Shadows and cryptology

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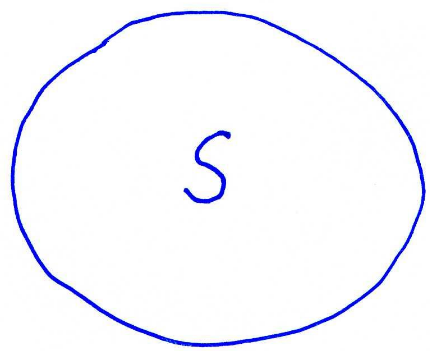
The practical problem

label,

impossible to copy because of its 3-dimensional nature

A mapping

$f :$



$\longrightarrow \{0,1\}^n$

space of all possible labels

$|S| >>> 2^n$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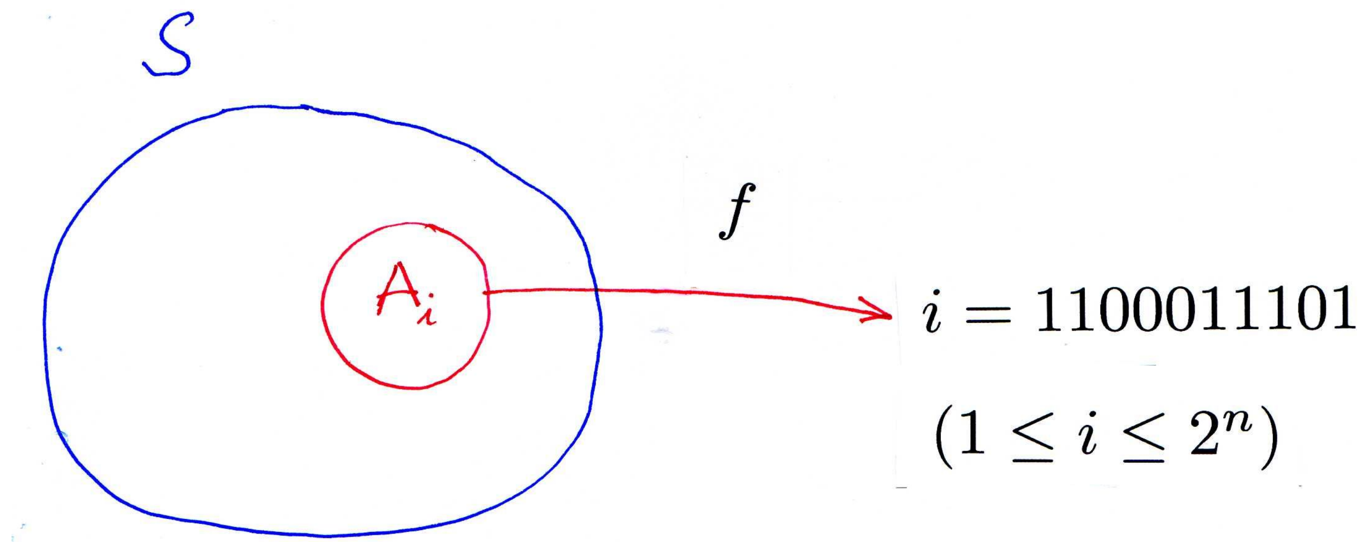
Its properties



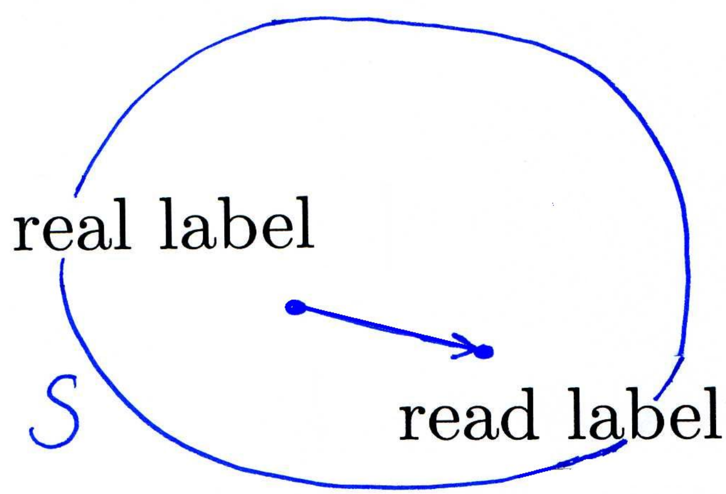
$$A_i \cap A_j = \emptyset \quad (i \neq j)$$

Reading with error

a distance is given on S

$$0 \leq d(a,b) \quad a,b \in S$$

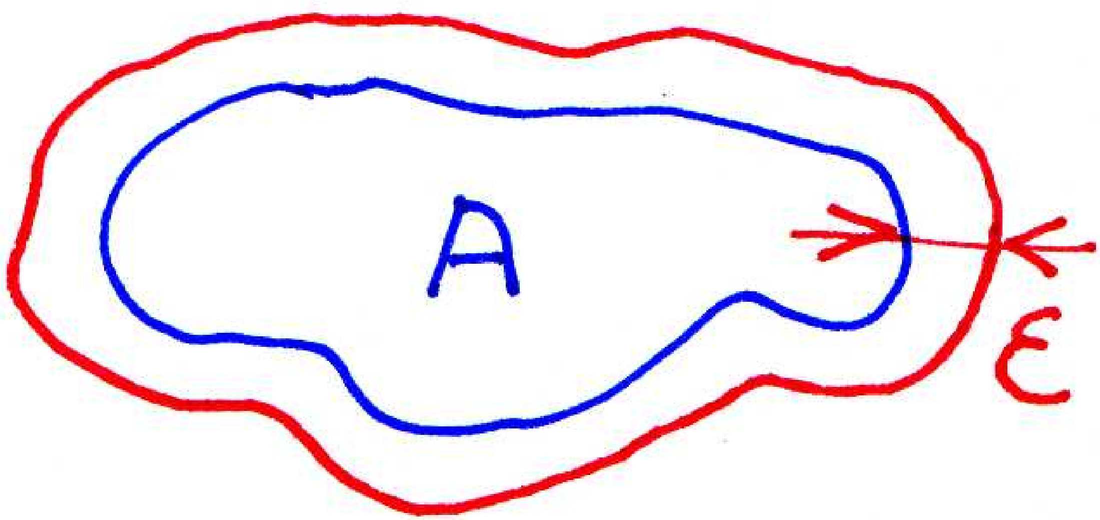
$$d(\text{real label},\ \text{read label}) \leq \varepsilon$$ where $0 < \varepsilon$ is given



