Toughness and Hamiltonicity of graphs on surfaces

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Partially Joint Work with

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Hamilton cycles

Hamilton cycle : Connection to TSP or other topics

Hamiltonicity of graphs on a surface

Tait (1884) :

$\exists$ Hamilton cycles in $\forall$ cubic map **False**

$\Downarrow$

$\exists$ 4-coloring in $\forall$ map **True (4 Color Thm)**

Hamilton cycles

G : Hamilton-connected

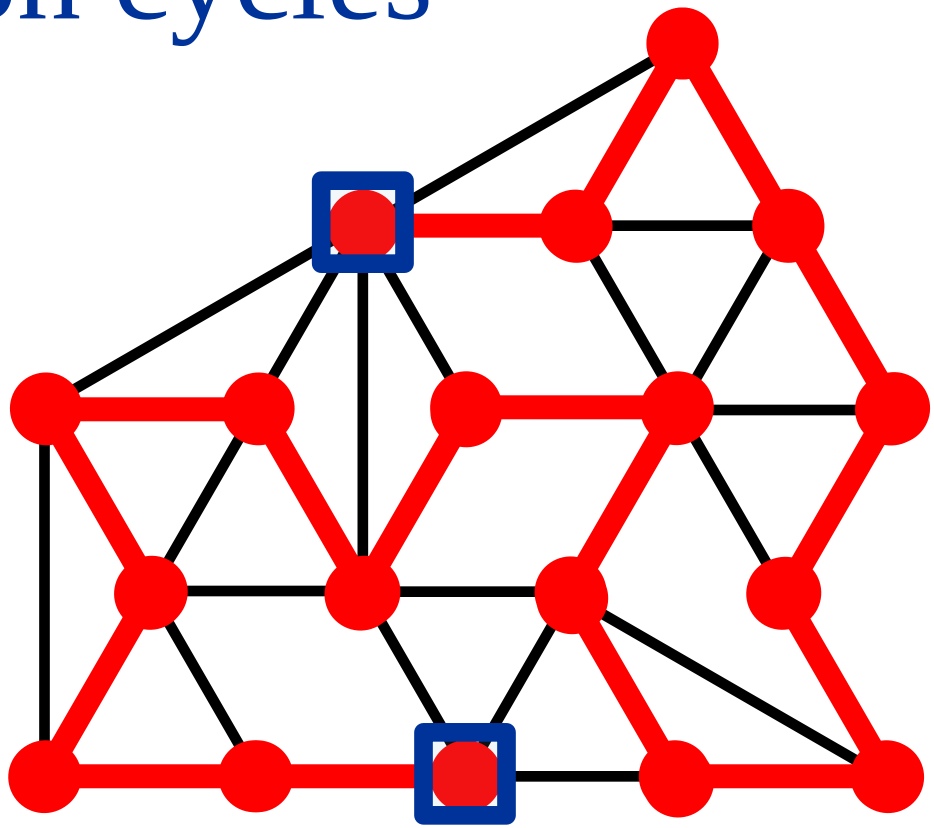
$\Updownarrow$

For $\forall$ pair of vertices,
$\exists$H-path between them

G : Hamilton-connected

$\Rightarrow$ $\exists$Hamilton cycle

$\Rightarrow$ $\exists$Hamilton path



$\bullet \in V(G)$ $\in E(G)$

Hamilton cycles

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For $\forall$ pair of vertices,
$\exists$H-path between them

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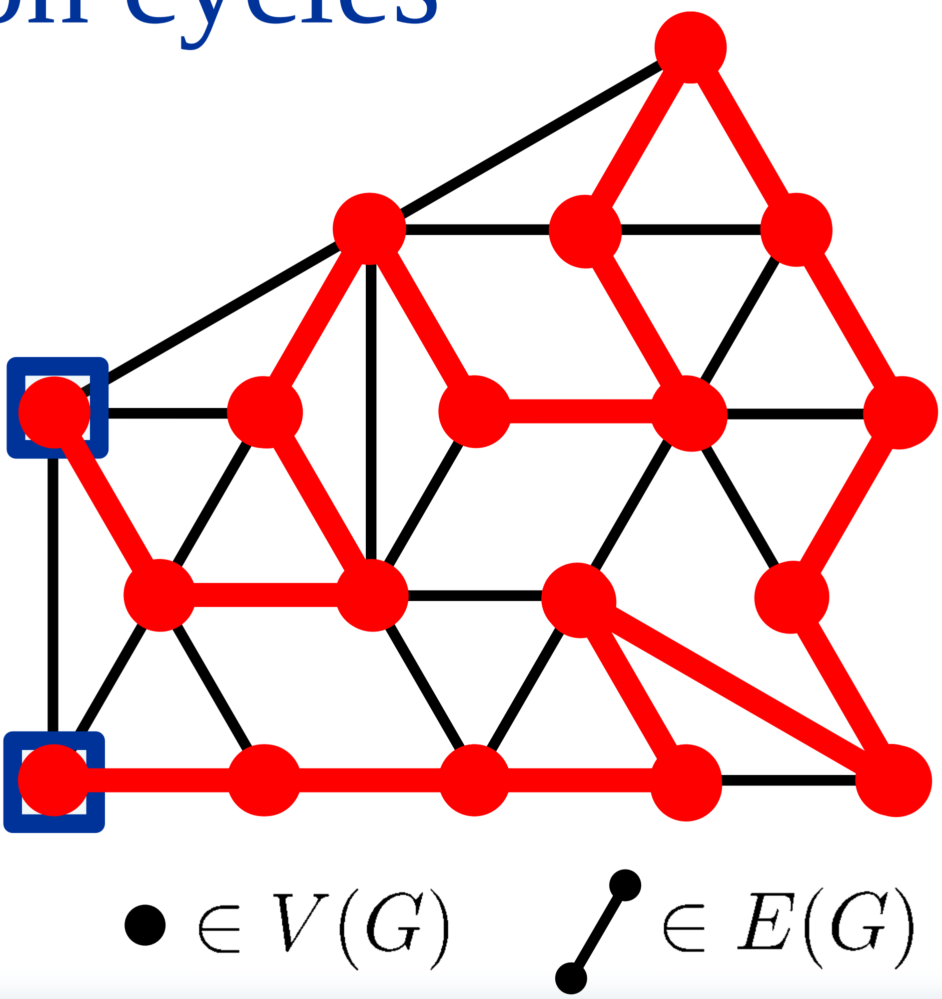


