# RANDOM GRAPHS

Joel Spencer *

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**Graph Theory Preliminaries** A graph $G$, formally speaking, is a pair $(V(G), E(G))$ where the elements $v \in V(G)$ are called vertices and the elements of $E(G)$, called edges, are two element subsets $\{v, w\}$ of $V(G)$. When $\{v, w\} \in E(G)$ we say $v, w$ are adjacent. (In standard graph theory terminology our graphs are undirected and have no loops and no multiple edges.) Pictorially, we often display the $v \in V(G)$ as points and draw an arc between $v$ and $w$ when they are adjacent. We call $V(G)$ the vertex set of $G$ and $E(G)$ the edge set of $G$. (When $G$ is understood we shall write simply $V$ and $E$ respectively. We also often write $v \in G$ or $\{v, w\} \in G$ instead of the formally correct $v \in V(G)$ and $\{v, w\} \in E(G)$ respectively.) A set $S \subseteq V$ is called a *clique* if all pairs $x, y \in S$ are adjacent. The clique number, denoted by $\omega(G)$, is the largest size of a clique in $G$. The complete graph on $k$ vertices, denoted by $K_k$, consists of a vertex set of size $k$ with all pairs $x, y$ adjacent.

## 1 Lecture 1: Random Graphs

### 1.1 What is a Random Graph

Let $n$ be a positive integer, $0 \leq p \leq 1$. The random graph $G(n, p)$ is a probability space over the set of graphs on the vertex set $\{1, \ldots, n\}$ determined by

$$\Pr[\{i, j\} \in G] = p \tag{1}$$

with these events mutually independent.

Random Graphs is an active area of research which combines probability theory and graph theory. The subject began in 1960 with the monumental paper *On the Evolution of Random Graphs* by Paul Erdős and Alfred Rényi. The book *Random Graphs* by Béla Bollobás is the standard source for the field. The book *The Probabilistic Method* by Noga Alon and this author contains much of the material in these notes, and more.

There is a compelling dynamic model for random graphs. For all pairs $i, j$ let $x_{i,j}$ be selected uniformly from $[0, 1]$, the choices mutually independent. Imagine $p$ going from 0 to 1. Originally, all potential edges are "off". The edge from $i$ to $j$ (which we may imagine as a neon light) is turned on when $p$ reaches $x_{i,j}$ and then stays on. At $p = 1$ all edges

*Courant Institute of Mathematical Sciences (New York). E-mail: spencer@cims.nyu.edu

are "on". At time $p$ the graph of all "on" edges has distribution $G(n,p)$. As $p$ increases $G(n,p)$ *evolves* from empty to full.

In their original paper Erdős and Rényi let $G(n,e)$ be the random graph with $n$ vertices and precisely $e$ edges. Again there is a dynamic model: Begin with no edges and add edges randomly one by one until the graph becomes full. Generally $G(n,e)$ will have very similar properties as $G(n,p)$ with $p \sim \frac{e}{\binom{n}{2}}$. We will work on the probability model exclusively.

## 1.2 Threshold Functions

The term "the random graph" is, strictly speaking, a misnomer. $G(n,p)$ is a probability space over graphs. Given any graph theoretic property $A$ there will be a probability that $G(n,p)$ satisfies $A$, which we write $\Pr[G(n,p) \models A]$. When $A$ is monotone $\Pr[G(n,p) \models A]$ is a monotone function of $p$. As an instructive example, let $A$ be the event "$G$ is triangle free". Let $X$ be the number of triangles contained in $G(n,p)$. Linearity of expectation gives

$$E[X] = \binom{n}{3} p^3 \tag{2}$$

This suggests the parametrization $p = c/n$. Then

$$\lim_{n\to\infty} E[X] = \lim_{n\to\infty} \binom{n}{3} p^3 = c^3/6 \tag{3}$$

We shall see that the distribution of $X$ is asymptotically Poisson. In particular

$$\lim_{n\to\infty} \Pr[G(n,p) \models A] = \lim_{n\to\infty} \Pr[X=0] = e^{-c^3/6} \tag{4}$$

Note that

$$\lim_{c\to 0} e^{-c^3/6} = 1 \tag{5}$$

$$\lim_{c\to\infty} e^{-c^3/6} = 0 \tag{6}$$

When $p = 10^{-6}/n$, $G(n,p)$ is very unlikely to have triangles and when $p = 10^6/n$, $G(n,p)$ is very likely to have triangles. In the dynamic view the first triangles almost always appear at $p = \Theta(1/n)$. If we take a function such as $p(n) = n^{-.9}$ with $p(n) \gg n^{-1}$ then $G(n,p)$ will almost always have triangles. Occasionally we will abuse notation and say, for example, that $G(n, n^{-.9})$ contains a triangle - this meaning that the probability that it contains a triangle approaches 1 as $n$ approaches infinity. Similarly, when $p(n) \ll n^{-1}$, for example, $p(n) = 1/(n \ln n)$, then $G(n,p)$ will almost always not contain a triangle and we abuse notation and say that $G(n, 1/(n \ln n))$ is trianglefree. It was a central observation of Erdős and Rényi that many natural graph theoretic properties become true in a very narrow range of $p$. They made the following key definition.

**Definition 1.** *$r(n)$ is called a* threshold function *for a graph theoretic property $A$ if*

1. *When $p(n) \ll r(n), \lim_{n\to\infty} \Pr[G(n,p) \models A] = 0$*
2. *When $p(n) \gg r(n), \lim_{n\to\infty} \Pr[G(n,p) \models A] = 1$*

*or visa versa.*

In our example, $1/n$ is a threshold function for $A$. Note that the threshold function, when one exists, is not unique. We could equally have said that $10/n$ is a threshold function for $A$.