neighborhood

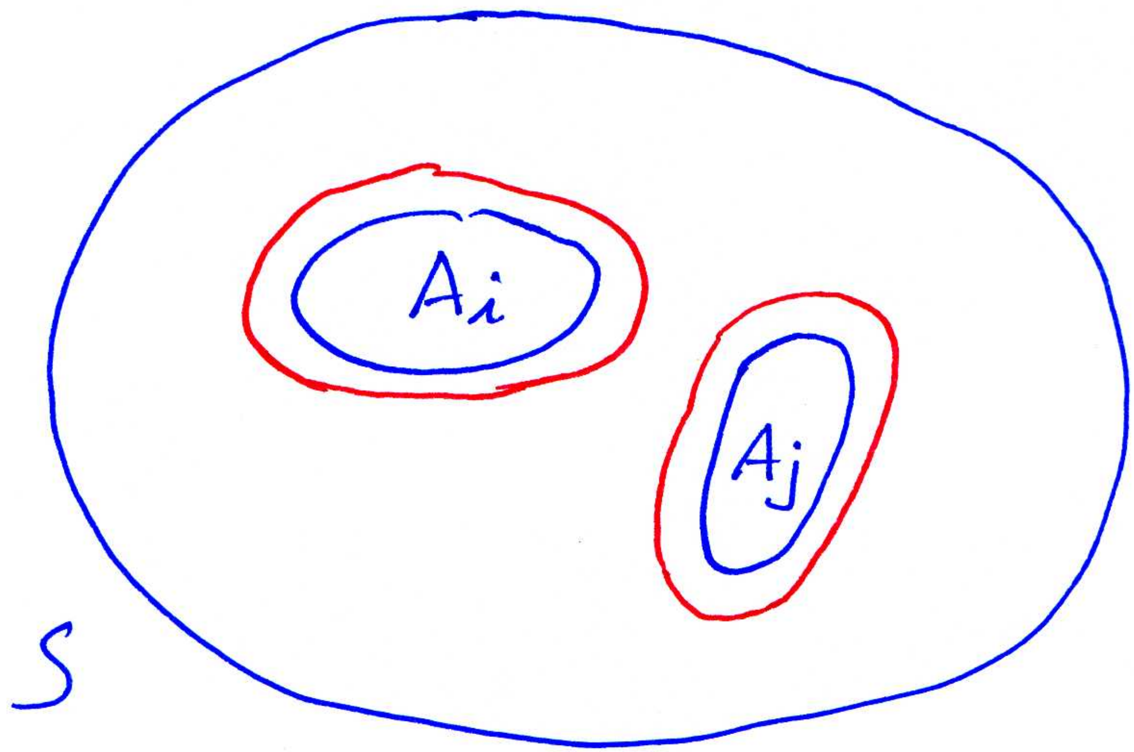
$A \subset S$

$n(A, \varepsilon) = \{x \in S : \ d(A, x) \leq \varepsilon\}$



neighborhoods are disjoint

$$n(A_i, \varepsilon) \cap n(A_j, \varepsilon) = \emptyset \ (i \neq j)$$



A_i cannot be large

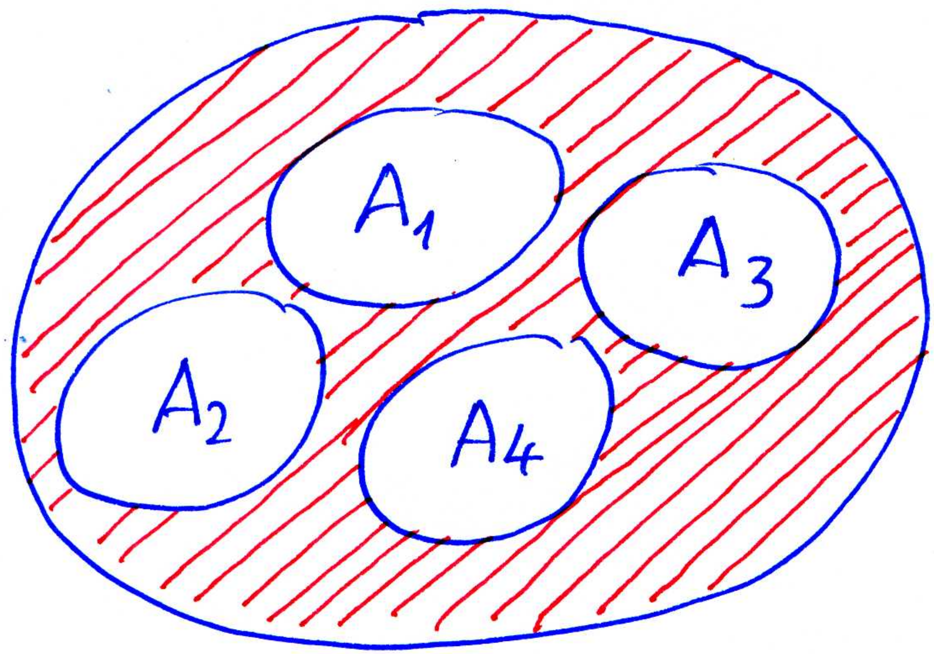
If A_i is large then the swindler chooses an $x \in S$ “randomly” and $x \in A_i$ will have a “large probability”.

This is why a measure μ must be considered on S.

$\mu(A_i) \leq \rho \; (1 \leq i \leq 2^n)$

where $0 < \rho$ is given.

waste



waste cannot be too large:

$\mu\left(\cup_{i=1}^{n} A_i\right) \geq \alpha$ where $0 < \alpha$ is not very small, say 0.1.

