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Topology and Lagrange-Hamilton mechanics of magnetic confinement fusion

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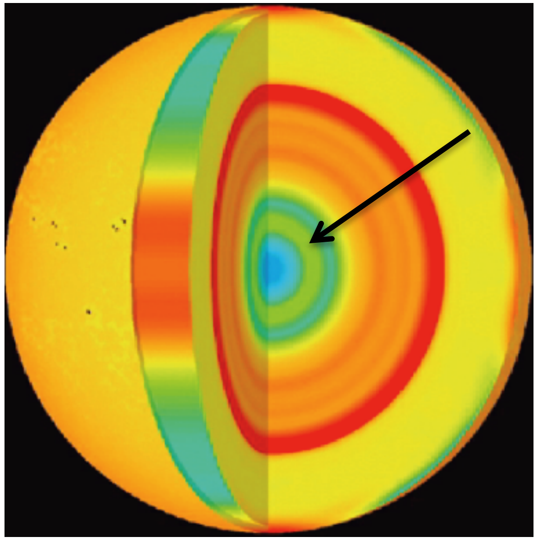
Topology and Lagrange-Hamilton mechanics of magnetic confinement fusion

M. Kikuchi, JAEA

1. Fundamental question : What is the shape of magnetic confinement bottle?

In the Sun, plasma is confined with gravitational field. It is a central force and the force acts in the direction of field line. For this reason, confinement bottle has a topology of "Sphere".

In the man-made fusion reactor, plasma is confined by the Lorentz force. The force acts in the direction perpendicular to the field line. For this reason, confinement bottle has a topology of "Torus".



Force along field

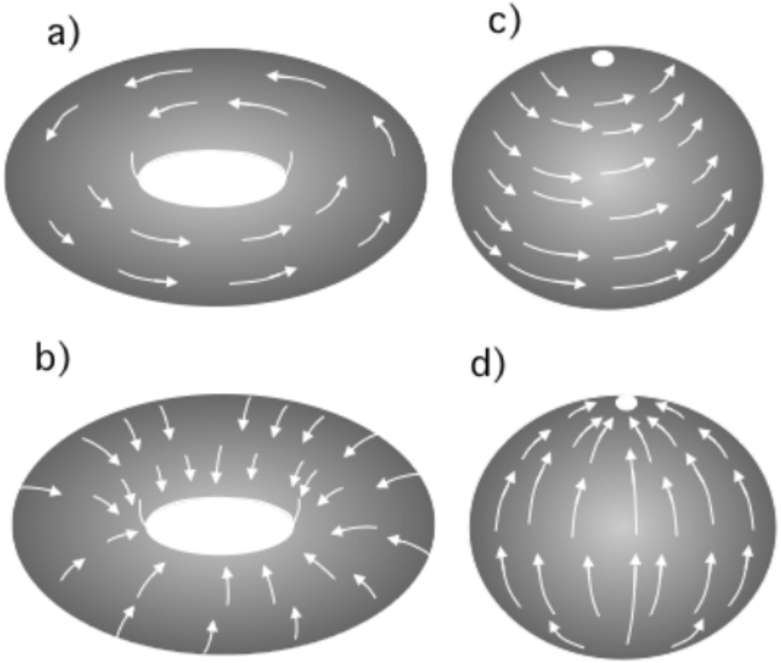
Force perpendicular to field

2. Topology : Closed surface without fixed point

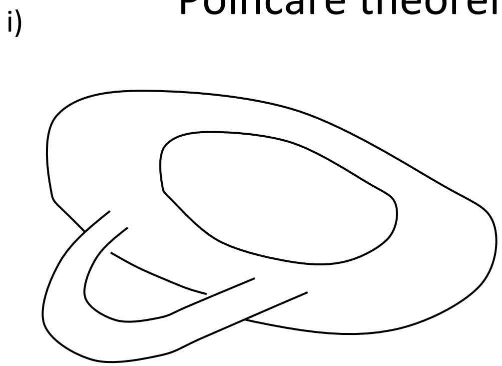
French mathematician **Henri Poincare** (1854-1912) proved a theorem "**Closed surface that can be covered with vector field without fixed point is restricted to a torus**." This is called "**Poincare theorem**"

For the magnetic confinement fusion, plasma must be confined inside the closed surface. If the magnetic field on this surface has some null point (fixed point), the plasma will leak through that region.

Hair whorl on the head



See my book for proof of Poincare theorem



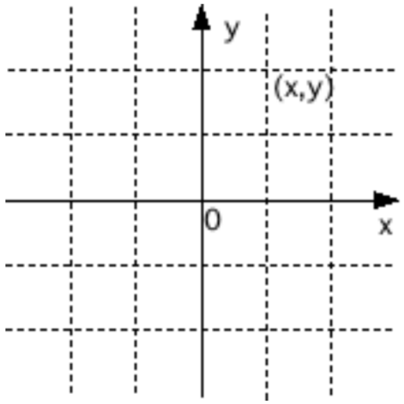
Bundle divertor is topologically not a torus.
So, it necessarily includes null point of vector field.