Toughness of a graph

Toughness (type) condition : **Necessary condition!!**

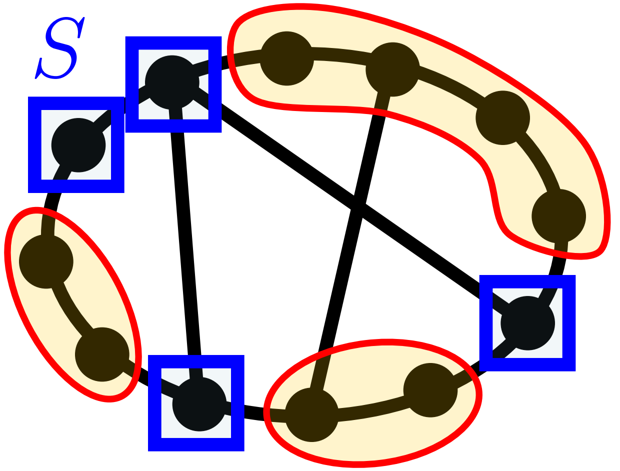
$(*) \forall S \subset V(G)$: cutset, (# of components in $G-S$) $\leq a|S|+b$

$\forall$ graph G satisfies $(*)$ with

$(a,b)=(1,1)$ if G : $\exists$ Hamilton path

$=(1,0)$ $\exists$ Hamilton cycle

$=(1,-1)$ Hamilton-connected



(# comp.s in $G-S$) $\leq |S|$

Toughness of a graph

Toughness (type) condition : **Necessary condition!!**

(*) $\forall S \subset V(G)$: cutset, (# of components in $G - S$) $\leq a|S| + b$

(a,b)	
$(1,-t)$	$\exists$ H-cycle if we delete $\forall$ t vertices
$(1,-t+1)$	t-leaf connected
$(1,-1)$	Hamilton-connected
$(1,0)$	$\exists$ Hamilton cycles
$(1,1)$	$\exists$ Hamilton path
$(1,t-1)$	$\exists$ Sp. tree with $\leq t$ leaves

(a,b)	
$(k,0)$	$\exists$ Sp. closed walk passing $\forall$ vrt. $\leq k$ times
$(k,0)$	$\exists$ vertex cover with $\leq k$ cycles
$(k,0)$	k-prism Hamiltonian
$(k-1,1)$	$\exists$ Sp. tree with max. deg. $\leq k$
$(k-1,t+1)$	$\exists$ Sp. tree with te $\leq t$ from k

Related to $\exists$ 1-factor, matching extension etc.

Toughness of a graph

Toughness (type) condition : **Necessary condition!!**

$(*) \forall S \subset V(G)$: cutset, (# of components in $G-S$) $\leq a|S|+b$

We expect that a graph satisfying $(*)$ has a good property

Ex. $\exists$ H-cycle, $\exists$ H-path, … $\Rightarrow$???

Conjecture (Chvátal `73)

$\exists\, t$: integer s.t. $\forall G$ satisfying $(*)$ with $(a,b)=(\frac{1}{t},0)$ has an H-cycle

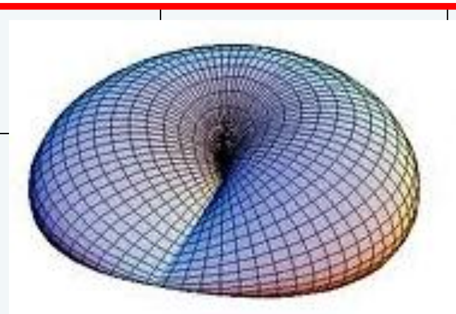
But how about graphs on surfaces??

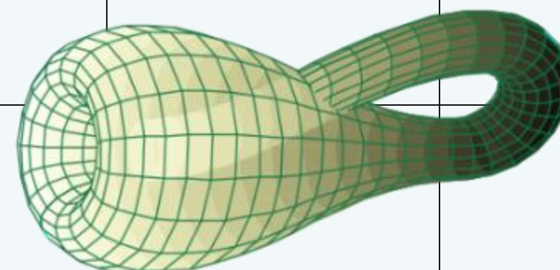
Graphs on surfaces

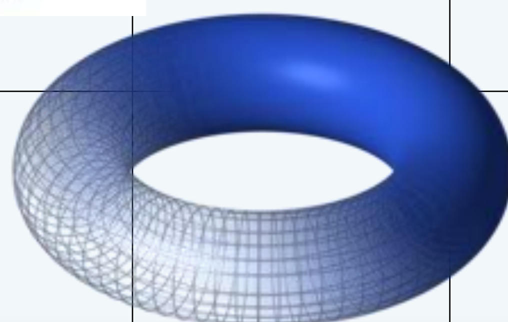
Euler characteristic

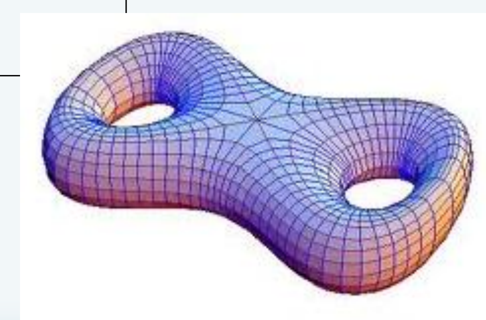
	Plane	Proj. Plane	Torus	Klein bottle	Others
	$\chi = 2$	$\chi = 1$	$\chi = 0$	$\chi = 0$	$\chi < 0$

