On the adjacency matrix of a block graph

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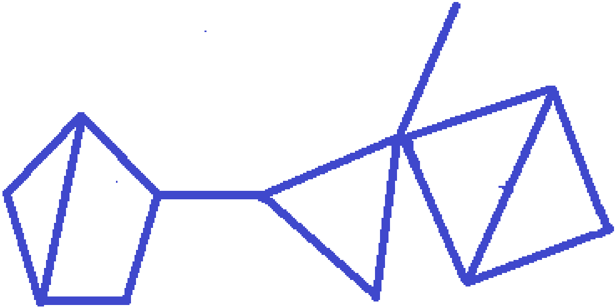
This talk is primarily based on joint work with Souvik Roy.

The later part contains related results and some recent work in progress with Ebrahim Ghorbani.

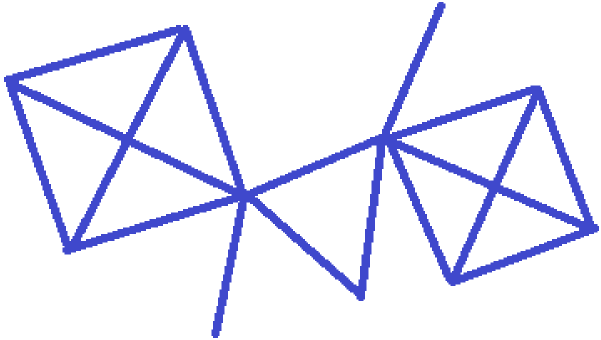
Block

A block is a maximal subgraph with no cut-vertex.



Block graph

A block graph is a graph in which each block is a complete graph.



A characterization of trees

A tree is a block graph.

A connected graph is a tree if and only if each edge is a block.

A block graph generalizes:

(i) tree

(ii) complete graph.

Motivation

The following classical results motivated the present work:

A tree is nonsingular if and only if it has a perfect matching.

When a tree is nonsingular, there is a formula for its inverse in terms of alternating paths.

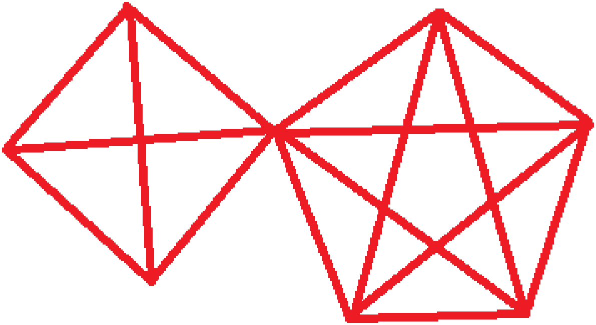
Adjacency matrix of a complete graph

If A is the adjacency matrix of K_n, then

$$\det A = (-1)^{n-1}(n-1).$$

Adjacency matrix of a block graph

Consider the block graph



Adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\det A = \sum (\alpha_1 - 1)(\alpha_2 - 1),$$

where the summation is over

$$0 \leq \alpha_1 \leq 4, 0 \leq \alpha_2 \leq 5, \alpha_1 + \alpha_2 = 8.$$

$$\det A = (4-1)(4-1) + (3-1)(5-1) = 17.$$

Theorem

Let G be a block graph with n vertices. Let $B_1, \ldots, B_k$ be the blocks of G and let $|V(B_i)| = n_i, i = 1, \ldots, k$. Let A be the adjacency matrix of G. Then

$$\det A = (-1)^{n-k} \sum (\alpha_1 - 1) \cdots (\alpha_k - 1)$$

where the summation is over all k-tuples $(\alpha_1, \ldots, \alpha_k)$ of nonnegative integers satisfying the following conditions:

(i) $\sum_{i=1}^{k} \alpha_i = n$

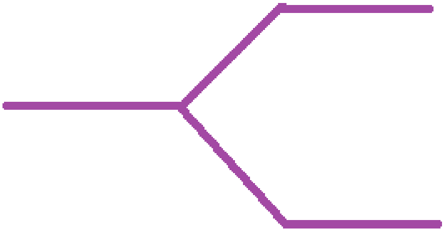
(ii) for any nonempty $S \subset \{1, \ldots, k\}$,

$$\sum_{i \in S} \alpha_i \leq |V(G_S)|,$$

where G_S is the subgraph induced by the blocks $B_i, i \in S$.

Nonsingular trees

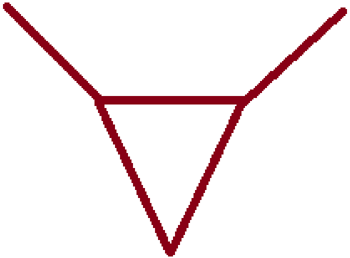
Corollary A tree is nonsingular if and only if it has a perfect matching.