J.B. Taylor, Bundle divertors and topology, CLM-R 132 (1974)

3. "Coordinates : Analytical Geometry of the Torus

Topology of the torus can be described without using coordinates, similar to situation in Euclidean Geometry.

But, quantitative understanding needs introduction of "Coordinates".



French philosopher **Rene Descartes** (1596-1650)

introduced coordinates to GEOMETRY

"Coordinates" is important element to understand physics of plasma confinement in torus.
Good works has been done in Fusion Research such as

Hamada Coordinates and Boozer Coordinates.

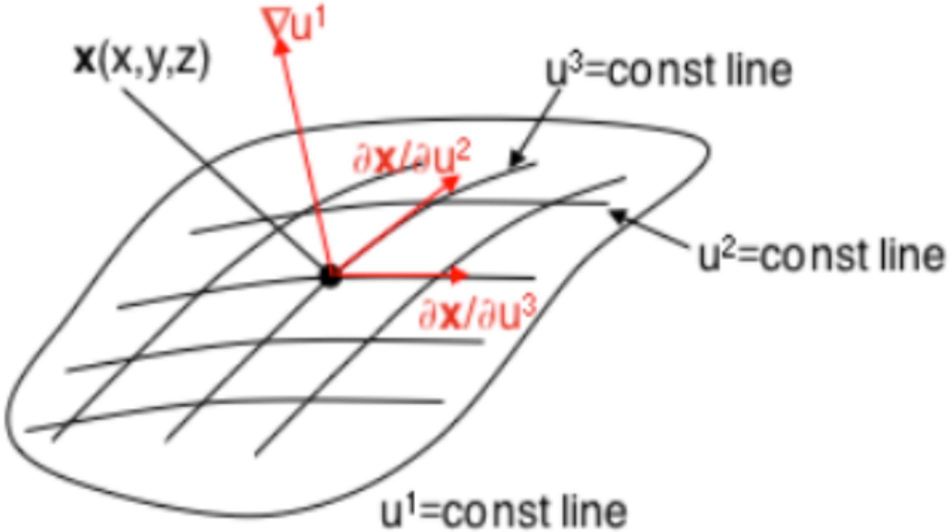
General curvilinear coordinates (u^1, u^2, u^3) : $u^1=u^1(x,y,z)$, $u^2=u^2(x,y,z)$, $u^3=u^3(x,y,z)$

Gradient vector : ∇u^i is normal to u^i*=constant* surface

$$\nabla u^i = \left(\frac{\partial u^i}{\partial x}\right) \boldsymbol{e}_x + \left(\frac{\partial u^i}{\partial y}\right) \boldsymbol{e}_y + \left(\frac{\partial u^i}{\partial z}\right) \boldsymbol{e}_z$$

Tangent vector : $\partial \mathbf{x}/\partial u^2$ is tangent to u^1*=constant* and u^3*=constant* surfaces

$$\frac{\partial \boldsymbol{x}}{\partial u^j} = \frac{\partial x}{\partial u^j}\boldsymbol{e}_x + \frac{\partial y}{\partial u^j}\boldsymbol{e}_y + \frac{\partial z}{\partial u^j}\boldsymbol{e}_z$$



"orthogonal relation."

$$\nabla u^i \cdot \frac{\partial \boldsymbol{x}}{\partial u^j} = \frac{\partial u^i}{\partial u^j} = \delta_{ij}$$

4. Hamilton dynamics of the magnetic field

Magnetic field is a vector field without source and sink, and therefore is incompressible as a flow field ($\nabla\cdot B=0$)

General vector potential A ($\nabla \times A = B$) is given by

$$A=\phi\nabla\theta-\psi\nabla\zeta+\nabla G$$

(θ and ζ are poloidal and toroidal angles (choice is arbitrary))

$$B=\nabla\times A=\nabla\times[\,\phi\nabla\theta-\psi\nabla\zeta+\nabla G\,]$$

$$\longrightarrow \boxed{B=\nabla\phi\times\nabla\theta-\nabla\psi\times\nabla\zeta \quad \text{(Symplectic form)}}$$