Lets approach the problem of $G(n, c/n)$ being trianglefree once more. For every set $S$ of three vertices let $B_S$ be the event that $S$ is a triangle. Then $\Pr[B_S] = p^3$. Then "trianglefreeness" is precisely the conjunction $\wedge\overline{B}_S$ over all $S$. If the $B_S$ were mutually independent then we *would* have

$$\Pr[\wedge\overline{B}_S] = \prod[\overline{B}_S] = (1-p^3)^{\binom{n}{3}} \sim e^{-\binom{n}{3}p^3} \to e^{-c^3/6} \tag{7}$$

The reality is that the $B_S$ are not mutually independent though when $|S \cap T| \leq 1$, $B_S$ and $B_T$ are mutually independent. This is quite a typical situation in the study of random graphs in which we must deal with events that are "almost", but not precisely, mutual independent.

### 1.3 Variance

Here we introduce the Variance in a form that is particularly suited to the study of random graphs. The expressions $\Delta$ and $\Delta^*$ defined in this section will appear often in these notes.

Let $X$ be a nonnegative integral valued random variable and suppose we want to bound $\Pr[X = 0]$ given the value $\mu = E[X]$. If $\mu < 1$ we may use the inequality

$$\Pr[X > 0] \leq E[X] \tag{8}$$

so that if $E[X] \to 0$ then $X = 0$ almost always. (Here we are imagining an infinite sequence of $X$ dependent on some parameter $n$ going to infinity. This is the standard situation with the random graph $G(n, p(n))$.) But now suppose $E[X] \to \infty$. It does *not* necessarily follow that $X > 0$ almost always. For example, let $X$ be the number of deaths due to nuclear war in the twelve months after reading this paragraph. Calculation of $E[X]$ can make for lively debate but few would deny that it is quite large. Yet we may believe - or hope - that $Pr[X \neq 0]$ is very close to zero. We can sometimes deduce $X > 0$ almost always if we have further information about $Var[X]$.

**Theorem 1.1.**

$$\Pr[X = 0] \leq \frac{Var[X]}{E[X]^2} \tag{9}$$

*Proof.* Set $\lambda = \mu/\sigma$ in Chebyschev's Inequality. Then

$$\Pr[X = 0] \leq \Pr[|X - \mu| \geq \lambda\sigma] \leq \frac{1}{\lambda^2} = \frac{\sigma^2}{\mu^2} \tag{10}$$

□

We generally apply this result in asymptotic terms.

**Corollary 1.2.** *If* $Var[X] = o(E[X]^2)$ *then* $X > 0$ *a.a.*

The proof of Theorem 1.3 actually gives that for any $\epsilon > 0$

$$\Pr[|X - E[X]| \geq \epsilon E[X]] \leq \frac{Var[X]}{\epsilon^2 E[X]^2} \tag{11}$$

and thus in asymptotic terms we actually have the following stronger assertion:

**Corollary 1.3.** *If* $Var[X] = o(E[X]^2)$ *then* $X \sim E[X]$ *a.a.*

Suppose again $X = X_1 + \ldots + X_m$ where $X_i$ is the indicator random variable for event $A_i$. For indices $i, j$ write $i \sim j$ if $i \neq j$ and the events $A_i, A_j$ are not independent. We set (the sum over ordered pairs)

$$\Delta = \sum_{i \sim j} \Pr[A_i \wedge A_j] \tag{12}$$

Note that when $i \sim j$

$$Cov[X_i, X_j] = E[X_i X_j] - E[X_i]E[X_j] \leq E[X_i X_j] = \Pr[A_i \wedge A_j] \tag{13}$$

and that when $i \neq j$ and not $i \sim j$ then $Cov[X_i, X_j] = 0$. Thus

$$Var[X] \leq E[X] + \Delta \tag{14}$$

**Corollary 1.4.** *If* $E[X] \to \infty$ *and* $\Delta = o(E[X]^2)$ *then* $X > 0$ *almost always. Furthermore* $X \sim E[X]$ *almost always.*

Let us say $X_1, \ldots, X_m$ are *symmetric* if for every $i \neq j$ there is an automorphism of the underlying probability space that sends event $A_i$ to event $A_j$. Examples will appear in the next section. In this instance we write

$$\Delta = \sum_{i \sim j} \Pr[A_i \wedge A_j] = \sum_i \Pr[A_i] \sum_{j \sim i} \Pr[A_j | A_i] \tag{15}$$

and note that the inner summation is independent of $i$. We set

$$\Delta^* = \sum_{j \sim i} \Pr[A_j | A_i] \tag{16}$$

where $i$ is any fixed index. Then

$$\Delta = \sum_i \Pr[A_i] \Delta^* = \Delta^* \sum_i \Pr[A_i] = \Delta^* E[X] \tag{17}$$

**Corollary 1.5.** *If* $E[X] \to \infty$ *and* $\Delta^* = o(E[X])$ *then* $X > 0$ *almost always. Furthermore* $X \sim E[X]$ *almost always.*

The condition of Corollary 1.4 has the intuitive sense that conditioning on any specific $A_i$ holding does not substantially increase the expected number $E[X]$ of events holding.

## 1.4 Appearance of Small Subgraphs

What is the threshold function for the appearance of a given graph $H$. This problem was solved in the original papers of Erdős and Rényi. We begin with an instructive special case.