Definition

Geometric identifying code in S

(endowed with a distance d and a measure μ) with

error tolerance ε (> 0),

waste rate α (> 0),

security ρ (> 0)

is a family of sets $A_1, \ldots, A_{2^n} \subset S$ such that

$n(A_i, \varepsilon) \cap n(A_j, \varepsilon) = \emptyset \ (1 \leq i < j \leq 2^n)$,

$\mu\left(\cup_{i=1}^n A_i\right) \geq \alpha$,

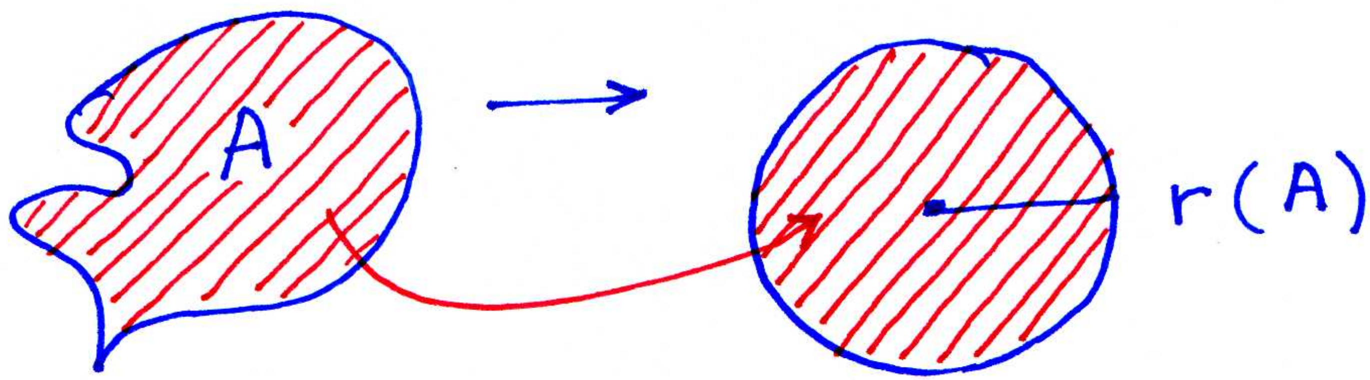
$\mu(A_i) \leq \rho \ (1 \leq i \leq 2^n)$.

Ball of radius r with center x: $b(x, r)$

Suppose:

$(*)$ $\mu(b(x, r))$ **independent of** x $(= \mu(r))$

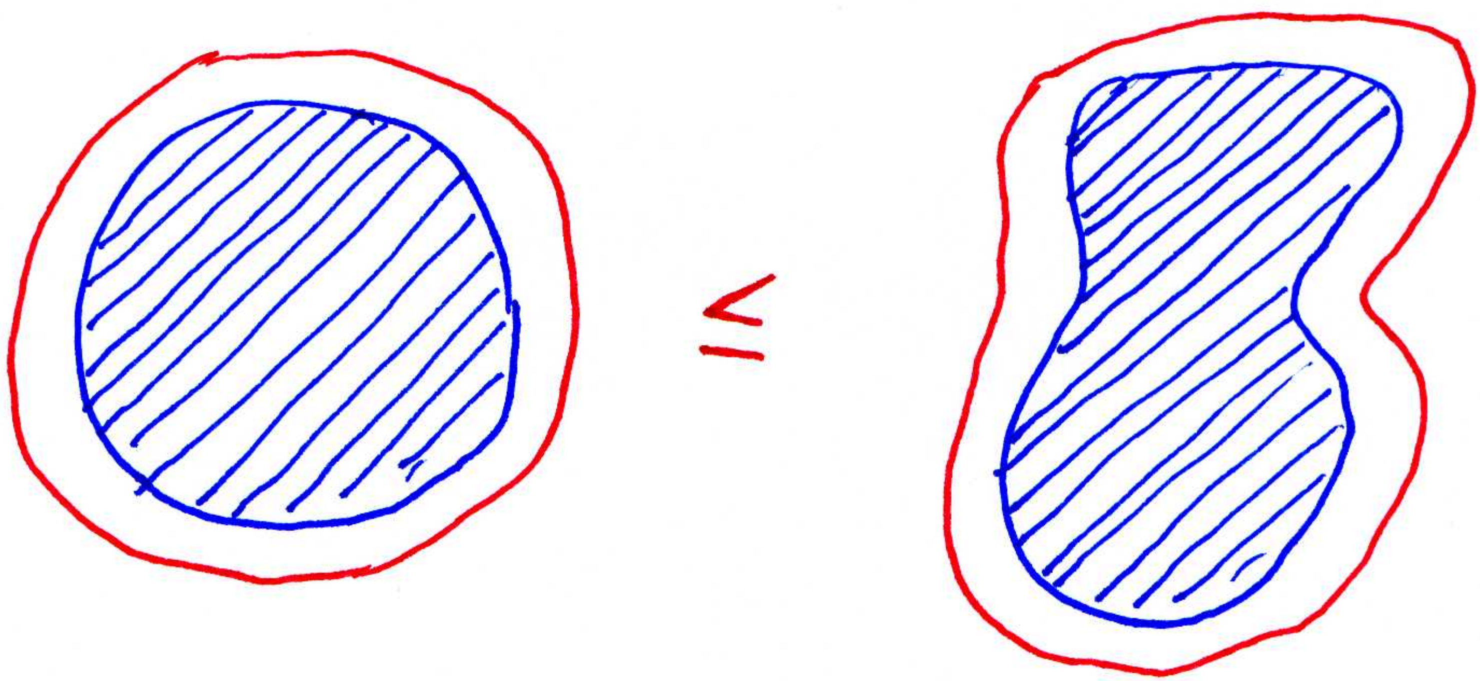
Notation: $r(A) =$ radius of a ball with measure $= \mu(A)$



Brunn-Minkowski property

Suppose:

$$(**) \qquad \mu(n(b(x,r(A))),\varepsilon) \leq \mu(n(A,\varepsilon))$$



$\mu^{-1}(x)$ is the **radius** of a ball with measure x.

Theorem (Csirmaz-Katona, 2003)

If $(*)$ and $(**)$ hold then

$$\varepsilon \le \mu^{-1}\left(\frac{\rho}{\alpha}\right) - \mu^{-1}(\rho)$$

for a geometric identifying code with parameters $\varepsilon, \alpha, \rho$.