Proof

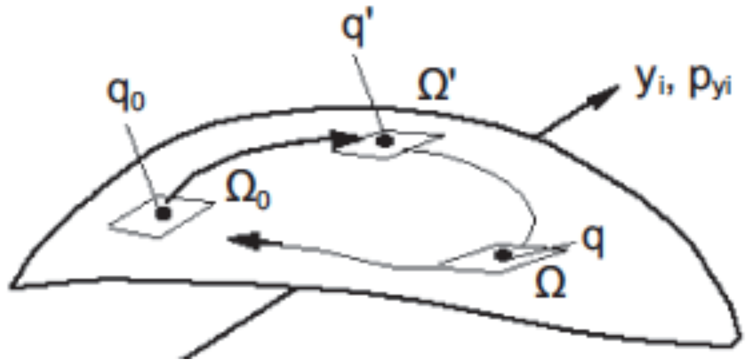
#1: Any vector A can be expressed as $A = A_u\nabla u + A_\theta\nabla\theta + A_\zeta\nabla\zeta$ in the general curvilinear coordinates (u, θ, ζ). If we define a scalar G by $G = \int A_u\, du$ ($\partial G/\partial u = A_u$) and consider $\nabla G = \partial G/\partial u\nabla u + \partial G/\partial\theta\nabla\theta + \partial G/\partial\zeta\nabla\zeta$, A can be expressed as $A = \nabla G + (A_\theta - \partial G/\partial\theta)\nabla\theta + (A_\zeta - \partial G/\partial\zeta)\nabla\zeta$. If we define $\phi = A_\theta - \partial G/\partial\theta$ and $\psi = -A_\zeta + \partial G/\partial\zeta$, we reach general expression for the vector potential $A = \phi\nabla\theta - \psi\nabla\zeta + \nabla G$.

Magnetic coordinates : (ϕ, θ, ζ)

Hamilton equation

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j}, \quad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j}$$

Flow on H=constant surface



Magnetic field trajectory: $\mathrm{d}u^i/\mathrm{d}s = \boldsymbol{b} \cdot \nabla u^i$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{\boldsymbol{B} \cdot \nabla\theta}{\boldsymbol{B} \cdot \nabla\zeta} = \frac{\partial\psi}{\partial\phi}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \frac{\boldsymbol{B} \cdot \nabla\phi}{\boldsymbol{B} \cdot \nabla\zeta} = -\frac{\partial\psi}{\partial\theta}$$

This is Hamilton equation in the dynamical system if we regard

ψ as **Hamiltonian**,

ψ θ as **canonical coordinate**,

ϕ as **canonical angular momentum**,

ϕ ζ as **time**.

Hint: $\boldsymbol{B}\cdot\nabla\theta = (\nabla\phi \mathbf{x} \nabla\theta - \nabla\psi \mathbf{x} \nabla\zeta)\cdot\nabla\theta$

$= -(\nabla\psi \mathbf{x} \nabla\zeta)\cdot\nabla\theta$

$= -\partial\psi/\partial\phi\, (\nabla\phi \mathbf{x} \nabla\zeta)\cdot\nabla\theta$

since $\nabla\psi = \partial\psi/\partial\phi\, \nabla\phi + \partial\psi/\partial\theta\, \nabla\theta + \partial\psi/\partial\zeta\, \nabla\zeta$

Variational principle for field lines

Hamilton equation can be derived by the variational principle in Hamilton form : $\delta S = 0$

$$S = \int [\boldsymbol{p} \cdot \mathrm{d}\boldsymbol{x}/\mathrm{d}t - H]\mathrm{d}t \longrightarrow \begin{array}{l} \mathrm{d}\boldsymbol{q}/\mathrm{d}t = \partial H/\partial \boldsymbol{p} \, , \\ \mathrm{d}\boldsymbol{p}/\mathrm{d}t = -\partial H/\partial \boldsymbol{q} \end{array}$$

q=x

$$\boldsymbol{p} \rightarrow \phi, \, \mathrm{d}\boldsymbol{x}/\mathrm{d}t \rightarrow \mathrm{d}\theta/\mathrm{d}\zeta, \, H \rightarrow \psi$$