**Theorem 1.6.** *The property $\omega(G) \geq 4$ has threshold function $n^{-2/3}$.*

*Proof.* For every 4-set $S$ of vertices in $G(n,p)$ let $A_S$ be the event "$S$ is a clique" and $X_S$ its indicator random variable. Then

$$E[X_S] = \Pr[A_S] = p^6 \tag{18}$$

as six different edges must all lie in $G(n,p)$. Set

$$X = \sum_{|S|=4} X_S \tag{19}$$

so that $X$ is the number of 4-cliques in $G$ and $\omega(G) \geq 4$ if and only if $X > 0$. Linearity of Expectation gives

$$E[X] = \sum_{|S|=4} E[X_S] = \binom{n}{4} p^6 \sim \frac{n^4 p^6}{24} \tag{20}$$

When $p(n) \ll n^{-2/3}$, $E[X] = o(1)$ and so $X = 0$ almost surely.

Now suppose $p(n) \gg n^{-2/3}$ so that $E[X] \to \infty$ and consider the $\Delta^*$ of Corollary 1.5. (All 4-sets "look the same" so that the $X_S$ are symmetric.) Here $S \sim T$ if and only if $S \neq T$ and $S, T$ have common edges - i.e., if and only if $|S \cap T| = 2$ or 3. Fix $S$. There are $O(n^2)$ sets $T$ with $|S \cap T| = 2$ and for each of these $\Pr[A_T|A_S] = p^5$. There are $O(n)$ sets $T$ with $|S \cap T| = 3$ and for each of these $\Pr[A_T|A_S] = p^3$. Thus

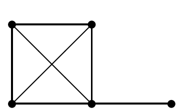
$$\Delta^* = O(n^2 p^5) + O(np^3) = o(n^4 p^6) = o(E[X])$$

since $p \gg n^{-2/3}$. Corollary 1.5 therefore applies and $X > 0$, i.e., there *does* exist a clique of size 4, almost always. □

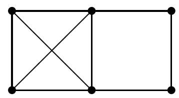
The proof of Theorem 1.6 appears to require a fortuitous calculation of $\Delta^*$. The following definitions will allow for a description of when these calculations work out.

**Definition 2.** *Let $H$ be a graph with $v$ vertices and $e$ edges. We call $\rho(H) = e/v$ the* density *of $H$. We call $H$* balanced *if every subgraph $H'$ has $\rho(H') \leq \rho(H)$. We call $H$* strictly balanced *if every proper subgraph $H'$ has $\rho(H') < \rho(H)$.*

`Examples.` $K_4$ and, in general, $K_k$ are strictly balanced. The graph



is not balanced as it has density $7/5$ while the subgraph $K_4$ has density $3/2$. The graph



is balanced but not strictly balanced as it and its subgraph $K_4$ have density 3/2.

**Theorem 1.7.** *Let $H$ be a balanced graph with $v$ vertices and $e$ edges. Let $A(G)$ be the event that $H$ is a subgraph (not necessarily induced) of $G$. Then $p = n^{-v/e}$ is the threshold function for $A$.*

*Proof.* We follow the argument of Theorem 1.6. For each $v$-set $S$ let $A_S$ be the event that $G|_S$ contains $H$ as a subgraph. Then

$$p^e \leq \Pr[A_S] \leq v! p^e \tag{21}$$

(Any particular placement of $H$ has probability $p^e$ of occuring and there are at most $v!$ possible placements. The precise calculation of $\Pr[A_S]$ is, in general, complicated due to the overlapping of potential copies of $H$.) Let $X_S$ be the indicator random variable for $A_S$ and

$$X = \sum_{|S|=v} X_S \tag{22}$$

so that $A$ holds if and only if $X > 0$. Linearity of Expectation gives

$$E[X] = \sum_{|S|=v} E[X_S] = \binom{n}{v} \Pr[A_S] = \Theta(n^v p^e) \tag{23}$$

If $p \ll n^{-v/e}$ then $E[X] = o(1)$ so $X = 0$ almost always.

Now assume $p \gg n^{-v/e}$ so that $E[X] \to \infty$ and consider the $\Delta^*$ of Corollary 1.5. (All $v$-sets look the same so the $X_S$ are symmetric.) Here $S \sim T$ if and only if $S \neq T$ and $S, T$ have common edges - i.e., if and only if $|S \cap T| = i$ with $2 \leq i \leq v-1$. Let $S$ be fixed. We split

$$\Delta^* = \sum_{T \sim S} \Pr[A_T | A_S] = \sum_{i=2}^{v-1} \sum_{|T \cap S| = i} \Pr[A_T | A_S] \tag{24}$$

For each $i$ there are $O(n^{v-i})$ choices of $T$. Fix $S, T$ and consider $\Pr[A_T | A_S]$. There are $O(1)$ possible copies of $H$ on $T$. Each has - since, critically, $H$ is balanced - at most $\frac{ie}{v}$ edges with both vertices in $S$ and thus at least $e - \frac{ie}{v}$ other edges. Hence

$$\Pr[A_T | A_S] = O(p^{e - \frac{ie}{v}}) \tag{25}$$

and

$$\Delta^* = \sum_{i=2}^{v-1} O(n^{v-i} p^{e - \frac{ie}{v}}) = \sum_{i=2}^{v-1} O((n^v p^e)^{1 - \frac{i}{v}}) \tag{26}$$

so that

$$\Delta^* == \sum_{i=2}^{v-1} o(n^v p^e) = o(E[X]) \tag{27}$$

as $n^v p^e \to \infty$. Hence Corollary 1.5 applies. □

**Theorem 1.8.** *In the notation of Theorem 1.7 if $H$ is* not *balanced then $p = n^{-v/e}$ is* not *the threshold function for $A$.*