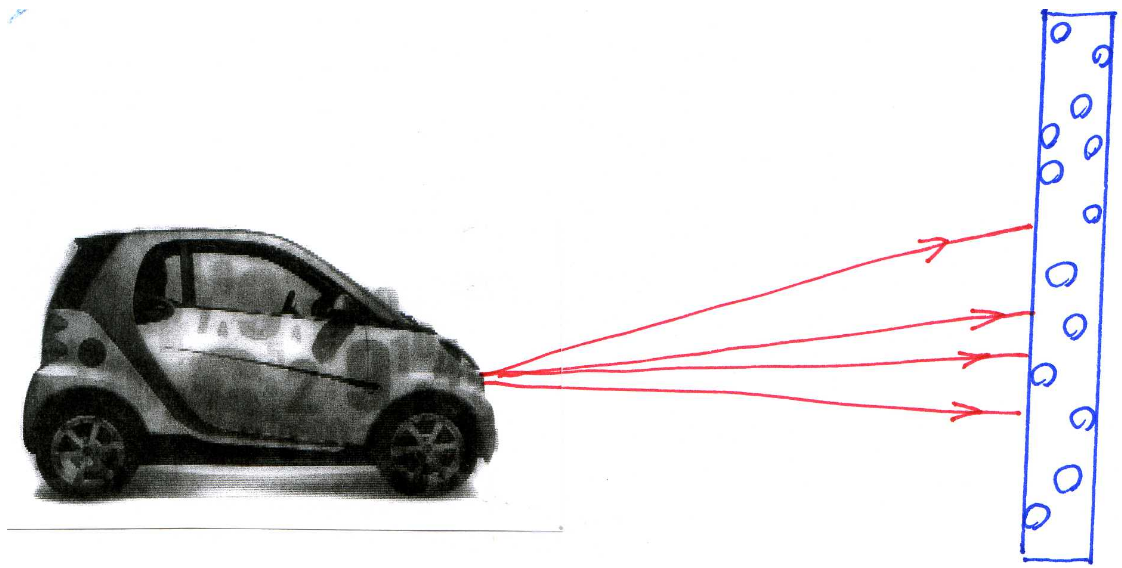
A functioning prototype was constructed by

Haraszti, Marsovszky (Hewlett Packard, Hungary)

and

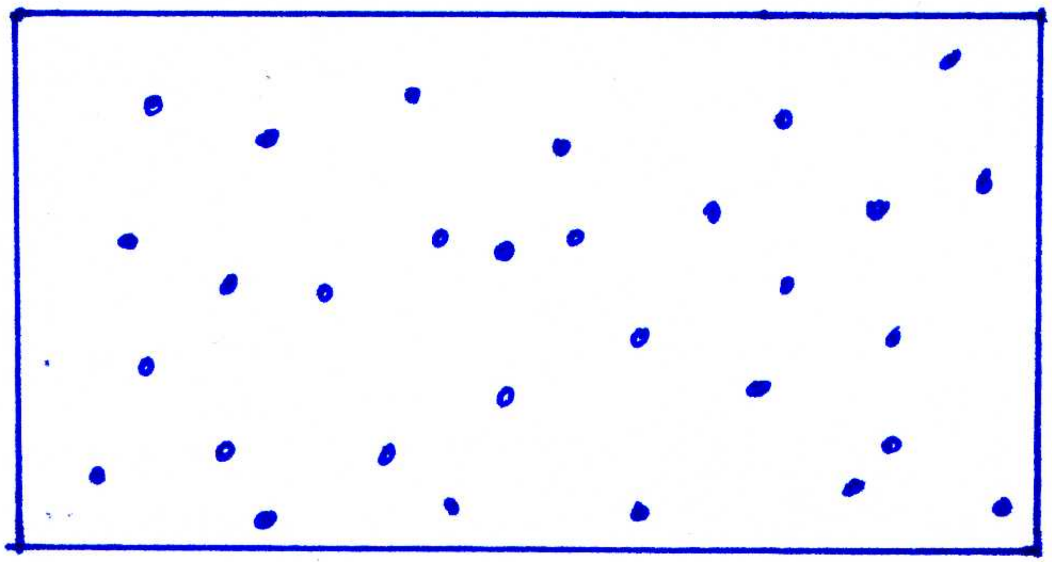
Csirmaz, Katona, Miklós, Nemetz (Rényi Institute, Budapest)

Label: reflecting foil



random, 3-dimensional

mathematically



a set of points (centers of the glass balls)

their number is between two bounds

unordered

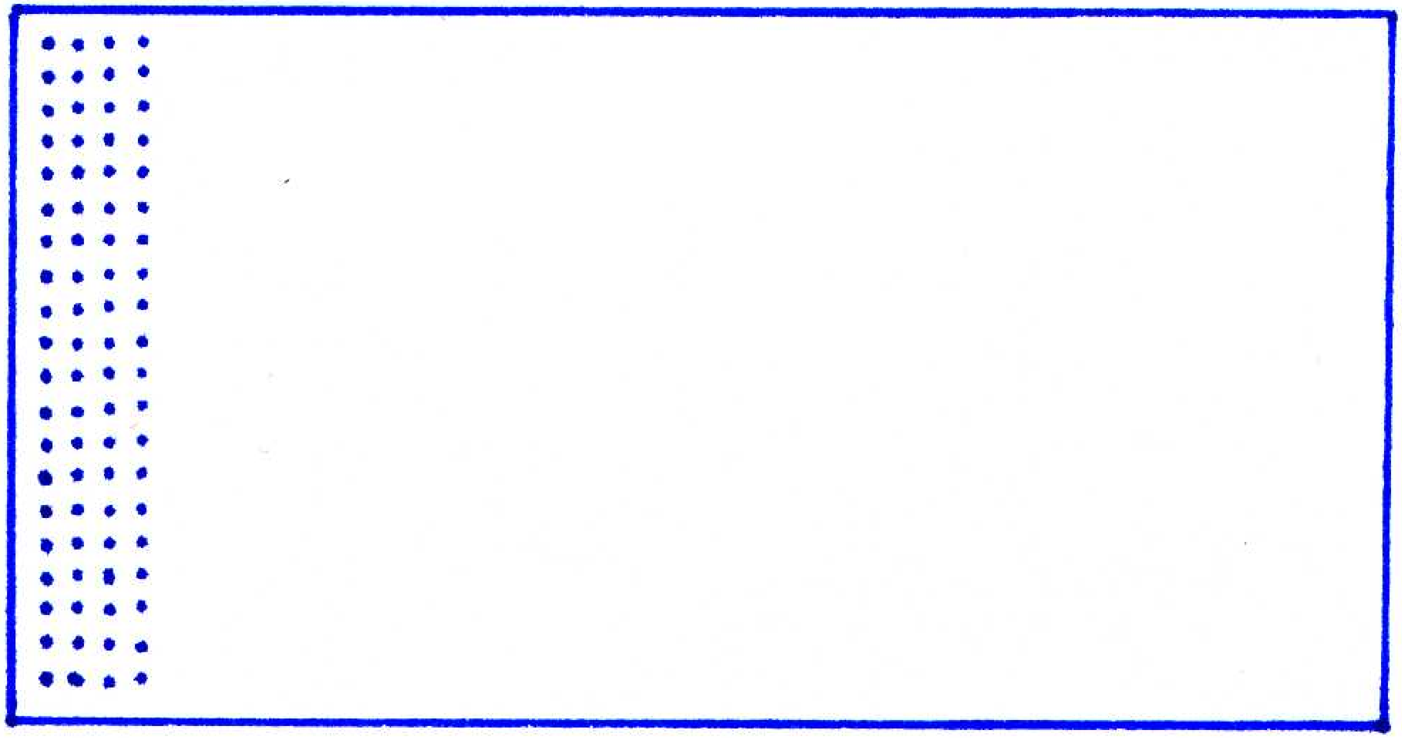
Big surprise: some points disappear at the reading