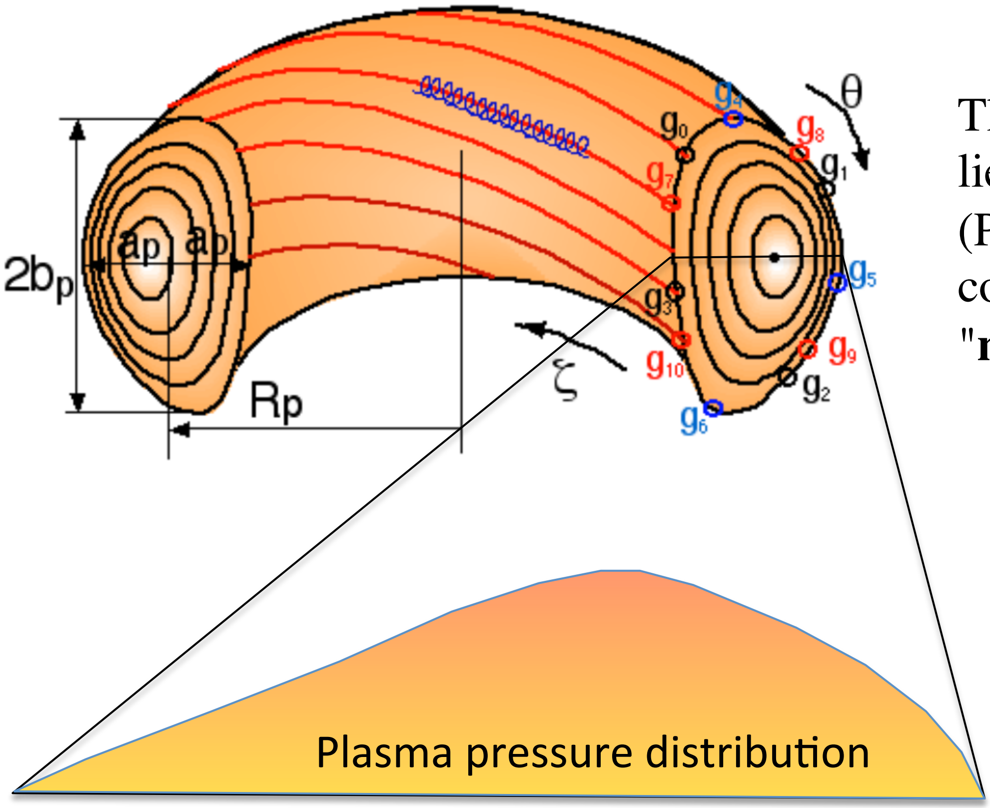
$$\begin{aligned} S &= \int [\phi \mathrm{d}\theta/\mathrm{d}\zeta - \psi]\mathrm{d}\zeta \\ &= \int [\phi \nabla\theta - \psi \nabla\zeta] \cdot \mathrm{d}\boldsymbol{x} \\ &= \int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x} \end{aligned}$$

So, $\delta \int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x} = 0$ gives magnetic field line trajectory

$$\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{\partial \psi}{\partial \phi} \, , \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = -\frac{\partial \psi}{\partial \theta}$$

5. "Magnetic Surface": Integrable magnetic field

In plasma force equilibrium, the plasma's expansion force (∇P) is balanced with the Lorentz force ($\boldsymbol{J} \times \boldsymbol{B}$). Here, $\boldsymbol{J}$ is the current flowing in the plasma, $\boldsymbol{B}$ is the magnetic field, P is the pressure of the plasma. This is the basic principle of the magnetic confinement fusion.



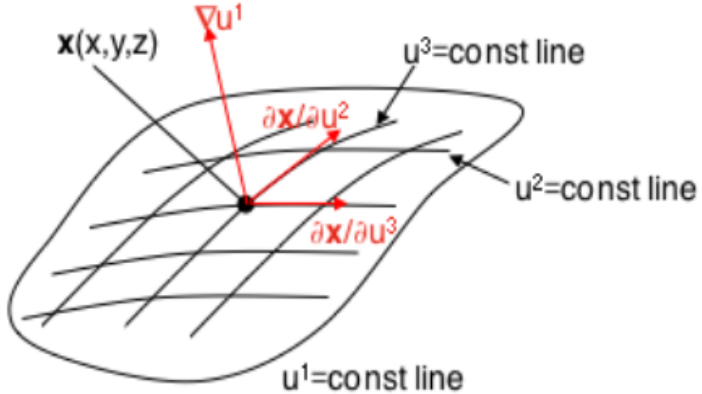
$$\boldsymbol{J} \times \boldsymbol{B} = \nabla P$$

The magnetic field **B** and current density **J** lies on constant pressure surface (P=constant) in force equilibrium. This constant pressure surface is called "**magnetic surface**" or "**flux surface**".

$$\boldsymbol{B} \cdot \nabla P = 0$$

$$\boldsymbol{J} \cdot \nabla P = 0$$

Stream function of magnetic field lines on a flux surface



(Flux surface)

Any vector on u^1=constant surface (u^1=u for simplicity and is flux surface) can be expressed by a combination of two tangent vectors.

$$\boldsymbol{B}=a_2\partial\boldsymbol{x}/\partial\theta+a_3\partial\boldsymbol{x}/\partial\zeta$$

Using the dual relation ($\partial\boldsymbol{x}/\partial u^i = J\nabla u^j \times \nabla u^k$),

$$\boldsymbol{B} = b_2\nabla\zeta \times \nabla u + b_3\nabla u \times \nabla\theta$$

Substituting this into $\nabla\cdot\boldsymbol{B} = 0$ $\quad \partial b_2/\partial\theta + \partial b_3/\partial\zeta = 0$

This is satisfied by introducing stream function : h

$$b_2 = -\frac{\partial h}{\partial\zeta}, \quad b_3 = \frac{\partial h}{\partial\theta} \qquad \boldsymbol{B}=\nabla u \times \nabla h$$