$\forall G$: graph on a surface, $|V(G)| - |E(G)| + |F(G)| \geq \chi$









Graphs on surfaces

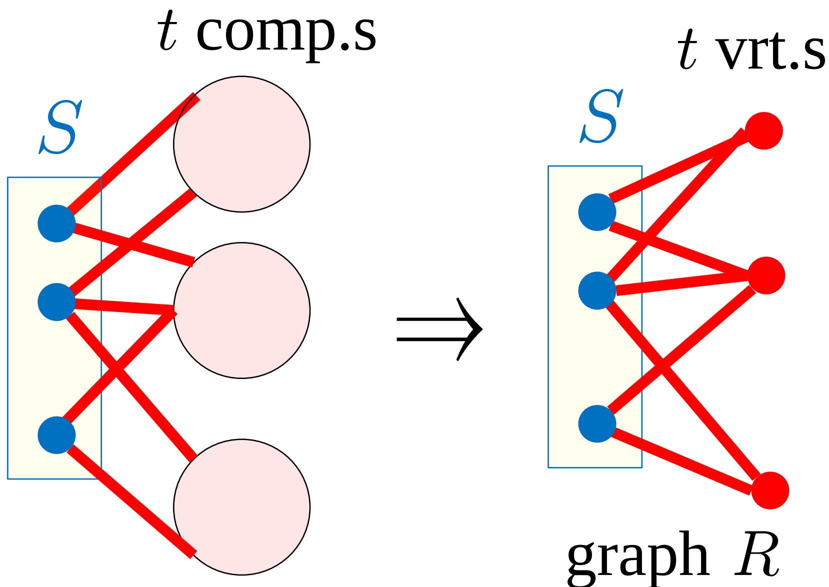
$\forall\, F^2$: a surface $\quad 3 \le k \le 5$

$\forall\, G$: k-conn. graph on F^2

$\forall\, S$: a vertex set of G

comp.s in $G - S$

$$\le \frac{2}{k-2}|S| + \frac{-2\chi(F^2)}{k-2}$$



$|E(R)| \ge kt$

$|E(R)| \le 2\big(|S| + t\big) - 2\chi$

G : k-conn. graph on F^2

(# comp.s in $G - S$) $\le a|S| + b$ holds for $(a, b) = \left(\frac{2}{k-2}, \frac{-2\chi}{k-2}\right)$

Graphs on surfaces

$^{\forall} F^2$: a surface $3 \le k \le 5$

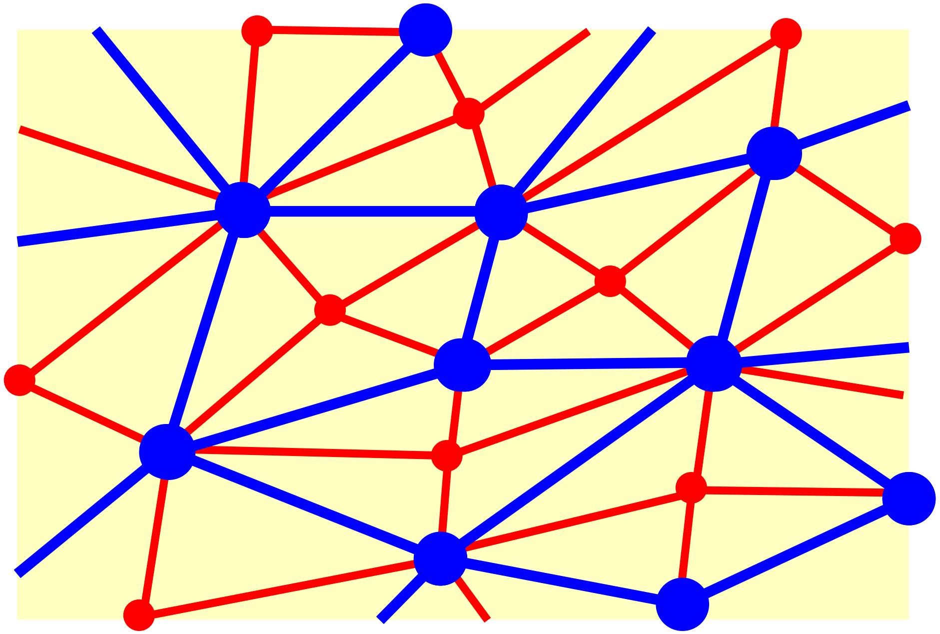
$\exists G$: k-conn. triangulation

$\exists$ S : a vertex set of G

s.t.

comp.s in $G - S$

$= |F(H)| = \frac{2}{k-2}|S| + \frac{-2\chi(F^2)}{k-2}$



H : Tri/quadr/pent-angulation of F^2

G : Face sub. of H $S = V(H)$

G : k-conn. graph on F^2

(# comp.s in $G - S$) $\le a|S| + b$ holds for $(a, b) = \left(\frac{2}{k-2}, \frac{-2\chi}{k-2}\right)$

Graphs on surfaces

- Graphs on surfaces, connectivity and toughness

G : k-conn. graph on F^2

(# comp.s in $G - S$) $\leq a|S| + b$ holds for $(a, b) = \left(\frac{2}{k-2}, \frac{-2\chi}{k-2}\right)$

This bound is best possible

- Properties concerning Hamiltonicity (and toughness)

Statement : $\forall$ k-conn. graph on a surface F^2 is [...]