*Proof.* Let $H_1$ be a subgraph of $H$ with $v_1$ vertices, $e_1$ edges and $e_1/v_1 > e/v$. Let $\alpha$ satisfy $v/e < \alpha < v_1/e_1$ and set $p = n^{-\alpha}$. The expected number of copies of $H_1$ is then $o(1)$ so almost always $G(n,p)$ contains no copy of $H_1$. But if it contains no copy of $H_1$ then it surely can contain no copy of $H$. □

The threshold function for the property of containing a copy of $H$, for general $H$, was examined in the original papers of Erdős and Rényi. Let $H_1$ be that subgraph with maximal density $\rho(H_1) = e_1/v_1$. (When $H$ is balanced we may take $H_1 = H$.) They showed that $p = n^{-v_1/e_1}$ is the threshold function. This will follow fairly quickly from the methods of theorem 1.7.

We finish this section with two strengthenings of Theorem 1.7.

**Theorem 1.9.** *Let $H$ be strictly balanced with $v$ vertices, $e$ edges and $a$ automorphisms. Let $X$ be the number of copies of $H$ in $G(n,p)$. Assume $p \gg n^{-v/e}$. Then almost always*

$$X \sim \frac{n^v p^e}{a} \tag{28}$$

*Proof.* Label the vertices of $H$ by $1, \ldots, v$. For each ordered $x_1, \ldots, x_v$ let $A_{x_1,\ldots,x_v}$ be the event that $x_1, \ldots, x_v$ provides a copy of $H$ in that order. Specifically we define

$$A_{x_1,\ldots,x_v} : \{i,j\} \in E(H) \Rightarrow \{x_i, x_j\} \in E(G) \tag{29}$$

We let $I_{x_1,\ldots,x_v}$ be the corresponding indicator random variable. We define an equivalence class on $v$-tuples by setting $(x_1, \ldots, x_v) \equiv (y_1, \ldots, y_v)$ if there is an automorphism $\sigma$ of $V(H)$ so that $y_{\sigma(i)} = x_i$ for $1 \le i \le v$.Then

$$X = \sum I_{x_1,\ldots,x_v} \tag{30}$$

gives the number of copies of $H$ in $G$ where the sum is taken over one entry from each equivalence class. As there are $(n)_v/a$ terms

$$E[X] = \frac{(n)_v}{a} E[I_{x_1,\ldots,x_v}] = \frac{(n)_v p^e}{a} \sim \frac{n^v p^e}{a} \tag{31}$$

Our assumption $p \gg n^{-v/e}$ implies $E[X] \to \infty$. It suffices therefore to show $\Delta^* = o(E[X])$. Fixing $x_1, \ldots, x_v$,

$$\Delta^* = \sum_{(y_1,\ldots,y_v)\sim(x_1,\ldots,x_v)} \Pr[A_{(y_1,\ldots,y_v)} | A_{(x_1,\ldots,x_v)}] \tag{32}$$

There are $v!/a = O(1)$ terms with $\{y_1, \ldots, y_v\} = \{x_1, \ldots, x_v\}$ and for each the conditional probability is at most one (actually, at most $p$), thus contributing $O(1) = o(E[X])$ to $\Delta^*$. When $\{y_1, \ldots, y_v\} \cap \{x_1, \ldots, x_v\}$ has $i$ elements, $2 \le i \le v-1$ the argument of Theorem 1.7 gives that the contribution to $\Delta^*$ is $o(E[X])$. Altogether $\Delta^* = o(E[X])$ and we apply Corollary 1.5 □

**Theorem 1.10.** *Let $H$ be* any *fixed graph. For every subgraph $H'$ of $H$ (including $H$ itself) let $X_{H'}$ denote the number of copies of $H'$ in $G(n,p)$. Assume $p$ is such that $E[X_{H'}] \to \infty$ for every $H'$. Then*

$$X_H \sim E[X_H] \tag{33}$$

*almost always.*

*Proof.* Let $H$ have $v$ vertices and $e$ edges. As in Theorem 4.4 it suffices to show $\Delta^* = o(E[X])$. We split $\Delta^*$ into a finite number of terms. For each $H'$ with $w$ vertice and $f$ edges we have those $(y_1, \ldots, y_v)$ that overlap with the fixed $(x_1, \ldots, x_v)$ in a copy of $H'$. These terms contribute, up to constants,

$$n^{v-w}p^{e-f} = \Theta\left(\frac{E[X_H]}{E[X_{H'}]}\right) = o(E[X_H])$$

to $\Delta^*$. Hence Corollary 1.5 does apply. □

## 1.5 Connectivity

In this section we give a relatively simple example of what we call the Poisson Paradigm: the rough notion that if there are many rare and nearly independent events then the number of events that hold has approximately a Poisson distribution. This will yield one of the most beautiful of the Erdős- Rényi results, a quite precise description of the threshold behavior for connectivity. A vertex $v \in G$ is *isolated* if it is adjacent to no $w \in V$. In $G(n,p)$ let $X$ be the number of isolated vertices.

