On Rank of Graphs

B. Tayfeh-Rezaie

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

(A joint work with E. Ghorbani and A. Mohammadian)

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Theorem

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Theorem (Omidi (2009)):

For any bipartite graph G with no cycle of length a multiple of 4 as a subgraph, rank$(G) \geqslant \frac{2e}{\triangle}$.

Theorem (Alon *et al.* (2010)):

For any G with no $K_{2,r}$ as a subgraph, rank$(G) \geqslant \frac{2e}{r\triangle}$.

Definition

Reduced graph: No isolated vertices and no two vertices with the same neighborhood.

A General Problem

C: A given class of reduced graphs

For any r, find the maximum order of graphs from C which are of rank r.

[or equivalently, for any n, find the maximum nullity of graphs of order n from C.]

Example

C:=The class of reduced P_4-free graphs (cographs)

Theorem (Royle (2003)):

Any graph of rank r from C has r vertices.

Example

C:=The class of reduced trees

Theorem (Ghorbani, Mohammadian and B. T):

Every reduced tree of rank r has at most $3r/2 - 1$ vertices.

Example

C:=The class of reduced bipartite graphs

Theorem (Ghorbani, Mohammadian and B. T):

Every reduced bipartite graph of rank r has at most $2^{r/2} + r/2 - 1$ vertices.

Example

C:=The class of reduced non-bipartite triangle-free graphs

Theorem (Ghorbani, Mohammadian and B. T):

Every reduced non-bipartite triangle-free graph of rank r has at most $3 \cdot 2^{[r/2]-2} + [r/2]$ vertices.

Main Problem

C:=The class of all reduced graphs

For any r, find the maximum order of reduced graphs with rank r.

It is straightforward to show that every reduced graph of rank r has at most $2^r - 1$ vertices.

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Kotlov and Lovász (1996):

There exists a constant c such that any reduced graph of rank r has at most $c \cdot 2^{r/2}$ vertices.

Let

$$m(r) = \begin{cases} 2^{(r+2)/2} - 2 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 2 & \text{if } r > 1 \text{ is odd.} \end{cases}$$

Conjecture (Akbari, Cameron, Khosrovshahi (200?)):

A reduced graph of rank r has at most $m(r)$ vertices.

Constructions for graphs of rank r and order $m(r)$

Construction 1 (Kotlov and Lovász):

Let G be a reduced graph of order n, adjacency matrix A and rank r. Then the graph with adjacency matrix

$$\begin{pmatrix} A & A & 1 & 0 \\ A & A & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is reduced of order $2n+2$ and rank $r+2$.

Construction 2 (Akbari, Cameron, Khosrovshahi):

Let G be a reduced regular graph of order n with degree $n/2$, adjacency matrix A and rank r. Then the graph with adjacency matrix

$$\begin{pmatrix} A & A & 1 & 0 \\ A & A & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

is reduced of order $2n+2$ and rank $r+2$.

Construction 3 (Haemers and Peeters):

Let G be a reduced graph of order n, adjacency matrix A and rank r. Then the graph with adjacency matrix

$$\begin{pmatrix} A & \overline{A} & 1 & 0 \\ \overline{A} & A & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

is reduced of order $2n + 2$ and rank $r + 2$.

Known results

(i) The conjecture is verified for graphs with rank at most 9.

(ii) A reduced graph of rank r and with an induced matching of size $r/2$ or a disjoint union of an induced matching of size $(r-3)/2$ and a triangle has at most $m(r)$ vertices. (Haemers and Peeters (2010))

(ii) The conjecture is true for specific families of graphs like cographs, trees, bipartite graphs, triangle-free graphs, line graphs, ...

Theorem (Ghorbani, Mohammadian and B. T):

Suppose that the conjecture is true for graphs of rank $r \leqslant 47$. Then the conjecture holds for all r.

Theorem (Ghorbani, Mohammadian and B. T):

A reduced graph of rank r has at most $8m(r) + 14$ vertices.

Notation

$\rho(G)$:= The minimum number of vertices whose removal results in a graph with a smaller rank.

$t(G)$:= min$\{|N(u)\triangle N(v)|\,|\,u,v\in V(G)$ and distinct$\}$.

$\tau(G)$:= min$\{|N(u)\triangle N(v)|\,|\,u,v\in V(G)$ and nonadjacent$\}$.

$$\boxed{\rho(G)\le t(G)\le \tau(G)}$$

Definition:

An (r, n, φ)-**spherical code** C is a set of n unit vectors of $\mathbb{R}^r$ such that

$$\langle x, y \rangle \leqslant \cos \varphi,$$

holds for any two distinct elements x and y of C.

Bounding the order of a graph by spherical codes

Let G be a reduced graph of order n and rank r.