$\Longrightarrow$

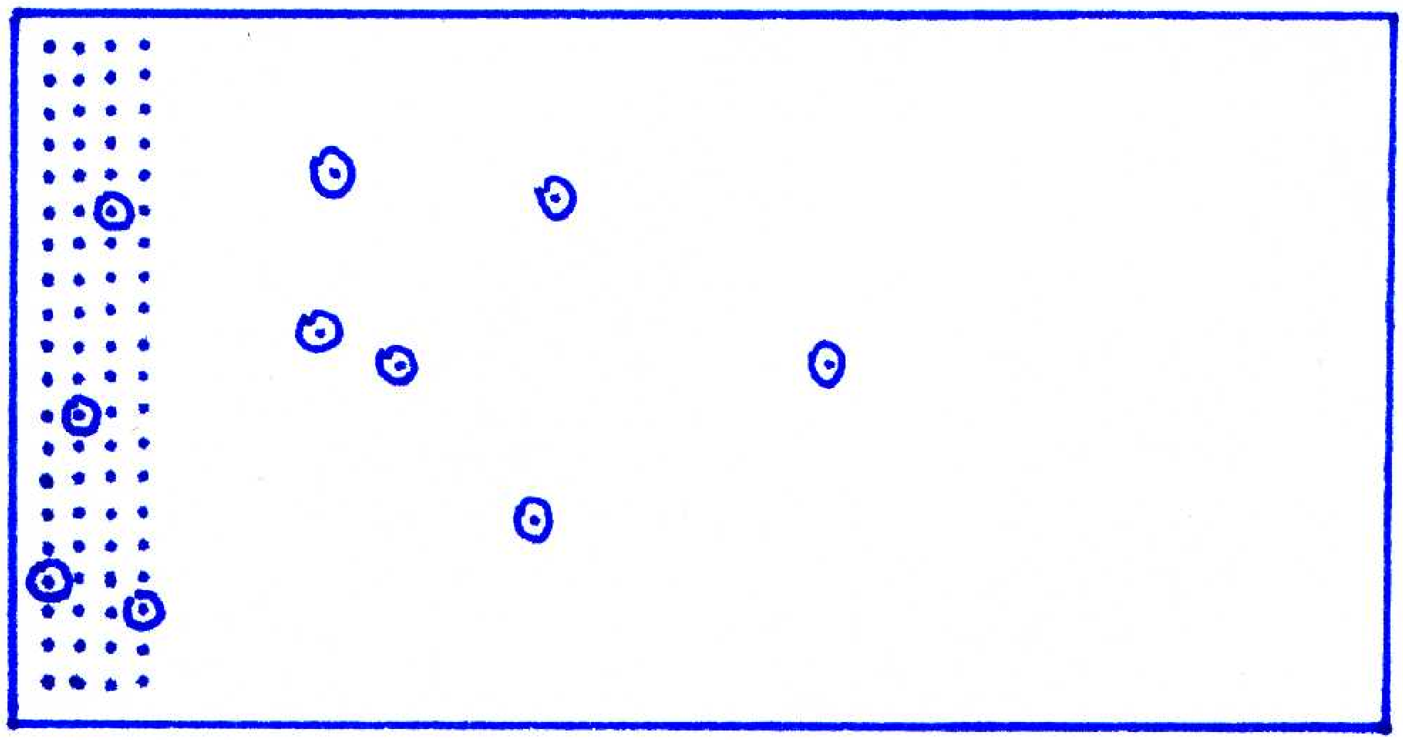
distance d between labels is defined not only by their geometry, but the also by the set distance

computational accuracy $\Longrightarrow$

number of possible points is finite



A **label** is a **subset** of the possible points



Our model in this particular case

Given a finite set of N points.

Space S consists of subsets of this N-element set X.

Illustrated by 0,1 sequences of length N.

Given $0 < k < N$, suppose that the number of elements (number of 1s in the sequence) is exactly k.

This is a condition to make the model mathematically treatable. The real condition would be: the number of 1s is between two bounds.