**Theorem 1.11.** *Let $p = p(n)$ satisfy $n(1-p)^{n-1} = \mu$. Then*

$$\lim_{n\to\infty} \Pr[X = 0] = e^{-\mu} \tag{34}$$

*Proof.* We let $X_i$ be the indicator random variable for vertex $i$ being isolated so that $X = X_1 + \ldots + X_n$. Then $E[X_i] = (1-p)^{n-1}$ so by linearity of expectation $E[X] = \mu$. Now consider the $r$-th factorial moment $E[(X)_r]$ ($(X)_r := \prod i = 0^{r-1}(X-i)$) for any fixed $r$. By the symmetry $E[(X)_r] = (n)_r E[X_1 \cdots X_r]$. For vertices $1, \ldots, r$ to all be isolated the $r(n-1) - \binom{r}{2}$ pairs $\{i, x\}$ overlapping $1, \ldots, r$ must all not be edges. Thus

$$E[(X)_r] = (n)_r(1-p)^{r(n-1)-\binom{r}{2}} \sim n^r(1-p)^{r(n-1)} \sim \mu^r \tag{35}$$

(That is, the dependence among the $X_i$ was asymptotically negligible.) All the moments of $X$ approach those of $P(\mu)$. This implies (a nonobvious result in probability theory) that $X$ approaches $P(\mu)$ in distribution. □

Now we give the Erdős-Rényi famous "double exponential" result.

**Theorem 1.12.** *Let*

$$p = p(n) = \frac{\log n}{n} + \frac{c}{n} + o(\frac{1}{n}) \tag{36}$$

*Then*

$$\lim_{n\to\infty} \Pr[G(n,p) \textit{ is connected}] = e^{-e^{-c}} \tag{37}$$

*Proof.* For such $p$, $n(1-p)^{n-1} \sim \mu = e^{-c}$ and by the above argument the probability that $X$ has no isolated vertices approaches $e^{-\mu}$. If $G$ has no isolated vertices but is not connected there is a component of $k$ vertices for some $2 \le k \le \frac{n}{2}$. Letting $B$ be this event

$$\Pr[B] \le \sum_{k=2}^{n/2} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-1)-\binom{k}{2}} \tag{38}$$

The first factor is the choice of a component set $S \subset V(G)$. The second factor is a choice of tree on $S$. The third factor is the probability that those tree pairs are in $E(G)$. The final factor is that there be no edge from $S$ to $V(G)-S$. Some calculation (which we omit but note that $k=2$ provides the main term) gives that $\Pr[B] = o(1)$ so that $X \ne 0$ and connectivity have the same limiting probability. □

### 1.6 The Janson Inequalities

In many instances we would like to bound the probability that none of a set of bad events $B_i, i \in I$ occur. If the events are mutually independent then

$$\Pr[\wedge_{i\in I}\overline{B_i}] = \prod_{i\in I} \Pr[\overline{B_i}] \tag{39}$$

When the $B_i$ are "mostly" independent the Janson Inequalities allow us, sometimes, to say that these two quantities are "nearly" equal.

Let $\Omega$ be a finite universal set and let $R$ be a random subset of $\Omega$ given by

$$\Pr[r \in R] = p_r, \tag{40}$$

these events mutually independent over $r \in \Omega$. (In application to $G(n,p)$, $\Omega$ is the set of pairs $\{i,j\}$, $i,j \in V(G)$ and all $p_r = p$ so that $R$ is the edge set of $G(n,p)$.) Let $A_i, i \in I$, be subsets of $\Omega$, $I$ a finite index set. Let $B_i$ be the *event* $A_i \subseteq R$. (That is, each point $r \in \Omega$ " flips a coin" to determine if it is in $R$. $B_i$ is the event that the coins for all $r \in A_i$ came up "heads".) Let $X_i$ be the indicator random variable for $B_i$ and $X = \sum_{i\in I} X_i$ the number of $A_i \subseteq R$. The event $\wedge_{i\in I}\overline{B_i}$ and $X = 0$ are then identical. For $i,j \in I$ we write $i \sim j$ if $i \ne j$ and $A_i \cap A_j \ne \emptyset$. Note that when $i \ne j$ and not $i \sim j$ then $B_i, B_j$ are independent events since they involve separate coin flips. Furthermore, and this plays a crucial role in the proofs, if $i \notin J \subset I$ and not $i \sim j$ for all $j \in J$ then $B_i$ is mutually independent of $\{B_j | j \in J\}$, i.e., independent of any Boolean function of those $B_j$. This is because the coin flips on $A_i$ and on $\cup_{j\in J} A_j$ are independent. We define

$$\Delta = \sum_{i\sim j} \Pr[B_i \wedge B_j] \tag{41}$$

Here the sum is over ordered pairs so that $\Delta/2$ gives the same sum over unordered pairs. (This will be the same $\Delta$ as defined earlier.) We set

$$M = \prod_{i\in I} \Pr[\overline{B_i}], \tag{42}$$

the value of $\Pr[\wedge_{i\in I}\overline{B_i}]$ if the $B_i$ were independent.

**Theorem 1.13** (The Janson Inequality)**.** *Let $B_i, i \in I$, $\Delta, M$ be as above and assume all $\Pr[B_i] \le \epsilon$. Then*

$$M \le \Pr[\wedge_{i\in I}\overline{B_i}] \le Me^{\frac{1}{1-\epsilon}\frac{\Delta}{2}} \tag{43}$$

Now set

$$\mu = E[X] = \sum_{i\in I} \Pr[B_i] \tag{44}$$

For each $i \in I$

$$\Pr[\overline{B_i}] = 1 - \Pr[B_i] \le e^{-\Pr[B_i]} \tag{45}$$

so, multiplying over $i \in I$,

$$M \le e^{-\mu} \tag{46}$$

It is often more convenient to replace the upper bound of Theorem 1.13 with

$$\Pr[\wedge_{i\in I}\overline{B_i}] \le e^{-\mu+\frac{1}{1-\epsilon}\frac{\Delta}{2}} \tag{47}$$

As an example, set $p = cn^{-2/3}$ and consider the probability that $G(n,p)$ contains no $K_4$. The $B_i$ then range over the $\binom{n}{4}$ potential $K_4$ - each being a 6-element subset of $\Omega$. Here, as is often the case, $\epsilon = o(1)$, $\Delta = o(1)$ (as calculated previously) and $\mu$ approaches a constant, here $k = c^6/24$. In these instances $\Pr[\wedge_{i\in I}\overline{B_i}] \to e^{-k}$. Thus we have the fine structure of the threshold function of $\omega(G) = 4$.

