}{2} - k + \frac{k^2}{2n} - \frac{\lambda k}{n^{1/3}} + \frac{\lambda c^2}{2} + o(1) \\ &= -\frac{c^3}{6} - \frac{\lambda^2 c}{2} + \frac{\lambda c^2}{2} + o(1) \end{aligned}$$

so that

$$E[X] \sim n^{-2/3} e^{-\frac{c^3}{6} - \frac{\lambda^2 c}{2} + \frac{\lambda c^2}{2}} c^{-5/2} (2\pi)^{-1/2}$$

For any particular such k $E[X] \rightarrow 0$ but if we sum k between $cn^{2/3}$ and $(c+dc)n^{2/3}$ we multiply by $n^{2/3}dc$. Going to the limit gives an integral: For any fixed a, b, λ let X be the number of tree components of size between $an^{2/3}$ and $bn^{2/3}$. Then

$$\lim_{n \rightarrow \infty} E[X] = \int_a^b e^{-\frac{c^3}{6} - \frac{\lambda^2 c}{2} + \frac{\lambda c^2}{2}} c^{-5/2} (2\pi)^{-1/2} dc$$

The large components are not all trees. E.M. Wright [1977] proved that for fixed l there are asymptotically $c_l k^{k-2 + \frac{3}{2}l}$ connected graphs on k points with $k-1+l$ edges, where c_l was given by a specific recurrence. Asymptotically in l ,

$$c_l \sim \left(\frac{e}{12l}(1+o(1))\right)^{l/2}.$$

The calculation for $X^{(l)}$, the number of such components on k vertices, leads to extra factors of $c_l k^{\frac{3}{2}l}$ and n^{-l} which gives $c_l c^{\frac{3}{2}l}$. For fixed a, b, λ, l the number $X^{(l)}$ of components of size between $an^{2/3}$ and $bn^{2/3}$ with $l - 1$ more edges than vertices satisfies

$$\lim_{n \rightarrow \infty} E[X^{(l)}] = \int_a^b e^{-\frac{c^3}{6} - \frac{\lambda^2 c}{2} + \frac{\lambda c^2}{2}} c^{-5/2} (2\pi)^{-1/2} (c_l c^{\frac{3}{2}l}) dc$$

and letting X^* be the total number of components of size between $an^{2/3}$ and $bn^{2/3}$

$$\lim_{n \rightarrow \infty} E[X^*] = \int_a^b e^{-\frac{c^3}{6} - \frac{\lambda^2 c}{2} + \frac{\lambda c^2}{2}} c^{-5/2} (2\pi)^{-1/2} g(c) dc$$

where

$$g(c) = \sum_{l=0}^{\infty} c_l c^{\frac{3}{2}l}$$

a sum convergent for all c , (here $c_0 = 1$). A component of size $\sim cn^{2/3}$ will have probability $c_l c^{\frac{3}{2}l} / g(c)$ of having $l - 1$ more edges than vertices, independent of λ . As $\lim_{c \rightarrow 0} g(c) = 1$, most components of size $\epsilon n^{2/3}$, $\epsilon \ll 1$, are trees but as c gets bigger the distribution on l moves inexorably higher.

3.4 The Emergence of Nation States

For any fixed λ the sizes of the largest components are of the form $cn^{2/3}$ with a distribution over the constant. For $\lambda = -10^6$ there is some positive limiting probability that the largest component is bigger than $10^6 n^{2/3}$ and for $\lambda = +10^6$ there is some positive limiting probability that the largest component is smaller than $10^{-6} n^{2/3}$, though both these probabilities are minuscule. The functions integrated have a pole at $c = 0$, reflecting the notion that for any λ there should be many components of size near $\epsilon n^{2/3}$ for $\epsilon = \epsilon(\lambda)$ appropriately small. When λ is large negative (e.g., -10^6) the largest component is likely to be $\epsilon n^{2/3}$, ϵ small, and there will be many components of nearly that size. The nontree components will be a negligible fraction of the tree components.

Now consider the evolution of $G(n, p)$ in terms of λ . Suppose that at a given λ there are components of size $c_1 n^{2/3}$ and $c_2 n^{2/3}$. When we move from λ to $\lambda + d\lambda$ there is a probability $c_1 c_2 d\lambda$ that they will merge. Components have a peculiar gravitation in which the probability of merging is proportional to their sizes. With probability $(c_1^2/2)d\lambda$ there will be a new internal edge in a component of size $c_1 n^{2/3}$ so that large components rarely remain trees. Simultaneously, big components are eating up other vertices.