We have to think about toughness type necessary condition

What else do we need?

Graphs on surfaces

Nothing???

There are NO known counterexample to the following type statement, EXCEPT for the toughness reason

I'll show several results and conjectures as ``evidences'' for it

- Properties concerning Hamiltonicity (and toughness)

Statement : $\forall$ k-conn. graph on a surface F^2 is [...]

We have to think about toughness type necessary condition

What else do we need?

For properties with $a = 1$

$k = 4$ Necessary condition

$-\chi \le b \Rightarrow$ OK.

$-\chi > b \Rightarrow$ Not satisfied

$(a, b) = (1, -\chi)$

- 4-conne[...]

G : k-conn.

(# comp.s in $G - S$) $\le a|S| + b$ holds for $(a, b) = \left(\frac{2}{k-2}, \frac{-2\chi}{k-2}\right)$

This bound is best possible

- Properties concerning Hamiltonicity (and toughness)

Statement : $\forall$ k-conn. graph on a surface F^2 is [...]

Ex. $\exists$ Hamilton cycle → $(a, b) = (1, 0)$ is required

Graphs on a surface with $-\chi \le 0$ satisfies the necessary condition

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$						×
H-cycle $b = 0$					×	×
H-conn $b = -1$			×	×	×	×

$k = 4$ <u>Necessary condition</u>

$-\chi \leq b \Rightarrow$ OK.

$-\chi > b \Rightarrow$ Not satisfied

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)			×
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)			×	×
H-conn $b = -1$	○ Thomassen ('83)		×	×	×	×

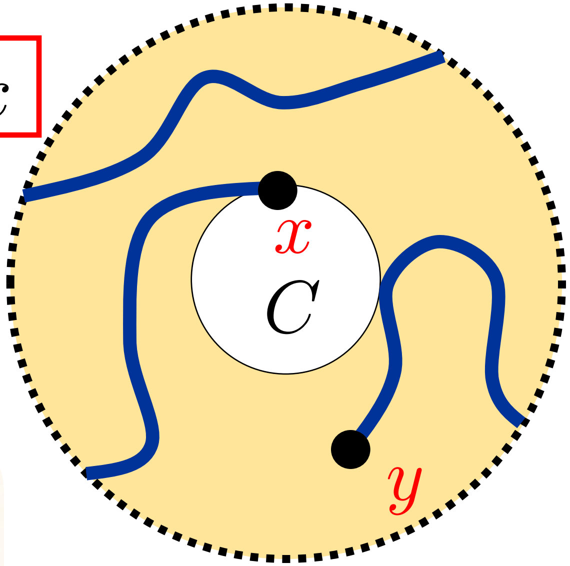
4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)	○ K.K. & Oz.('12+)	?	×
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)	? Grunbaum('70) Nash-Williams('73)	?	×	×
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+) Dean ('90)	×	×	×	×

Idea of the proof

Condition G ~~: 4-conn~~. 2-conn

$x, y \in V(G)$ | C : face containing x



Want to find :

T : Hamilton path between x, y

T : C-Tutte path

$\Leftrightarrow$ For $^{\forall} B$: T-bridge,

\# of attachments of B on T is ≤ 3

and ≤ 2 if B contains an edge in C

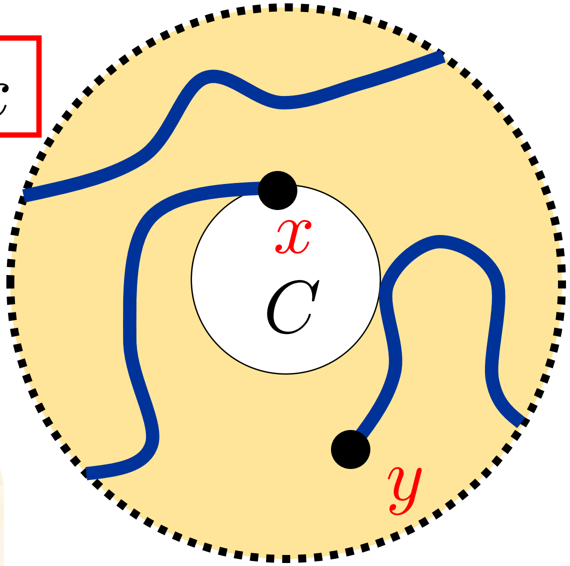
Idea of the proof

Condition G ~~: 4-conn~~. 2-conn

$x, y \in V(G)$ C : face containing x

Want to find :

T : C-Tutte path between x, y



T : C-Tutte path

$\Leftrightarrow$ For $^{\forall} B$: T-bridge,

# of attachments of B on T is ≤ 3

and ≤ 2 if B contains an edge in C

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)	○ K. K. & Oz.('12+)	?	✕
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)	? Grunbaum('70 Nash-Williams('73)	?	✕	✕
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	✕	✕	✕	✕

4-conn. graphs on surfaces $(a = 1)$

Conjecture (Grunbaum '70, Nash-Williams '73)

$\forall G$: 4-conn graph on the torus $\Rightarrow \exists$ H-cycle

Why is this conjecture difficult?

- A surface (and graphs on it) is more complicated
- To find an H-cycle ($b = 0$)

  we usually think the property for $b = -1$

  Ex. H-conn, $\exists$ H-cycle through a given edge….