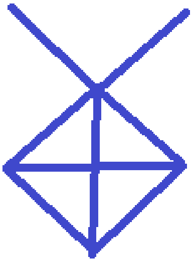
Singular block graphs

Trees with no perfect matching are examples of singular block graphs. There are other examples.

A singular block graph with an odd number of vertices:



A singular block graph with an even number of vertices



A class of singular graphs

Let T be a singular tree and let $S \subset V(T)$ be the set of vertices corresponding to a zero in the null vector.

Let G be the graph obtained from T by attaching an arbitrary graph at each vertex in S.

Then G is singular.

An open problem

Characterize nonsingular block graphs.

Adjacency matrix over GF(2)

Lemma Let G be a graph with n vertices and let A be the adjacency matrix of G. If n is odd then $\det A$ is even.

In particular, A is singular over GF(2).

A reduction procedure

Let G be a graph with blocks $B_1, \ldots, B_k$. Let B_1 be pendant and let v be the cut-vertex of B_1.

(i) If $|V(B_1)|$ is even, then G is nonsingular if and only if $G \setminus B_1$ is nonsingular.

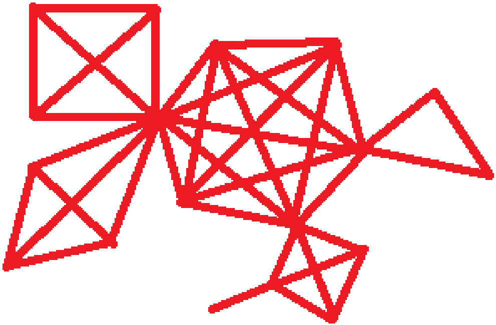
(i) If $|V(B_1)|$ is odd, then G is nonsingular if and only if $G \setminus (B_1 \setminus v)$ is nonsingular.

Example

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Example

Using the previous result we conclude that this graph is singular:



Lemma Let G be a block graph with adjacency matrix A. Let v be a vertex of G such that $G \setminus v$ has at least two odd components. Then $\det A$ is an even integer. In particular, A is singular over GF(2).

Nonsingular block graphs over GF(2)

Theorem Let G be a block graph and let A be the adjacency matrix of G. Then A is nonsingular over GF(2) if and only if for any vertex v, $G \setminus v$ has exactly one odd component.

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Corollary 1 Let G be a block graph with n vertices and let A be the adjacency matrix of G. If n is odd, then A is singular over GF(2).

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Corollary 1 Let G be a block graph with n vertices and let A be the adjacency matrix of G. If n is odd, then A is singular over GF(2).

Corollary 2 Let T be a tree and let A be the adjacency matrix of G. If T has no perfect matching then A is singular over GF(2).

A formula for the inverse

Theorem Let G be a block graph and let A be the adjacency matrix of G. Let A be nonsingular over GF(2) and let $B = A^{-1}$. Then $b_{ii} = 0, i = 1, \ldots, n$. Moreover, if $i \neq j$, then the following conditions are equivalent:

(i) $b_{ij} = 1$ (ii) $\det A(i|j) = 1$

(iii) $\det A(i,j|i,j) = 1$

(iv) $G \setminus \{i,j\}$ is nonsingular

(v) For any k other than i, j, $G \setminus \{i,j,k\}$ has exactly one odd component.

Equivalence of (ii) and (iii) follows from the Sylvester identity:

$$\det A(i|i) \det A(j|j) - \det A(i|j) \det A(j|i) = (\det A) \det A(i,j|i,j)$$

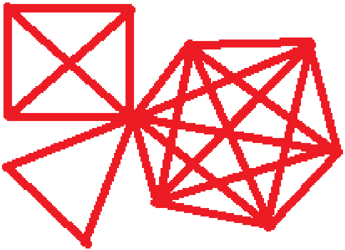
Example

For the block graph seen earlier,

$$A^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Flower

A flower is a block graph with only one cut-vertex.



Determinant of the adjacency matrix

Theorem Let F be a flower with n vertices and let $B_1, \ldots, B_k$ be the blocks of F. Let $\beta_i = |V(B_i)| - 2, i = 1, \ldots, k$. Let A be the adjacency matrix of F. Then

$$\det A = (-1)^{n-k} \left(k \prod_{i=1}^{k} \beta_i + \sum_{i=1}^{k} \prod_{j \neq i} \beta_j \right).$$

Singularity over GF(2)

Corollary Let F be a flower and let A be the adjacency matrix of F. Then A is singular over GF(2) if and only if F has exactly one even block.