With $\lambda = -10^6$, say, we have feudalism. Many small components (castles) are each vying to be the largest. As λ increases the components increase in size and a few large components (nations) emerge. An already large France has much better chances of becoming larger than a smaller Andorra. The largest components tend strongly to merge and by $\lambda = +10^6$ it is very likely that a giant component, Roman Empire, has emerged. With high probability this component is nevermore challenged for supremacy but continues absorbing smaller components until full connectivity - One World - is achieved.

3.5 Computer Simulation

It is instructive to simulate the above with a computer on, say, $n = 50000$ vertices. Start with the empty graph and randomly add edges $\{x, y\}$. When there are E edges let λ be defined by

$$\frac{E}{\binom{n}{2}} = \frac{1}{n} + \lambda n^{-4/3} \quad (118)$$

and look particularly closely in the range $-4 \leq \lambda \leq +4$. Keep track of the component sizes and their complexity using a Merge and Find algorithm. Also, keep track of the, say, top ten components. While $\lambda = 0$ has no special properties attached to it you will see ($1 - 10^{-50}$ of the time) a significant difference between $\lambda = -4$ and $\lambda = +4$ and you will “see” the largest components merging and the dominating component being formed.

3.6 A Continuous Model

There seems to be a continuous model underlying the asymptotic behavior of $G(n, p)$ with $p = n^{-1} + \lambda n^{-4/3}$. For fixed λ and k arbitrarily large but fixed one can look at the k largest components of $G(n, p)$ and parametrize them $x_1 n^{2/3}, \dots, x_k n^{2/3}$ in decreasing order. One can give explicitly a limiting distribution function $H(x_1, \dots, x_k)$ for these values. Now one can go to the limit with k and consider the “state” $P(\lambda)$ at “time” λ to be an infinite sequence $x_1 > x_2 > \dots$ of decreasing reals. There will be a distribution over the possible sequences. The sequences must be well-behaved; one can show, for example, that the number of x_i bigger than c must be asymptotic to $\frac{2}{3}(2\pi)^{-1/2}c^{-3/2}$ as $c \rightarrow 0$. (There is further information concerning the nature of the components - e.g., are they trees, unicyclic, . . . - that could also be added.) Now the intriguing thing is the “gravity” that defines $P(\lambda + d\lambda)$ in terms of $P(\lambda)$ in an appropriate limiting sense. If $P(\lambda)$ has terms x_i, x_j then with probability $x_i x_j d\lambda$ they will “merge” and form a single term with value $x_i + x_j$. This corresponds to certain coagulation models in physics though in the physical world the probability of coagulation depends on the surface area (and perhaps other invariants) of the objects while here it depends only on their sizes. So it seems there should be a probability space whose elements are histories - i.e., the value of $P(\lambda)$ for all real λ - where the change from $P(\lambda)$ to $P(\lambda + d\lambda)$ is governed by these coagulation laws and where further there have to be some appropriate entry laws so that each $P(\lambda)$ has the appropriate distribution.

3.7 Near Critical Branching Process

Let us reexamine the branching process of Section 2.1. Let T_ϵ^* denote the total size of the tree in the branching process with birth distribution Poisson with mean $1 + \epsilon$. This is defined for all $-1 \leq \epsilon < +\infty$ but are interest here will be in $|\epsilon|$ small. Set $T^* = T_0^*$ for convenience. Note that when ϵ is positive T_ϵ^* is infinite with nonzero probability. The exact distributions are known:

$$\Pr[T^* = u] = e^{-u} u^{u-1} / u! \quad (119)$$

$$\Pr[T_\epsilon^* = u] = \Pr[T^* = u] (1 + \epsilon)^{u-1} e^{-u\epsilon} \quad (120)$$

Our interest is more in the asymptotics.

$$\Pr[T^* = u] \sim (2\pi)^{-1/2} u^{-3/2} \quad (121)$$

Theorem 3.2. *Let $\epsilon \rightarrow 0$ and $u = u(\epsilon)$.*

1. *If $u = o(\epsilon^{-2})$ then $\Pr[T_\epsilon^* = u] \sim \Pr[T^* = u]$.*
2. *If $u = O(\epsilon^{-2})$ then $\Pr[T_\epsilon^* = u] = \Theta(\Pr[T^* = u])$.*

We see this from the Taylor Series $\ln(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3)$ so that

$$(1 + \epsilon)^{u-1} e^{-u\epsilon} \sim [(1 + \epsilon)e^{-\epsilon}]^u = e^{-\epsilon^2 u/2 + O(\epsilon^3 u)} \quad (122)$$

Roughly speaking, T_ϵ^* and T^* look the same when $u = o(\epsilon^{-2})$ and nearly the same when $u = O(\epsilon^{-2})$. When $u \gg \epsilon^{-2}$ the similarity ends and $\Pr[T_\epsilon^* = u]$ drops exponentially. When $\epsilon > 0$ let $x = \Pr[T_\epsilon = \infty]$. The exact formula is given by

$$1 - x = e^{-(1+\epsilon)x} \quad (123)$$

We are interested in the asymptotics.