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)	○ K.K. & Oz.('12+)	?	✕
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)	? Grunbaum('70) Nash-Williams('73)	?	✕	✕
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	✕	✕	✕	✕

4-conn. graphs on surfaces $(a = 1)$

Conjecture (Grunbaum '70, Nash-Williams '73)

$\forall G$: 4-conn graph on the torus $\Rightarrow \exists$ H-cycle

G : 4-conn. graph on the torus $\Rightarrow$ (# of comp.s in $G - S$) $\leq |S|$

Theorem (Fujisawa, Nakamoto, Oz. `12)

$\forall G$: 4-conn graph on the torus

$\exists S$: the equality in the above holds $\Rightarrow$ $\exists$ H-cycle

4-conn. graphs on surfaces $(a = 1)$

Conjecture (Grunbaum '70, Nash-Williams '73)

$\forall G$: 4-conn graph on the torus $\Rightarrow \exists$ H-cycle

Theorem (Kawarabayashi, Oz. `12+)

$\forall G$: 4-conn triangulation on the torus

$\Rightarrow \exists$ H-cycle

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$			○	○ ...12+)	?	×
H-cycle $b = 0$	Tutte ('56)	Thomas & Yu ('94)	Nash-Williams('73)		×	×
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	×	×	×	×
$b = -2$						

$k = 4$ Necessary condition

$-\chi \leq b \Rightarrow$ OK.

$-\chi > b \Rightarrow$ Not satisfied

4-conn. graphs on surfaces $(a = 1)$

2-H : $\forall\ x, y \in V(G)$, $\exists$ H-cycle in $G - \{x, y\}$

1-H-conn : $\forall\ x \in V(G)$, $G - x$ is H-conn

2-e-H-conn : $\forall e, f \in \binom{V}{2}$, $\exists$H-cycle through e, f in $G + \{e, f\}$

Necessary condition

(# of comp.s in $G - S$) $\leq |S| - 2$

G : 2-e-H-conn $\not\Leftrightarrow$ G : 2-H

$\Downarrow$ $\not\Uparrow$

G : 1-H-conn

- $\forall$4-conn. plane graph is 2-H (Tomas & Yu, \`94)
- $\forall$4-conn. plane graph is 1-H-conn. (Sanders, \`97)
- $\forall$4-conn. plane graph is 2-e-H-conn. (Oz. & Vrána \`12+)

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$			○	○ [illegible] 2+)	?	×
H-cycle $b = 0$	Tutte ('56)	Thomas & Yu ('94)	Nash-Williams('73)		×	×
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	×	×	×	×
$b = -2$	○ Oz.,V(`12+)	×				

$k = 4$ Necessary condition

$-\chi \leq b \Rightarrow$ OK.

$-\chi > b \Rightarrow$ Not satisfied

4-conn. graphs on surfaces $(a = 1)$

$\exists$ Spanning tree with (# of leaves) $\leq t$

(# of comp.s in $G - S$) $\leq a|S| + b$ $\quad (a, b) = (1, t-1)$

$k = 4$ Necessary condition

$-\chi \leq b \Rightarrow$ OK.

$-\chi > b \Rightarrow$ Not satisfied

Problem (Oz.`12+)

$\forall F^2$: a surface $\quad \chi = \chi(F^2) < 0$

$\forall G$: 4-conn. graph on F^2,

$\Rightarrow \exists$ Spanning tree with (# of leaves) $\leq -\chi + 1$

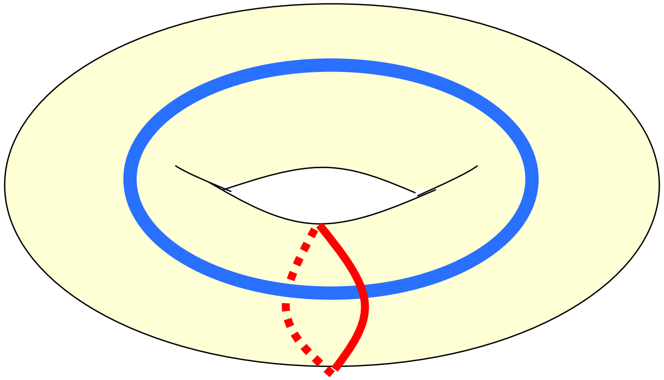
Representativity

G : graphs on a surface on F^2

Rep : sufficiently large $\Leftrightarrow$ Locally planar

Representativity of G (rep of G) :

$$= \min\{|G \cap \gamma| : \gamma \text{ is a non-contractible curve on } F^2\}$$



$\exists$ Graphs

Ex : $K_{n,m}$ $(n \neq m)$

s.t. connectivity : large

rep : small

$\nexists$ H-cycle

4-conn. graphs on surfaces

- 4-connected case ($k = 4$)

$(a, b) = (1, -\chi)$

↑

G : 4-conn. graph on F^2

(# comp.s in $G - S$) $\leq a|S| + b$ holds for $(a, b) = \left(\frac{2}{k-2}, \frac{-2\chi}{k-2}\right)$

$O(-\chi)$ is needed (AHL \`96)

This bound is best possible

This implies $|S|$: large

Rep : sufficiently large $\Longrightarrow$ It holds for $a > 1$ and $b \geq -4a + 2$

A necessary condition is OK if $a > 1$, and rep: large

Ex. $\exists$ Sp. tree with max. deg. ≤ 3 → $(a, b) = (2, 1)$ is needed

This holds for $\forall$ 4-conn. graph

on a surface with rep : large (Ellingham & Gao \`94, Yu \`97)