As $\Delta$ becomes large the Janson Inequality becomes less precise. Indeed, when $\Delta \ge 2\mu(1-\epsilon)$ it gives an upper bound for the probability which is larger than one. At that point (and even somewhat before) the following result kicks in.

**Theorem 1.14.** *(Generalized Janson Inequality) Under the assumptions of Theorem 1.13 and the further assumption that $\Delta \ge \mu(1-\epsilon)$*

$$\Pr[\wedge_{i\in I}\overline{B_i}] \le e^{-\frac{\mu^2(1-\epsilon)}{2\Delta}} \tag{48}$$

The Generalized Janson Inequality (when it applies) often gives a much stronger result than Chebyschev's Inequality as used earlier. We can bound $Var[X] \le \mu + \Delta$ so that

$$\Pr[\wedge_{i\in I}\overline{B_i}] = \Pr[X=0] \le \frac{Var[X]}{E[X]^2} \le \frac{\mu+\Delta}{\mu^2} \tag{49}$$

Suppose $\epsilon = o(1)$, $\mu \to \infty$, $\mu \ll \Delta$, and $\gamma = \frac{\mu^2}{\Delta} \to \infty$. Chebyschev's upper bound on $\Pr[X=0]$ is then roughly $\gamma^{-1}$ while Janson's upper bound is roughly $e^{-\gamma}$.

## 1.7 The Proofs

The original proofs of Janson are based on estimates of the Laplace transform of an appropriate random variable. The proof presented here follows that of Boppana and Spencer [1989]. We shall use the inequalities

$$\Pr[B_i| \wedge_{j\in J} \overline{B_j}] \le \Pr[B_i] \tag{50}$$

valid for all index sets $J \subset I, i \notin J$ and

$$\Pr[B_i | B_k \wedge \bigwedge_{j \in J} \overline{B_j}] \leq \Pr[B_i | B_k] \tag{51}$$

valid for all index sets $J \subset I, i, k \notin J$. The first follows from general Correlation Inequalities. The second is equivalent to the first since conditioning on $B_k$ is the same as assuming $p_r = \Pr[r \in R] = 1$ for all $r \in A_k$. We note that Janson's Inequality actually applies to any set of events $B_i$ and relation $\sim$ for which (50,51) apply.

*Proof.* (Thm. 1.13) The lower bound follows immediately. Order the index set $I = \{1, \ldots, m\}$ for convenience. For $1 \leq i \leq m$

$$Pr[B_i | \wedge_{1 \leq j < i} \overline{B_j}] \leq \Pr[B_i] \tag{52}$$

so

$$Pr[\overline{B_i} | \wedge_{1 \leq j < i} \overline{B_j}] \geq Pr[\overline{B_i}] \tag{53}$$

and

$$\Pr[\wedge_{i \in I} \overline{B_i}] = \prod_{i=1}^{m} Pr[\overline{B_i} | \wedge_{1 \leq j < i} \overline{B_j}] \geq \prod_{i=1}^{m} Pr[\overline{B_i}] \tag{54}$$

Now the upper bound. For a given $i$ renumber, for convenience, so that $i \sim j$ for $1 \leq j \leq d$ and not for $d+1 \leq j < i$. We use the inequality $\Pr[A|B \wedge C] \geq \Pr[A \wedge B|C]$, valid for any $A, B, C$. With $A = B_i$, $B = \overline{B_1} \wedge \ldots \wedge \overline{B_d}$, $C = \overline{B_{d+1}} \wedge \ldots \wedge \overline{B_{i-1}}$

$$Pr[B_i | \wedge_{1 \leq j < i} \overline{B_j}] = \Pr[A | B \wedge C] \geq Pr[A \wedge B | C] = Pr[A|C] Pr[B | A \wedge C] \tag{55}$$

From the mutual independence $Pr[A|C] = Pr[A]$. We bound

$$\Pr[B | A \wedge C] \geq 1 - \sum_{j=1}^{d} Pr[B_j | B_i \wedge C] \geq 1 - \sum_{j=1}^{d} Pr[B_j | B_i] \tag{56}$$

from the Correlation Inequality. Thus

$$Pr[B_i | \wedge_{1 \leq j < i} \overline{B_j}] \geq \Pr[B_i] - \sum_{j=1}^{d} Pr[B_j \wedge B_i] \tag{57}$$

Reversing

$$Pr[\overline{B_i} | \wedge_{1 \leq j < i} \overline{B_j}] \leq \Pr[\overline{B_i}] + \sum_{j=1}^{d} Pr[B_j \wedge B_i] \tag{58}$$

$$\leq \Pr[\overline{B_i}] \left( 1 + \frac{1}{1-\epsilon} \sum_{j=1}^{d} Pr[B_j \wedge B_i] \right) \tag{59}$$

since $\Pr[\overline{B_i}] \geq 1 - \epsilon$. Employing the inequality $1 + x \leq e^x$,

$$Pr[\overline{B_i} | \wedge_{1 \leq j < i} \overline{B_j}] \leq \Pr[\overline{B_i}] e^{\frac{1}{1-\epsilon} \sum_{j=1}^{d} Pr[B_j \wedge B_i]} \tag{60}$$