Theorem 3.3. *Let $\epsilon \rightarrow 0^+$. Then*

$$\Pr[T_\epsilon = +\infty] \sim 2\epsilon$$

3.8 ... and Random Graphs

Let us fix $p = \frac{1}{n}$. (The results here will also hold at any spot inside the critical window, i.e., $p = \frac{1}{n} + \lambda n^{-2/3}$ for any fixed real number λ . The value of λ will substantially effect the constants hidden in the $O - \Theta$ notations.) Let T be the size of the component of a fixed vertex v in $G(n, p)$. v has $B[n-1, p]$ neighbors and then its neighbors have neighbors and this continues. We compare this process with the Poisson mean one process. We could add “artificial points” to the graph process so that at each time we have $B[n-1, p]$ neighbors. (The difference between $B[n-1, p]$ and Poisson mean one turns out to be negligible and we ignore it.) This would give T^* vertices. If the actual process give $\geq u$ vertices then the artifical process would give $\geq u$ vertices so that

$$\Pr[T \geq u] \leq \Pr[T^* \geq u] = O(u^{-1/2}) \quad (124)$$

Let Y be the number of vertices in components of size $\geq u$. Let X be the number of components of size $\geq u$, so $X \leq Y/u$. Then

$$E[X] \leq \frac{E[Y]}{u} = \frac{n \Pr[T \geq u]}{u} = \Theta(nu^{-3/2}) \quad (125)$$

For $u = Kn^{2/3}$ we have $E[X] = \Theta(1)$, and the constant goes like $K^{-3/2}$ as K gets large. We deduce: Almost surely all components have size $O(n^{2/3})$

Now we turn this around and give a lower bound for $\Pr[T \geq u]$ where $u = Kn^{2/3}$. Compare T with the Branching Process where the number of children is $B[n - u, p]$. As the tree is growing, until it has u or more vertices the number of potential children (what we think of as the reservoir) is at least $n - u$. Now $p(n - u) = 1 + \epsilon$ where $\epsilon = -Kn^{-1/3}$. So the tree process is T_ϵ^* . Note, critically, that $u = O(\epsilon^{-2})$. We are in the region where the difference between n and $n - u$ vertices in the reservoir has an effect by a constant factor. Note that for K very small positive this effect will be small. We have

$$\Pr[T \geq u] > \Pr[T_\epsilon^* \geq u] = \Theta(\Pr[T^* \geq u]) = \Theta(u^{-1/2}) \quad (126)$$

Let Y_u be the number of vertices in $G(n, p)$ in components of size $\geq u$. Let W_u the the number of components of $G(n, p)$ of size $\geq u$. When $u = Kn^{2/3}$, $E[Y_u] = \Theta(nu^{-1/2})$ and for all u , $E[Y_u] = O(nu^{-1/2})$. For $u = Kn^{2/3}$,

$$E[W_u] \leq E\left[\frac{1}{u}Y_u\right] = O(nu^{-3/2}) \quad (127)$$

For $u = Kn^{2/3}$ select $L > K$, $u' = Ln^{2/3}$ with $E[Y_{u'}] \leq \frac{1}{2}E[Y_u]$. Then

$$O(1) \geq E[W_u] \geq E\left[\frac{1}{u'}(Y_u - Y_{u'})\right] = \Omega(nu^{-3/2}) = \Omega(1) \quad (128)$$

Theorem 3.4. *For any fixed positive K the probability that $G(n, p)$, $p = \frac{1}{n}$, has a component of size at least $Kn^{2/3}$ is bounded from below by a positive constant, independent of n .*

Proof. By Markov, $\Pr[W_u \geq 2E[W_u]] \leq \frac{1}{2}$. In $G(n, p)$ generate components one by one. If you get $2E[W_u]$ components of size $\geq Kn^{2/3}$ then start over. Indeed, you have fewer vertices so that the situation is more difficult. To get $2aE[W_u]$ this would have to happen a times and so $\Pr[W_u \geq 2aE[W_u]] \leq 2^{-a}$ for each integer $a \geq 1$. So W_u has positive limiting expectation and its distribution drops exponentially so the weight can't be all in the infinite tail and so $\Pr[W_u = 0] \neq 1 - o(1)$. \square

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