4-conn. graphs on surfaces $(a = 1)$

Locally planar

Conjecture (Mohar '95)

$\forall F^2$: a surface $\chi = \chi(F^2) < 0$ $\exists\, r(F^2)$: constant
s.t. $\forall G$: 4-conneced graph on F^2 with rep $> r(F^2)$
$\Rightarrow$ $\exists$ Spanning tree with max. deg ≤ 3
s.t. (# of leaves) $= O(-\chi)$

Problem (Oz. `12+)

$\forall F^2$: a surface $\chi = \chi(F^2) < 0$
$\forall G$: 4-conn. graph on F^2,
$\Rightarrow$ $\exists$ Spanning tree with (# of leaves) $\leq -\chi + 1$

4-conn. graphs on surfaces $(a = 1)$

Locally planar

Conjecture (Mohar '95)

$\forall F^2$: a surface $\chi = \chi(F^2) < 0$ $\exists\, r(F^2)$: constant
s.t. $\forall G$: 4-conneced graph on F^2 with rep $> r(F^2)$
$\Rightarrow$ $\exists$ Spanning tree with max. deg ≤ 3
s.t. (# of leaves) $= O(-\chi)$

Theorem (Kawarabayashi, Oz. `12+)

The above conjecture is true if G is a triangulation

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$			○	○ 2+)	?	×
H-cycle $b = 0$	Tutte ('56)	Thomas & Yu ('94)	Nash-Williams('73)		×	×
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	×	×	×	×
$b = -2$	○ Oz.,V(\`12+)	×				

$k = 4$ Necessary condition

$-\chi \leq b \Rightarrow$ OK.

$-\chi > b \Rightarrow$ Not satisfied

5-conn. graphs on surfaces $(a = \frac{2}{3})$

- 5-connected case $(k = 5)$

$$(a, b) = \left(\frac{2}{3}, \frac{-2}{3}\chi\right)$$

G : k-conn. graph on F^2

(# comp.s in $G - S$) $\leq a|S| + b$

holds for $(a, b) = \left(\frac{2}{k-2}, \frac{-2\chi}{k-2}\right)$

This bound is best possible

This implies $|S|$: large

Rep : sufficiently large $\Longrightarrow$ It holds for $a > \frac{2}{3}$ and $b \geq -5a + 2$

A necessary condition is OK if $a > \frac{2}{3}$, and rep: large

Ex. Hamiltonicity of graphs $\rightarrow$ $(a, b) = (1, 0)$ is required

We do not need any assumption on rep for large χ

5-conn. graphs on surfaces $(a = \frac{2}{3})$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)	○ K.K. & Oz.('12+)	?	?
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)	○ Thomas & Yu ('97)	○ K.K & Mohar ('12+)	?	?
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	?	?	?	?
$b = -2$	○					

5-conn. graphs on surfaces $(a = \frac{2}{3})$

Locally planar

Conjecture (Thomassen `94)

$\forall F^2$: a surface $\chi = \chi(F^2) < 0$ $\exists\, r(F^2)$: constant

s.t. $\forall G$: 5-connected graphs on F^2,

rep $> r(F^2) \Rightarrow \exists$ Hamilton cycles

G : 5-connected graphs on F^2 with rep : large enough

$\exists$ Hamilton cycle if G is a triangulation (Yu '95)

$\exists$ Hamilton cycle (Kawarabayashi, Oz.`11+)

if ≥ 3 triangles are incident with $\forall$ vertex

5-conn. graphs on surfaces $(a = \frac{2}{3})$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)	○ K.K. & Oz.('12+)	?	?
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)	○ Thomas & Yu ('97)	○ K.K & Mohar ('12+)	?	? Thomassen('94+)
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	○ K.K. & Oz.('12+)	○ K.K. & Oz.('12+)	?	?
$b = -2$	○					

3-conn. graphs on surfaces $(a = 2)$

G : 3-conn. graph on F^2

(# comp.s in $G-S$) $\leq 2|S| - 2\chi$

$(k-1, 1)$	$\exists$ Sp. tree with max. deg. $\leq k$
$(k-1, t+1)$	$\exists T$: Sp. tree with $\mathrm{te}(T, k) \leq t$