For each $1 \leq i \leq m$ we plug this inequality into

$$\Pr[\wedge_{i\in I}\overline{B_i}] = \prod_{i=1}^{m} Pr[\overline{B_i} | \wedge_{1\leq j<i} \overline{B_j}] \tag{61}$$

The terms $\Pr[\overline{B_i}]$ multiply to $M$. The exponents add: for each $i, j \in I$ with $j < i$ and $j \sim i$ the term $Pr[B_j \wedge B_i]$ appears once so they add to $\Delta/2$. □

*Proof.* (Theorem 1.14) As discussed earlier, the proof of Theorem 1.13 gives

$$\Pr[\wedge_{i\in I}\overline{B_i}] \leq e^{-\mu + \frac{1}{1-\epsilon}\frac{\Delta}{2}} \tag{62}$$

which we rewrite as

$$-\ln[\Pr[\wedge_{i\in I}\overline{B_i}]] \geq \sum_{i\in I} \Pr[B_i] - \frac{1}{2(1-\epsilon)} \sum_{i\sim j} \Pr[B_i \wedge B_j] \tag{63}$$

For any set of indices $S \subset I$ the same inequality applied only to the $B_i, i \in S$ gives

$$-\ln[\Pr[\wedge_{i\in S}\overline{B_i}]] \geq \sum_{i\in S} \Pr[B_i] - \frac{1}{2(1-\epsilon)} \sum_{i,j\in S, i\sim j} \Pr[B_i \wedge B_j] \tag{64}$$

Let now $S$ be a random subset of $I$ given by

$$\Pr[i \in S] = p \tag{65}$$

with $p$ a constant to be determined, the events mutually independent. (Here we are using probabilistic methods to prove a probability theorem!)

$$E\left[-\ln[\Pr[\wedge_{i\in S}\overline{B_i}]]\right] \geq E\left[\sum_{i\in S} \Pr[B_i]\right] - \frac{1}{2(1-\epsilon)} E\left[\sum_{i,j\in S, i\sim j} \Pr[B_i \wedge B_j]\right] \tag{66}$$

Each term $\Pr[B_i]$ then appears with probability $p$ and each term $\Pr[B_i \wedge B_j]$ with probability $p^2$ so that

$$E\left[-\ln[\Pr[\wedge_{i\in S}\overline{B_i}]]\right] \geq p\mu - \frac{1}{1-\epsilon} p^2 \frac{\Delta}{2} \tag{67}$$

We set

$$p = \frac{\mu(1-\epsilon)}{\Delta} \tag{68}$$

so as to maximize this quantity. The added assumption of Theorem 1.14 assures us that the probability $p$ is at most one. Then

$$E\left[-\ln[\Pr[\wedge_{i\in S}\overline{B_i}]]\right] \geq \frac{\mu^2(1-\epsilon)}{2\Delta} \tag{69}$$

Therefore there is a specific $S \subset I$ for which

$$-\ln[\Pr[\wedge_{i\in S}\overline{B_i}] \geq \frac{\mu^2(1-\epsilon)}{2\Delta} \tag{70}$$

That is,

$$\Pr[\wedge_{i\in S}\overline{B_i}] \le e^{-\frac{\mu^2(1-\epsilon)}{2\Delta}} \tag{71}$$

But

$$\Pr[\wedge_{i\in I}\overline{B_i}] \le \Pr[\wedge_{i\in S}\overline{B_i}] \tag{72}$$

completing the proof. □

## 1.8 Appearance of Small Subgraphs Revisited

Generalizing the fine threshold behavior for the appearance of $K_4$ we find the fine threshold behavior for the appearance of any strictly balanced graph $H$.

**Theorem 1.15.** *Let $H$ be a* strictly *balanced graph with $v$ vertices, $e$ edges and $a$ automorphisms. Let $c > 0$ be arbitrary. Let $A$ be the property that $G$ contains no copy of $H$. Then with $p = cn^{-v/e}$,*

$$\lim_{n\to\infty} \Pr[G(n,p) \models A] = exp[-c^e/a] \tag{73}$$

*Proof.* Let $A_\alpha, 1 \le \alpha \le \binom{n}{v}v!/a$, range over the edge sets of possible copies of $H$ and let $B_\alpha$ be the event $G(n,p) \supseteq A_\alpha$. We apply Janson's Inequality. As

$$\lim_{n\to\infty} \mu = lim_{n\to\infty}\binom{n}{v}v!p^e/a = c^e/a \tag{74}$$

we find

$$\lim_{n\to\infty} M = exp[-c^e/a] \tag{75}$$

Now we examine (similar to Theorem 1.7)

$$\Delta = \sum_{\alpha\sim\beta} \Pr[B_\alpha \wedge B_\beta] \tag{76}$$

We split the sum according to the number of *vertices* in the intersection of copies $\alpha$ and $\beta$. Suppose they intersect in $j$ vertices. If $j = 0$ or $j = 1$ then $A_\alpha \cap A_\beta = \emptyset$ so that $\alpha \sim \beta$ cannot occur. For $2 \le j \le v$ let $f_j$ be the maximal $|A_\alpha \cap A_\beta|$ where $\alpha \sim \beta$ and $\alpha, \beta$ intersect in $j$ vertices. As $\alpha \ne \beta$, $f_v < e$. When $2 \le j \le v - 1$ the critical observation is that $A_\alpha \cap A_\beta$ is a subgraph of $H$ and hence, as $H$ is strictly balanced,

$$\frac{f_j}{j} < \frac{e}{v} \tag{77}$$

There are $O(n^{2v-j})$ choices of $\alpha, \beta$ intersecting in $j$ points since $\alpha, \beta$ are determined, except for order, by $2v - j$ points. For each such $\alpha, \beta$

$$\Pr[B_\alpha \wedge B_\beta] = p^{|A_\alpha \cup A_\beta|} = p^{2e-|A_\alpha \cap A_\beta|} \le p^{2e-f_j} \tag{78}$$