Nullity over GF(2)

Theorem Let F be a flower and let A be the adjacency matrix of F. If F has t even blocks, then the nullity of A over GF(2) is $|t-1|$.

Line graph of a tree

The line graph of a tree is a block graph.

Moreover, each cut-vertex in the block graph is adjacent to at most two blocks.

The converse is also true.

Nonsingularity of the line graph of a tree

Theorem Let T be a tree with $n+1$ vertices. Then the line graph $L(T)$ is nonsingular (over GF(2)) if and only if n is even.

A formula for the inverse

Theorem Let T be a tree with $n+1$ vertices, n even. Let A be the adjacency matrix of $L(T)$ and let $B = A^{-1}$. Then $b_{ii} = 0, i = 1, \ldots, n$ and for $i \neq j$, $b_{ij} = 1$ if and only if $L(T) \setminus \{i, j\}$ has no odd component.

Nullity of the line graph of a tree

Theorem Let T be a tree with $n+1$ vertices, n odd. Then the incidence vector of the edges which produce an even-even partition is the unique null vector of the adjacency matrix of $L(T)$.

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Over reals, the nullity of $L(T)$ is at most one ...

Sample results over reals

(Gutman and Sciriha, 2001) For a tree T, the nullity of $L(T)$ is at most one.

(RBB, 2011) If G is a graph with an odd number of spanning trees, then the nullity of $L(G)$ is at most one.

(Ghorbani, 2012) If G is a graph with an odd number of vertices and an odd number of spanning trees, then $L(G)$ is nonsingular.

Inverse of a nonsingular tree

Buckley, Doty, Harary (1998), Pavlikova, Krc-Jediny (1990):

Theorem Lat T be a tree with a perfect matching, let A be the adjacency matrix of T and let $B = A^{-1}$. Then $b_{ij} = \pm 1$ if and only if there is an alternating path from i to j.

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Barik, Neumann and Pati (2006) show that the same formula holds for the inverse in case of a bipartite graph with a unique perfect matching.

If G is a bipartite graph with a unique perfect matching, then after a relabeling of vertices, the adjacency matrix A of G has the form

$$\begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix},$$

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$$\begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & (X')^{-1} \\ X^{-1} & 0 \end{pmatrix}.$$

Inverse of a lower-triangular matrix

In recent joint work with E. Ghorbani we have proved a combinatorial formula for the inverse of a lower-triangular matrix.

Bipartite graph associated with a matrix

A : $n \times n$ matrix

G_A : Bipartite graph with vertex set $\{R_1, \ldots, R_n\} \cup \{C_1, \ldots, C_n\}$

There is an edge from R_i to C_j if and only if $a_{ij} \neq 0$ with weight a_{ij}.

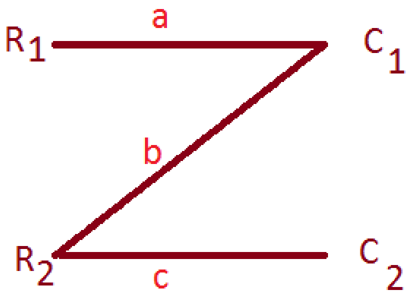
A formula for the inverse

Theorem Let A be a lower triangular $n \times n$ matrix with nonzero diagonal elements and let $B = A^{-1}$. Let $\mathcal{M}$ be the unique perfect matching in G_A consisting of the edges from R_i to $C_i, i = 1, \ldots, n$. Then for $1 \leq j \leq i \leq n$,

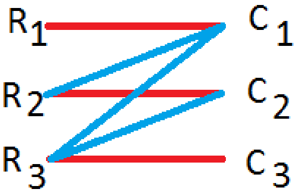
$$b_{ij} = \sum \epsilon(P) w(P),$$

where the summation is over all alternating paths P from R_j to C_i in G_A.

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}, \ A^{-1} = \begin{pmatrix} 1/a & 0 \\ -b/ac & 1/c \end{pmatrix}$$



$$A = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \text{ If } B = A^{-1}, \text{ then } b_{31} = -d/af + be/acf$$



Remark

The formula for the inverse of a bipartite graph with a unique perfect matching (and the inverse of a nonsingular tree) follow as special cases.

Conclusion

- A formula for the determinant of the adjacency matrix of a block graph.
- Necessary and sufficient condition for a block graph to be nonsingular over GF(2).
- A formula for the inverse of the adjacency matrix of a block graph over GF(2).
- A formula for the inverse of a lower triangular matrix.

Thank You!