$\exists$ Sp. tree with bounded total excess

3-conn. graphs on surfaces $(a = 2)$

$\mathrm{te}(T, k) := \sum_{x \in V(T)} \max\{d_T(x) - k, 0\}$

Total excess of a sp. tree T from k

$\mathrm{te}(T, k) = 0$

$\Updownarrow$

T : sp. tree with max. deg. $\leq k$

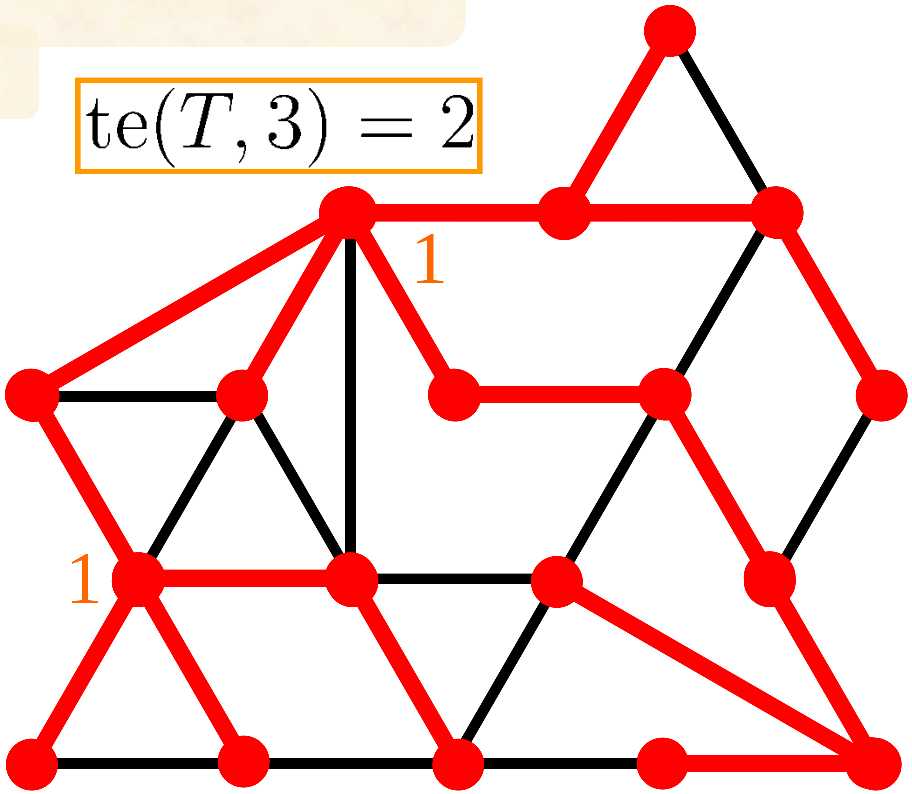
$\exists$ T : Sp. tree with $\mathrm{te}(T, k) \leq t$

$\Downarrow$

(# comp.s in $G - S$)

$\leq (k-1)|S| + t - 1$

$\mathrm{te}(T, 3) = 2$



3-conn. graphs on surfaces $(a = 2)$

$$\mathrm{te}(T,k) := \sum_{x \in V(T)} \max\{d_T(x) - k, 0\}$$

Total excess of a sp. tree T from k

$\mathrm{te}(T,k) = 0$

$\Updownarrow$

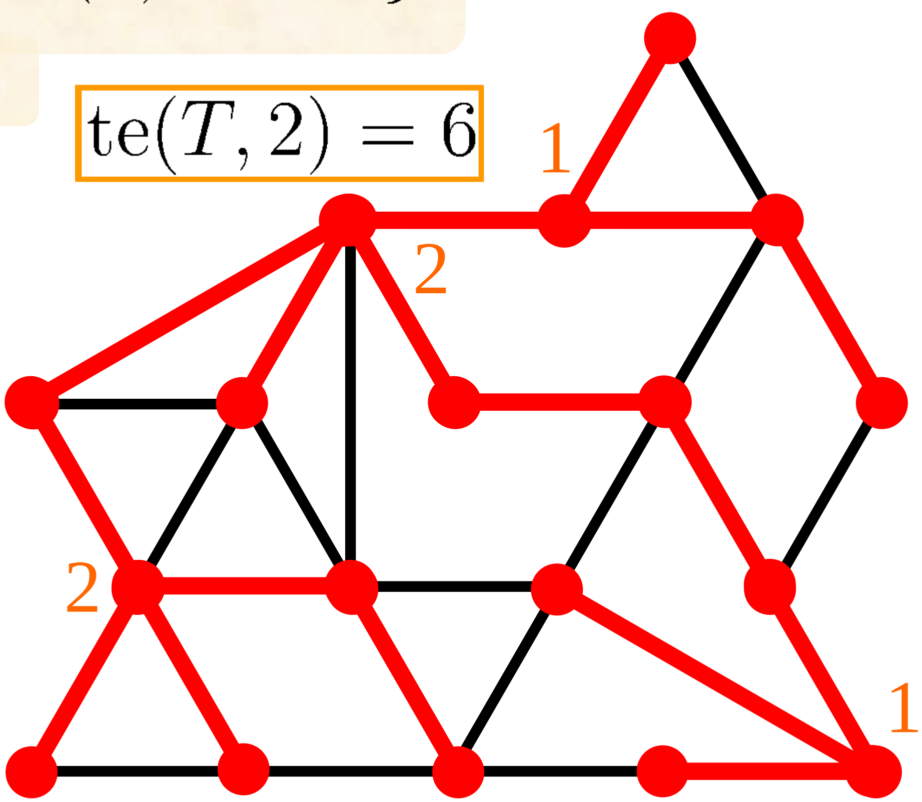
T : sp. tree with max. deg. $\leq k$

$\exists$ T : Sp. tree with $\mathrm{te}(T,k) \leq t$

$\Downarrow$

(# comp.s in $G - S$)

$\leq (k-1)|S| + t - 1$



3-conn. graphs on surfaces $(a = 2)$

$(k-1, 1)$	$\exists$Sp. tree with max. deg. $\leq k$
$(k-1, t+1)$	$\exists T$: Sp. tree with te$(T,k) \leq t$

G : 3-conn. graph on F^2

(# comp.s in $G - S$) $\leq 2|S| - 2\chi$

$\chi \geq 0$ $\exists$Sp. tree with max. deg. ≤ 3 (Barnette \`66, \`92)

$\chi \leq -1$ $\exists T$: Sp. tree with te$(T,3) \leq -2\chi - 1$ (Oz. \`12+)

(# comp.s in $G - S$) $\leq 3|S| + 1$ If rep : sufficiently large

$\exists$Sp. tree with max. deg. ≤ 4 (Yu \`97)

$\exists$Sp. tree with max. deg. ≤ 4 and te$(T,3) \leq -2\chi - 1$

(Kawarabayashi, Nakamoto, Ota \`97)

(# comp.s in $G - S$) $\leq \lceil \frac{-2\chi+5}{3} \rceil |S| + 1$ $\because |S| \geq 3$

$\exists$Sp. tree with max. deg. $\leq \lceil \frac{-2\chi+8}{3} \rceil$ (Sanders, Zhao \`01, Ota, Oz.\`12)

Graphs on surfaces

- Graphs on surfaces, connectivity and toughness

G : k-conn. graph on F^2

(# comp.s in $G - S$) $\leq a|S| + b$ holds for $(a, b) = \left(\frac{2}{k-2}, \frac{-2\chi}{k-2}\right)$

This bound is best possible

- Properties concerning Hamiltonicity (and toughness)

Statement : $\forall$ k-conn. graph on a surface F^2 is [...]