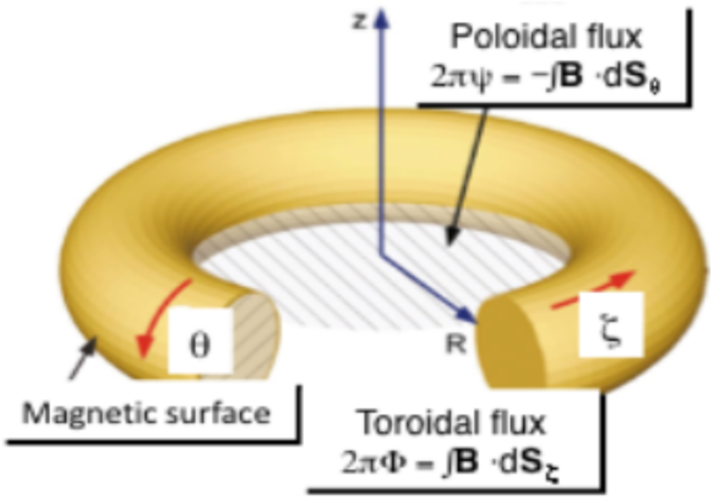
$\nabla\cdot\boldsymbol{B}=\nabla\cdot[b_2\nabla\zeta\times\nabla u + b_3\nabla u\times\nabla\theta]=(\nabla\zeta\times\nabla u)\cdot\nabla b_2+(\nabla u\times\nabla\theta)\cdot\nabla b_3=J^{-1}[\partial b_2/\partial\theta+\partial b_3/\partial\zeta]=0$

Since $\nabla\cdot(\phi\boldsymbol{a}) =\phi\nabla\cdot\boldsymbol{a}+\boldsymbol{a}\nabla\phi$, $\nabla\cdot(\nabla a\times\nabla b) =0$, $\nabla b=(\partial b/\partial u^j)\nabla u^j$

Periodicity requirement on stream function and flux coordinates

$\boldsymbol{B} = \nabla u \times \nabla h$ must satisfy periodic condition in θ and ζ.

Since $\boldsymbol{B} = \nabla u \times [(\partial h/\partial \theta)\nabla\theta + (\partial h/\partial \zeta)\nabla\zeta]$, $\partial h/\partial\theta$ and $\partial h/\partial\zeta$ should be periodic.



Result :ψ and ϕ are function of u.

$$\boldsymbol{B} = \nabla\phi \times \nabla\theta_{\mathrm{m}} - \nabla\psi \times \nabla\zeta$$
$$= \nabla\phi \times \nabla(\theta_{\mathrm{m}} - \zeta/q)$$

$$q = \mathrm{d}\phi(u)/\mathrm{d}\psi(u)$$

$$h(u,\theta,\zeta) = h_2(u)\theta + h_3(u)\zeta + \tilde{h}(u,\theta,\zeta)$$

toroidal magnetic flux : $2\pi\phi(u) = \int \boldsymbol{B} \cdot d\boldsymbol{a}_{\zeta}$,

poloidal magnetic flux : $2\pi\psi(u) = -\int \boldsymbol{B} \cdot d\boldsymbol{a}_{\theta}$

$$\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \int \boldsymbol{B} \cdot \nabla u^k J \,\mathrm{d}u^i \,\mathrm{d}u^j$$

$\tilde{h}(u,\theta,\zeta)$ is a periodic function of θ and ζ

Define $\lambda = \tilde{h}(u,\theta,\zeta)/h_2(u)$

Coordinate transformation : $\theta_m = \theta + \lambda$

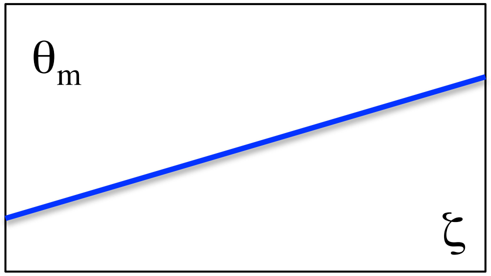
$$h(u,\theta_{\mathrm{m}},\zeta) = h_2(u)\theta_{\mathrm{m}} + h_3(u)\zeta$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = -h_3(u)\,, \quad \frac{\mathrm{d}\phi}{\mathrm{d}u} = h_2(u)$$

Flux coordinates is straight field line coordinates

Using $\boldsymbol{B} = \nabla\phi \times \nabla\theta_{\mathrm{m}} - \nabla\psi \times \nabla\zeta$, we see field line is straight in the flux coordinates,

$$\frac{\mathrm{d}\theta_{\mathrm{m}}}{\mathrm{d}\zeta} = \frac{\boldsymbol{B}\cdot\nabla\theta_{\mathrm{m}}}{\boldsymbol{B}\cdot\nabla\zeta} = \frac{1}{q(\psi)}$$



Flux coordinates (ϕ,θ_{m},ζ)

Using $\boldsymbol{B} = \nabla\phi \times \nabla\theta_{\mathrm{m}} - \nabla\psi \times \nabla\zeta$, we see

$$\boldsymbol{A} = \phi\,\nabla\theta_m - \psi\nabla\zeta.$$

Action integral S is given as $S = \int \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{x} = \int [\phi\,\mathrm{d}\theta_{\mathrm{m}} - \psi\,\mathrm{d}\zeta]$

Lagrangian L is given as $L = \phi\dfrac{\mathrm{d}\theta_{\mathrm{m}}}{\mathrm{d}\zeta} - \psi(\phi)$

The coordinate θ_{m} becomes a cyclic coordinate.