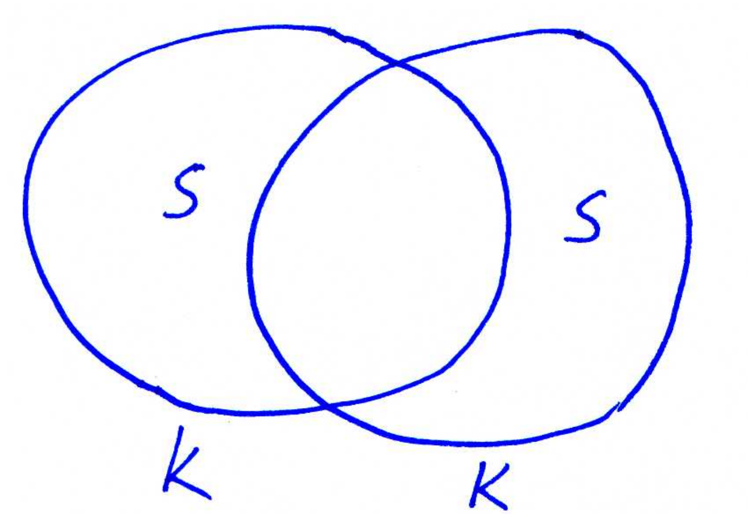
The size of the space is

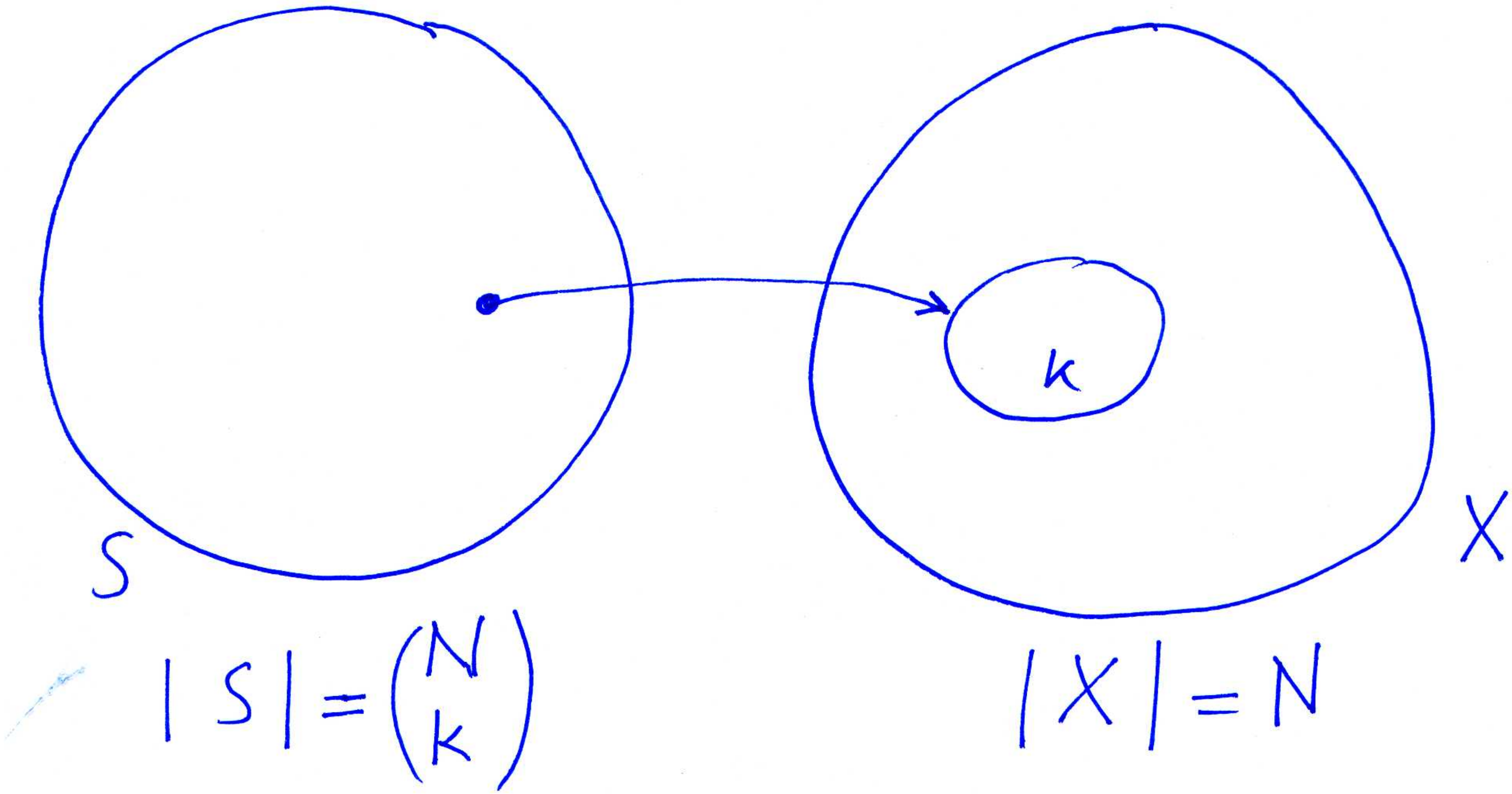
$$|S| = \binom{N}{k}.$$

Distance d of two sequences in S $=$ Hamming distance

(= number of different digits = size of the symmetric difference)

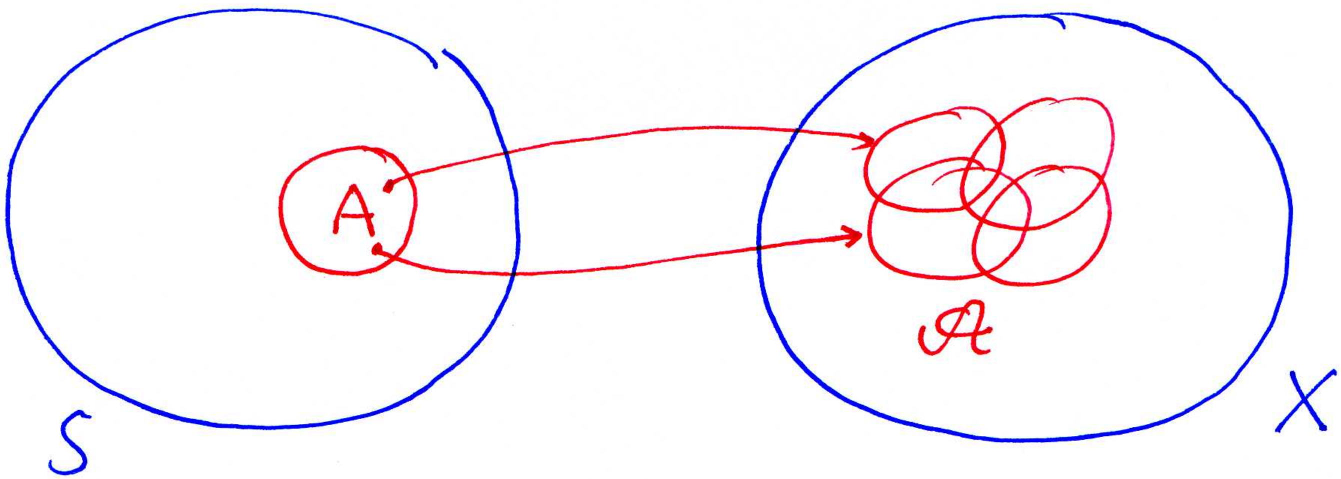
This is a condition to make the model mathematically treatable. The real distance would be a combination of this with the geometrical distance. For instance, the distance of two such sets is $2s$ if:





One element of S is a k-element subset of (the N-element) X.

A set A of elements of S is a family of k-element sets. It will be denoted by $\mathcal{A}$.



Its measure:

$$\mu(\mathcal{A}) = \frac{|\mathcal{A}|}{\binom{N}{k}}.$$

- **security:**

$$\mu(\mathcal{A}_i) = \frac{|\mathcal{A}_i|}{\binom{N}{k}} < \rho,$$

that is,

$$|\mathcal{A}_i| < \rho \binom{N}{k} = \binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}$$

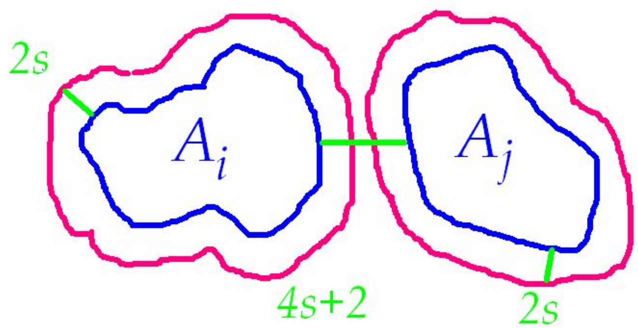
for some positive real number $x > 0$.