We have to think about toughness type necessary condition

What else do we need?

4-conn. graphs on surfaces $(a = 1)$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)	○ K.K. & Oz.('12+)	?	✕
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)	? Grunbaum('70) Nash-Williams('73)	?	✕	✕
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	✕			

$b = -2$ ○ Oz.,V(`12+) ✕

$k = 4$ Necessary condition

$-\chi \leq b \Rightarrow$ OK.

$-\chi > b \Rightarrow$ Not satisfied

5-conn. graphs on surfaces $(a = \frac{2}{3})$

	Plane $\chi = 2$	Proj. plane $\chi = 1$	Torus $\chi = 0$	K-bottle $\chi = 0$	N_3 $\chi = -1$	Others $\chi < -1$
H-path $b = 1$	○	○	○ Thomas, Yu & Zang ('05)	○ K.K. & Oz.('12+)	?	?
H-cycle $b = 0$	○ Tutte ('56)	○ Thomas & Yu ('94)	○ Thomas & Yu ('97)	○ K.K & Mohar ('12+)	?	?
H-conn $b = -1$	○ Thomassen ('83)	○ K.K. & Oz.('12+)	○ K.K. & Oz.('12+)	○ K.K. & Oz.('12+)	?	?
$b = -2$	○					

3-conn. graphs on surfaces $(a = 2)$

$(k-1, 1)$	$\exists$Sp. tree with max. deg. $\leq k$
$(k-1, t+1)$	$\exists T$: Sp. tree with te$(T, k) \leq t$

G : 3-conn. graph on F^2

(# comp.s in $G - S$) $\leq 2|S| - 2\chi$

$\chi \geq 0$ $\exists$Sp. tree with max. deg. ≤ 3 (Barnette \`66, \`92)

$\chi \leq -1$ $\exists T$: Sp. tree with te$(T, 3) \leq -2\chi - 1$ (Oz. \`12+)

(# comp.s in $G - S$) $\leq 3|S| + 1$ If rep : sufficiently large

$\exists$Sp. tree with max. deg. ≤ 4 (Yu \`96)

$\exists$Sp. tree with max. deg. ≤ 4 and te$(T, 3) \leq -2\chi - 1$

(Kawarabayashi, Nakamoto, Ota \`97)

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$\exists$Sp. tree with max. deg. $\leq \lceil \frac{-2\chi+8}{3} \rceil$ (Sanders, Zhao \`01, Ota, Oz.\`12)

$K_{3,p}$-minor free graphs

$(k-1,1)$	$\exists$ Sp. tree with max. deg. $\leq k$
$(1,t-1)$	$\exists$ Sp. tree with $\leq t$ leaves

$p \geq 3$

G : 3-conn. $K_{3,p}$-minor-free graphs

(# comp.s in $G - S$) $\leq a|S| + b$

These bounds are best possible

holds for $(a,b) = (p-1, -2p+2)$ if p: odd

(Chen, Egawa, Kawarabayashi, Mohar, Ota, `11)

holds for $(a,b) = (p-2, -2p+6)$ if p: even

(Ota, Oz., `12+)

$\exists$ Sp. tree with max. deg. $\leq p$ if p : odd

$\exists$ Sp. tree with max. deg. $\leq p-1$ if p : even

(Ota, Oz. `12)

$K_{3,p}$-minor free graphs

$p \geq 3$

$(k-1, 1)$	$\exists$Sp. tree with max. deg. $\leq k$
$(1, t-1)$	$\exists$Sp. tree with $\leq t$ leaves

G : 4-conn. $K_{3,p}$-minor-free graphs

(# comp.s in $G - S$) $\leq a|S| + b$

Is there good toughness bound? (I expect $a = 1$.)

→ $\exists$ H-cycle or $\exists$ sp. tree with bounded # of leaves?

G : 3-conn. $K_{2,p}$-minor-free graphs

(# comp.s in $G - S$) $\leq a|S| + b$ holds for $(a, b) = (1, \exists t)$

→ $\exists$ sp. tree with bounded # of leaves (Oz., `12+)

$K_{3,p}$-minor free graphs

$(k-1,1)$	$\exists$ Sp. tree with max. deg. $\leq k$
$(1,t-1)$	$\exists$ Sp. tree with $\leq t$ leaves

$p \geq 3$

G : 4-conn. $K_{3,p}$-minor-free graphs

(# comp.s in $G-S$) $\leq a|S|+b$

Is there good toughness bound? (I expect $a=1$.)

→ $\exists$ H-cycle or $\exists$ sp. tree with bounded # of leaves?

How about 4-conn. $K_{4,p}$-minor-free graphs??

Toughness bound holds for $(a,b)=(3p-3,-9p+9)$

(Chen, Egawa, Kawarabayashi, Mohar, Ota, `11)

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- Graphs on surfaces, connectivity and toughness

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This bound is best possible

- Properties concerning Hamiltonicity (and toughness)

Statement : $\forall$ k-conn. graph on a surface F^2 is […]

We have to think about toughness type necessary condition

What else do we need?

Thank you for your attention