In the derivation of the flux coordinate system, no geometrical symmetry is assumed for the torus plasma. But if we assume the existence of force equilibrium, double periodicity of the torus leads to “hidden symmetry” and hence becomes integrable.

3.7 Ergodicity: Field Line densely covers the Torus

"Poincaré mapping."

$$g : \Theta \longrightarrow \Theta\ , \quad g\theta_0 = \theta_0 + 2\pi/q \text{ for } \theta_0 \in \Theta$$

$$\{g^j\theta_0\} \quad \theta^j = 2\pi j/q + \theta_0$$

Rational q : $q=m/n$

mapping g^m is given

by $\theta^m = 2\pi m/q + \theta_0 = 2n\pi + \theta_0$

identity mapping

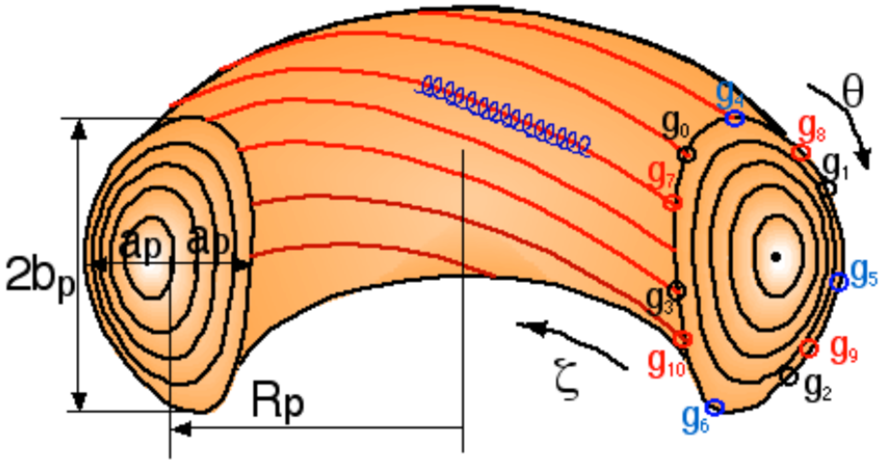
Irrational q :

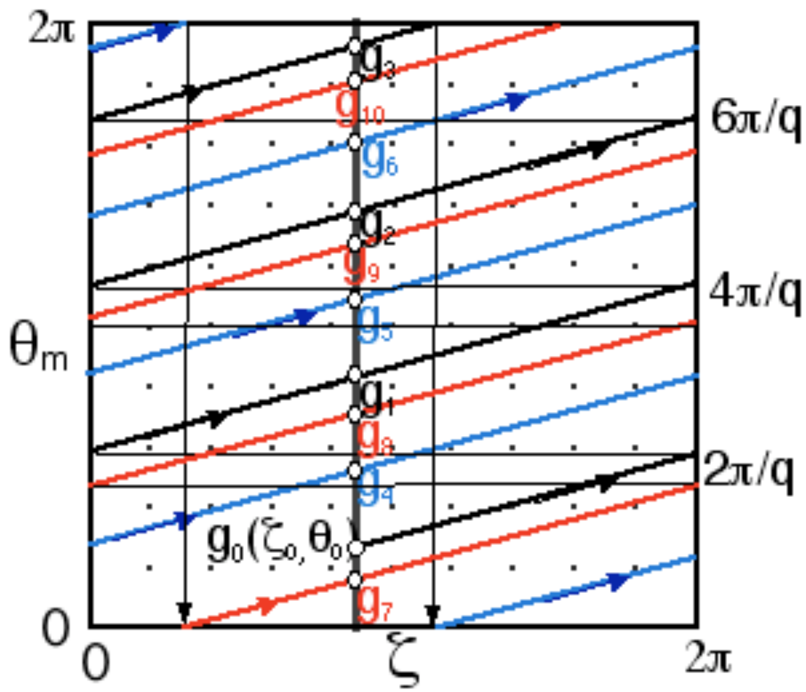
Mapping points : $\{g^j\theta_0\}$

never becomes identity mapping and will be different forever.

Mapping of neighborhoods : $\{g^i U\}$

should have common set and "densely cover" the torus surface.