Neighbor of $\mathcal{A}$

= $n(\mathcal{A}, 2s)$: all k-element sets of distance at most $2s$ from a member of $\mathcal{A}$, that is, the k-element sets obtained from a member by deleting at most s elements and adding the same number of elements.

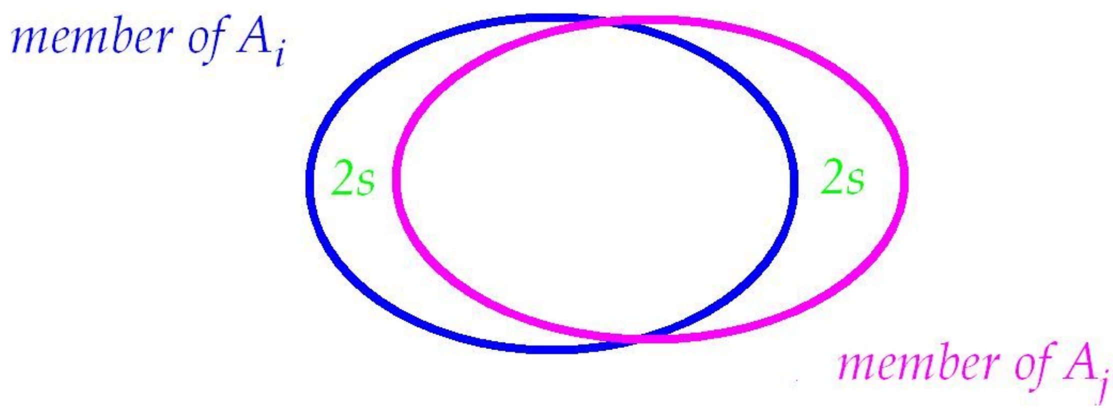
$n(\mathcal{A}_i, 2s) \cap n(A_j, 2s) = \emptyset$ iff $d(\mathcal{A}_i, \mathcal{A}_j) \geq 4s + 2$.

Illustration:

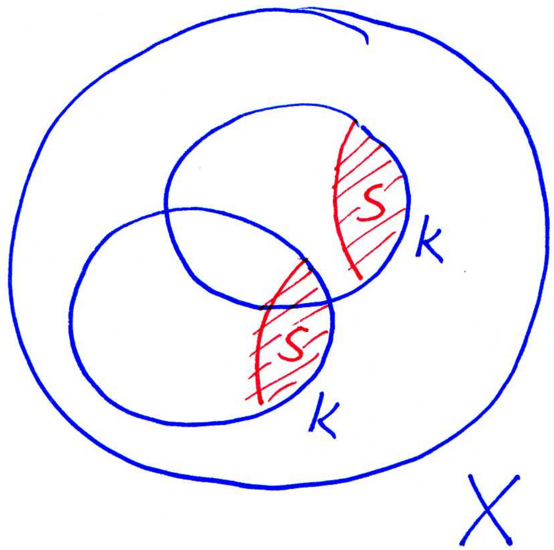


In other words, deleting $2s$ elements from a member of $\mathcal{A}_i$ and deleting $2s$ elements from a member of $\mathcal{A}_j$, two different $k-2s$-element sets are obtained.

This is **forbidden**:



Definition. The s-shadow $\sigma_s(\mathcal{A})$ of $\mathcal{A}$ is the family of all $k - s$-element sets obtained from the members of $\mathcal{A}$ by deleting s elements.



$$\sigma_s(\mathcal{A}) = \{B : \ |B| = k - s, B \subset A \in \mathcal{A}\}$$

- **error tolerance** $2s$

$$n(A_i, 2s) \cap n(A_j, 2s) = \emptyset \ (1 \leq i < j \leq 2^n),$$

or

$$\sigma_{2s}(\mathcal{A}_i) \cap \sigma_{2s}(\mathcal{A}_j) = \emptyset$$

- **waste rate** $\alpha\ (> 0)$

$$\mu\left(\cup_{i=1}^{n}\mathcal{A}_i\right) = \frac{\sum_{i=1}^{n}|\mathcal{A}_i|}{\binom{N}{k}} \geq \alpha,$$

Given the security ρ and the error tolerance $2s$, maximize α.

Combinatorial problem. Given N, maximize $\sum_{i=1}^{n}|\mathcal{A}_i|$ under the conditions

$$|\mathcal{A}_i| \leq \binom{x}{k}$$

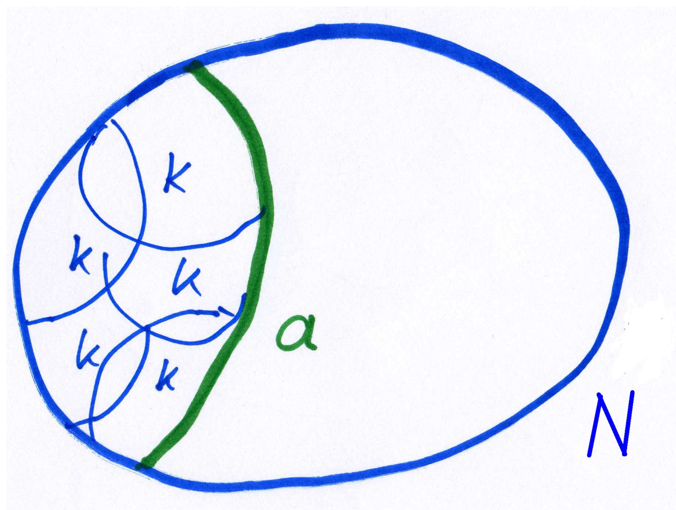
and

$$\sigma_{2s}(\mathcal{A}_i) \cap \sigma_{2s}(\mathcal{A}_j) = \emptyset.$$

Shadow problem

Given N, k and $|\mathcal{F}|$, **minimize** $|\sigma(\mathcal{F})|$.

If lucky then $|\mathcal{F}| = \binom{a}{k}$ holds for an integer a then the best construction is



min$|\sigma(\mathcal{F})| = \binom{a}{k-1}$

Shadow problem

Lemma If $0 < k, m$ are integers then one can find integers $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ such that

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

and they are unique.