Thus

$$\Delta = \sum_{j=2}^{v} O(n^{2v-j})O(n^{-\frac{v}{e}(2e-f_j)}) \tag{79}$$

But

$$2v - j - \frac{v}{e}(2e - f_j) = \frac{vf_j}{e} - j < 0 \tag{80}$$

so each term is $o(1)$ and hence $\Delta = o(1)$. By Janson's Inequality

$$\lim_{n\to\infty} \Pr[\wedge \overline{B}_\alpha] = \lim_{n\to\infty} M = exp[-c^e/a] \tag{81}$$

completing the proof. □

The fine threshold behavior for the appearance of an arbitrary graph $H$ has been worked out but it can get quite complicated.

### 1.9 Some Very Low Probabilities

Let $A$ be the property that $G$ does not contain $K_4$ and consider $\Pr[G(n,p) \models A]$ as $p$ varies. (Results with $K_4$ replaced by an arbitrary $H$ are discussed at the end of this section.) We know that $p = n^{-2/3}$ is a threshold function so that for $p \gg n^{-2/3}$ this probability is $o(1)$. Here we want to estimate that probability. Our estimates here will be quite rough, only up to a $o(1)$ additive factor in the hyperexponent, though with more care the bounds differ by "only" a constant factor in the exponent. If we were to consider all potential $K_4$ as giving mutually independent events then we would be led to the estimate $(1-p^6)^{\binom{n}{4}} = e^{-n^{4+o(1)}p^6}$. For $p$ appropriately small this turns out to be correct. But for, say, $p = \frac{1}{2}$ it would give the estimate $e^{-n^{4+o(1)}}$. This must, however, be way off the mark since with probability $2^{-\binom{n}{2}} = e^{-n^{2+o(1)}}$ the graph $G$ could be empty and hence trivially satisfy $A$.

Rather than giving the full generality we assume $p = n^{-\alpha}$ with $\frac{2}{3} > \alpha > 0$. The result is:

$$\Pr[G(n,p) \models A] = e^{-n^{4-6\alpha+o(1)}} \tag{82}$$

for $\frac{2}{3} > \alpha \geq \frac{2}{5}$ and

$$\Pr[G(n,p) \models A] = e^{-n^{2-\alpha+o(1)}} \tag{83}$$

for $\frac{2}{5} \geq \alpha > 0$.

The upper bound follows from the inequality

$$\Pr[G(n,p) \models A] \geq \max\left[(1-p^6)^{\binom{n}{4}}, (1-p)^{\binom{n}{2}}\right] \tag{84}$$

This is actually two inequalities. The first comes from the probability of $G$ not containing a $K_4$ being at most the probability as if all the potential $K_4$ were independent. The second is the same bound on the probability that $G$ doesn't contain a $K_2$ - i.e., that $G$ has no edges. Calculation shows that the "turnover" point for the two inequalities occurs when $p = n^{-2/5+o(1)}$.

The upper bound follows from the Janson inequalities. For each four set $\alpha$ of vertices $B_\alpha$ is that that 4-set gives a $K_4$ and we want $\Pr[\wedge \overline{B_\alpha}]$. We have $\mu = \Theta(n^4p^6)$ and $-\ln M \sim \mu$ and (as shown earlier) $\Delta = \Theta(\mu\Delta^*)$ with $\Delta^* = \Theta(n^2p^5 + np^3)$. With $p = n^{-\alpha}$ and $\frac{2}{3} > \alpha > \frac{2}{5}$ we have $\Delta^* = o(1)$ so that

$$\Pr[\wedge \overline{B_\alpha}] \leq e^{-\mu(1+o(1))} = e^{-n^{4-6\alpha+o(1)}}$$

When $\frac{2}{5} > \alpha > 0$ then $\Delta^* = \Theta(n^2 p^5)$ (somewhat surprisingly the $np^3$ never is significant in these calculations) and the extended Janson inequality gives

$$\Pr[\wedge \overline{B_\alpha}] \le e^{-\Theta(\mu^2/\Delta)} = e^{-\Theta(\mu/\Delta^*)} = e^{-n^{2-\alpha}}$$

The general result has been found by T. Łuczak, A. Rucinski and S. Janson (1990). Let $H$ be any fixed graph and let $A$ be the property of not containing a copy of $H$. For any subgraph $H'$ of $H$ the correlation inequality gives

$$\Pr[G(n,p) \models A] \le e^{-E[X_{H'}]} \tag{85}$$

where $X_{H'}$ is the number of copies of $H'$ in $G$. Now let $p = n^{-\alpha}$ where we restrict to those $\alpha$ for which $p$ is past the threshold function for the appearance of $H$. Then

$$\Pr[G(n,p) \models A] = e^{n^{o(1)}} \min_{H'} e^{-E[X_{H'}]} \tag{86}$$

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