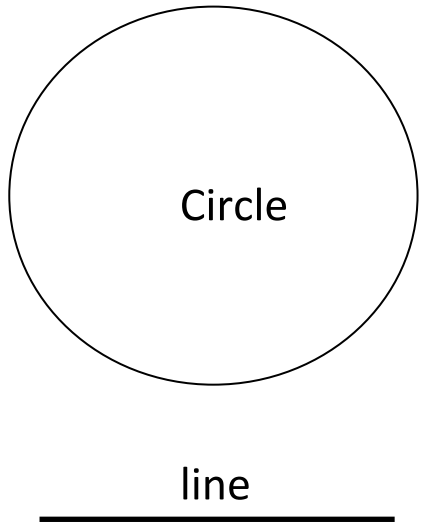


Set theory: G. Cantor is founder of set theory

Cantor

Wonder of infinity :
  Countable number (Denumerable):
natural number, integer, rational number
->
all can have one to one correspondence.
  But rational number can densely cover the line (any closest distance, we can find rational number). "Measure" of rational number is zero.

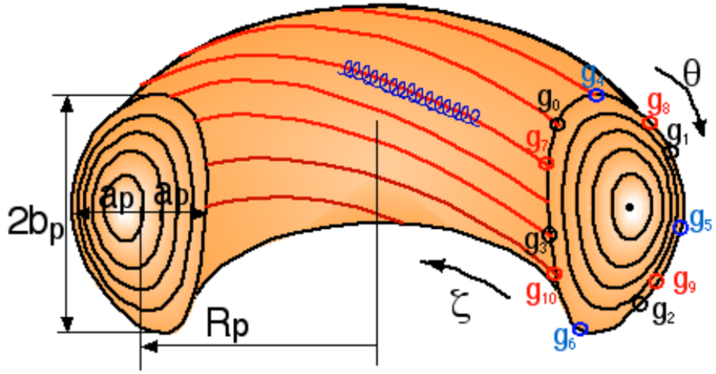
On the other hand, irrational number is much denser than denumerable. It has non-zero "measure".
Set theory : see Kolmogorov-Fomin, Chapter I,V



3.8 Apparent symmetry : Axisymmetric equilibrium

Variational principle *δS=0* to give Grad-Shafranov equation

$$S = \int L \, \mathrm{d}R \, \mathrm{d}Z = \int R \left(\frac{B_p^2}{2\mu_0} - \frac{B_\xi^2}{2\mu_0} - P \right) \mathrm{d}R \, \mathrm{d}Z$$



Euler-Lagrange equation is

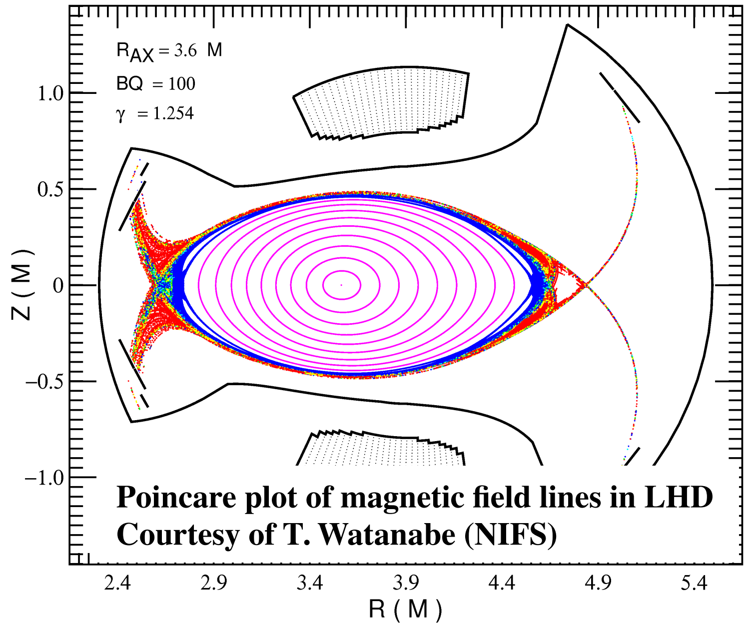
$$\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial R}\frac{\partial L}{\partial \psi_R} - \frac{\partial}{\partial Z}\frac{\partial L}{\partial \psi_Z} = 0$$

This gives Grad-Shafranov equation

$$[R\partial / \partial R(R^{-1}\partial / \partial R) + \partial^2 / \partial Z^2]\psi = -\mu_0 R^2 P'(\psi) + FF'(\psi)$$

Integrable system in dynamical system and flux surface

Simple closed line can't confine the plasma with pressure difference. Closed "surface" should be formed by the magnetic field "line" and torus-shaped plasma have to be confined in it. Thus the problem of covering the surface with magnetic field line becomes important. If the magnetic field line trajectory is on a surface, it is called "**integrable**".