This is called the **canonical form** of m.

Shadow problem

Shadow Theorem (Kruskal-K) If N, k and $|\mathcal{F}|$ are given,

the canonical form of $|\mathcal{F}|$ is

$$|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

then

$$\min |\sigma(\mathcal{F})| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_t}{t-1}.$$

another formulation

characteristic vector of the set $A \subset [N]$

(0,	0,	1,	1,	0,	...	1,	...,	0)
1	2	3	4	5	...	k	...	n

the kth coordinate is 1 iff $k \in A$

another formulation

The previous construction with $k = 3, a = 5$, that is $|\mathcal{F}| = \binom{5}{3} = 10$

(1,	1,	1,	0,	0,	0,	0,	...	)
(1,	1,	0,	1,	0,	0,	0,	...	)
(1,	0,	1,	1,	0,	0,	0,	...	)
(0,	1,	1,	1,	0,	0,	0,	...	)
(1,	1,	0,	0,	1,	0,	0,	...	)
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(1,	0,	0,	1,	1,	0,	0,	...	)
(0,	1,	0,	1,	1,	0,	0,	...	)
(0,	0,	1,	1,	1,	0,	0,	...	)
1	2	3	4	5	6	7	...	

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another formulation

Arabically lexicographical ordering

Version of the **Shadow Theorem**

If N, k and$|\mathcal{F}|$ are given, then

$|\sigma(\mathcal{F})|$ is minimum for

the lexicographically first $|\mathcal{F}|$ subsets containing k elements.

s-shadow

Shadow Theorem **(Kruskal-K)** If N, k and $|\mathcal{F}|$ are given,

the canonical form of $|\mathcal{F}|$ is

$$|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

then

$$\min |\sigma_s(\mathcal{F})| = \binom{a_k}{k-s} + \binom{a_{k-s}}{k-1-s} + \ldots + \binom{a_t}{t-s}.$$

Only an estimate

If x is a real number, $\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}$.

Theorem. (Lovász' version of the Shadow theorem) If $\mathcal{A}$ is a family of k-element sets,

$$|\mathcal{A}| = \binom{x}{k}$$

then

$$|\sigma_s(\mathcal{A})| \geq \binom{x}{k-s}.$$

Theorem. If $\mathcal{A}_i$ are families of k-element subsets of an N-element set,

$$|\mathcal{A}_i| \le \binom{x}{k} \ (1 \le i \le n)$$

and

$$\sigma_{2s}(\mathcal{A}_i) \cap \sigma_{2s}(\mathcal{A}_j) = \emptyset \ (1 \le i < j \le n)$$

hold, then

$$\frac{\sum_{i=1}^{n} |\mathcal{A}_i|}{\binom{N}{k}} \le \left(\frac{x}{N}\right)^{2s}.$$

Unfortunately this is **not sharp**.

A very special case, modified

$k = 3, s = 1, n = 2$

$\mathcal{A}, \mathcal{B}$ are families of 3-element subsets of an N-element set.

If $A \in \mathcal{A}, B \in \mathcal{B}$ then $|A \cap B| \leq 1$.

$|\mathcal{A}| = |\mathcal{B}|$

Find $\max |\mathcal{A}|$.

A weak estimate from Shadow Theorem

Choose x in this way: $|\mathcal{A}| = |\mathcal{B}| = \binom{x}{3}$

By the Shadow Theorem: $\binom{x}{2} \leq |\sigma(\mathcal{A})|, |\sigma(\mathcal{B})|$

$$2\binom{x}{2} \leq |\sigma(\mathcal{A})| + |\sigma(\mathcal{B})| \leq \binom{N}{2}$$

From here, asymptotically

$$|\mathcal{A}| = \binom{x}{3} \leq \frac{1}{2\sqrt{2}}\binom{N}{3}(1 + o(1))$$

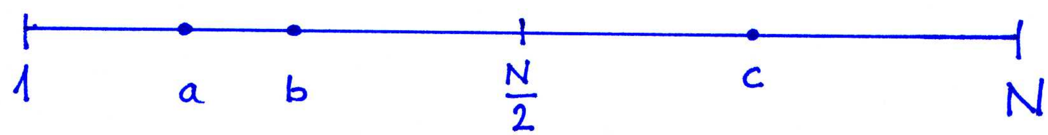
Trivial construction gives:

$$|\mathcal{A}| = \frac{N^3}{48}(1 + o(1)).$$

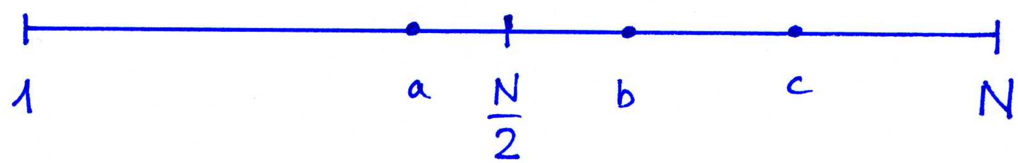
$$\mathcal{A} = \left\{(a, b, c) : a < b < c \leq \frac{N}{2}\right\}, \mathcal{B} = \left\{(a, b, c) : \frac{N}{2} \leq a < b < c\right\}.$$

A better construction: $|\mathcal{A}| = \frac{N^3}{24}(1 + o(1))$.

$$\mathcal{A} = \left\{(a, b, c) : \frac{b+c}{2} \leq \frac{N}{2}\right\}$$



$$\left\{B = \{(a, b, c) : \frac{N}{2} < \frac{a+b}{2}\right\}.$$



Theorem (Frankl-Kato-Katona-Tokushige, 2012+)

$$\max |\mathcal{A}| = \kappa \binom{N}{3} (1 + o(1))$$

where κ is the unique real root in the (0,1)-interval of the equation

$$z^3 = (1 - z)^3 + 3z(1 - z)^2.$$

Theorem (Frankl-Kato-Katona-Tokushige, 2012+)

If $\mathcal{A} \subset \binom{[N]}{k}$ then

$$\max |\mathcal{A}| = \mu_k \binom{N}{k} (1 + o(1))$$

where μ_k is the unique real root in the (0,1)-interval of the equation

$$z^k = (1-z)^k + kz(1-z)^{k-1}.$$

Badly needed generalizations

s-shadows rather than $s = 1$.

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Combining this distance with the **geometric distance**.

Thanks

Köszönöm

Grazie


تشکر میکنم


谢谢