Then

there is a $(r+1, n, \varphi)$-**spherical code** with

$$\varphi = \text{arccos}(\frac{n - 2t(G)}{n}).$$

Theorem

Let $r \geqslant 46$ and $\varphi = \arccos(\sqrt{2} - 1)$. If there exists an $(r+1, n, \varphi)$-spherical code, then

$$n < 5 \cdot 2^{(r-3)/2} - 2.$$

Recall that $m(r) = \begin{cases} 2^{(r+2)/2} - 2 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 2 & \text{if } r > 1 \text{ is odd.} \end{cases}$

Theorem

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Corollary

Let G be graph of order n and rank $r \geq 46$. If $n \geq 5 \cdot 2^{\frac{r-3}{2}} - 2$, then $t(G) < (1 - \frac{\sqrt{2}}{2})n$.

Lemma. *Let G be a reduced graph and H be an induced subgraph of G with the maximum possible order subject to H has duplication classes. Assume that* rank$(H) \geqslant$ rank$(G) - 3$ *Then*

(i) *If w is an isolated vertex of H, then $N(w) = V(G) \setminus V(H)$.*

(ii) *Each duplication class of H has two elements and H has at most one isolated vertex.*

(iii) *If H is not reduced and $\{v_1, v_1'\}, \ldots, \{v_s, v_s'\}$ are all the duplication classes of H, then there exist two disjoint sets T_1, T_2 such that $V(G) \setminus V(H) = T_1 \cup T_2$, $T_1 \subseteq N(v_i) \setminus N(v_i')$ and $T_2 \subseteq N(v_i') \setminus N(v_i)$, for $i = 1, \ldots, s$.*

Lemma. *Let G be a counterexample to the conjecture with the minimum possible order and let $r = \text{rank}(G)$. Let H be an induced subgraph of G with the maximum possible order subject to H has duplication classes. Assume that* $\text{rank}(H) \geqslant r-3$. *Let S be the graph induced on $v_1, \ldots, v_s$ as of the previous lemma. Then H has no isolated vertices and one of the following holds.*

(i) $S = K_1$ *and* $|V(G) \setminus V(H)| \geqslant m(r-2)+2$.

(ii) $S = K_2$ *and* $|V(G) \setminus V(H)| \geqslant m(r-2)+1$.

(iii) $S = K_3$ *and* $|V(G) \setminus V(H)| = m(r-2)$.

Definition

G: A graph of order n

μ : An eigenvalue with multiplicity k

μ**-rank of** G is defined as $n - k$.

Or equivalently

μ**-rank of** G is defined as the rank of $\mu I - A$.

Let

$$m(r) = \begin{cases} 2^{(r+2)/2} - 2 & \text{if } r \text{ is even,} \\ 5 \cdot 2^{(r-3)/2} - 2 & \text{if } r > 1 \text{ is odd.} \end{cases}$$

Conjecture

A coreduced graph of (-1)-rank r has at most $m(r)$ vertices. Moreover, there is a unique graph meeting this bound.

What about $\mu \neq 0, -1$?

Theorem (Bell and Rowlinson, 2003)

Let G be a graph of order $n > 4$ and μ-rank t. If $\mu \notin \{-1, 0\}$, then

$$n \leq \frac{1}{2}t(t+1).$$

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The bound is attained for example when G is the graph obtained from $L(K_9)$ by switching with respect to a clique of order 8; here $\mu = -2$, $t = 8$ and $n = 36$.

Theorem (Bell and Rowlinson, 2003)

Let G be an r-regular graph of order n and μ-rank $t > 2$. If $\mu \notin \{-1, 0, r\}$, then

$$n \leq \frac{1}{2}t(t+1) - 1.$$

The regular graphs attaining the bound are precisely the extremal strongly regular graphs.

Theorem

Let G be a strongly regular graph which is not extremal, pentagon, Clebsch, complete multipartite or their complements.

If G has μ-rank t, then

$$n \leq \frac{1}{2}t(t-1).$$

Our Problem

For any r, find the maximum order of reduced graphs with exactly r positive and negative eigenvalues.

Similar Problems

For any r, find the maximum order of reduced graphs with exactly r negative eigenvalues (Torgasev, 1985).

Torgasev proved that the maximum $M(r)$ is finite and $M(2) = 6, M(3) = 14, M(4) = 30$.

We conjecture that $M(r) = 2^{r+1} - 2$.

If we take positive eigenvalues instead of negative eigenvalues, then the maximum is infinite.

Similar Problems

For any r, find the maximum order of graphs with exactly r negative and zero eigenvalues (Charles, Farber, Johnson, Kennedy-Shaffer, 2011).

They proved that the maximum $N(r)$ is finite and $N(1) = 2, N(2) = 5, N(3) = 9, N(4) = 15.$

They also showed that

$$\binom{r+1}{2} \leqslant N(r) < R(r+1, r+2).$$