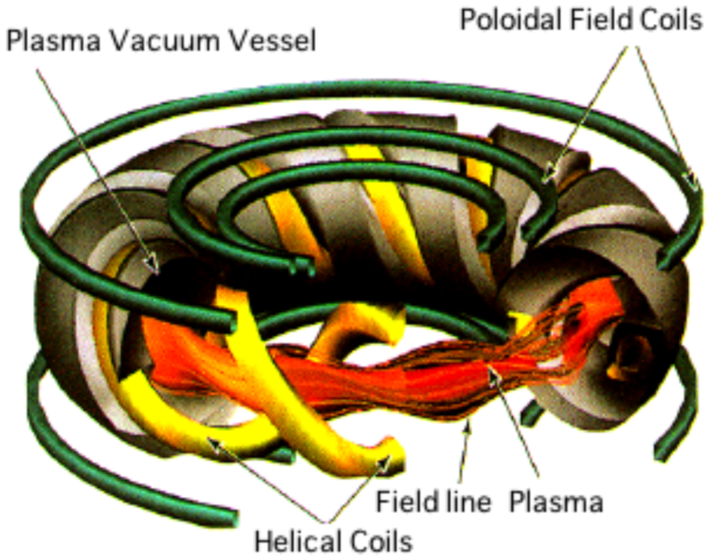


Poincare plot of magnetic field lines in LHD
Courtesy of T. Watanabe (NIFS)

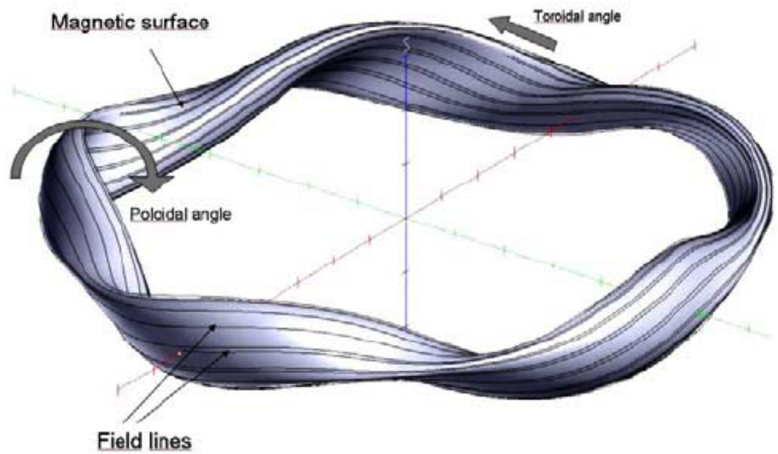
"Integrable" is a term in classical mechanics, having its origin in many-body-problem of celestial mechanics, and is plainly explained in Diacu&Homes [3-3]. French mathematician **J. Liouville** (1809-1882) gave its mathematical definition.

3 Dimensional magnetic confinement geometries

Schematic view of helical device LHD showing how twisted magnetic field lines are formed without toroidal plasma current using helical coils (left) and magnetic surface of advanced helical device W7-X (right) with 5 fold symmetry (right)



Large Helical Device (NIFS, Japan)



Wendelstein VII-X (IPP, Germany)

Exercise for Lecture 2

1. *Show that any vector in* (u,θ,ζ) *coordinate can be expressed as* $\boldsymbol{A}=\phi\boldsymbol{\nabla}\theta-\psi\boldsymbol{\nabla}\zeta+\boldsymbol{\nabla}G$

2. *Derive Hamilton equation of magnetic field line trajectory from the variational principle.*

$$\delta S = \delta \int \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{x} = 0$$

3. *At irrational flux surface,* $\boldsymbol{B}$ *will be wound indefinitely around the torus. Since signal can only propagate with a speed less than speed of light, how we can recognize magnetic field structure is irrational?*

End of Lecture 2

Below : appendix

A-1: Dual relation

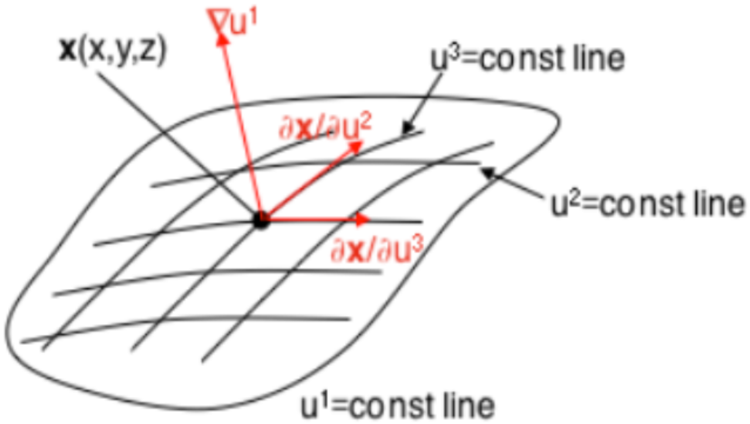
$\partial \boldsymbol{x} / \partial u^3$, $\partial \boldsymbol{x} / \partial u^2$ is orthogonal to ∇u^1

$$\nabla u^1 = J^{-1} (\partial \boldsymbol{x} / \partial u^2 \times \partial \boldsymbol{x} / \partial u^3)$$

$$\partial \boldsymbol{x} / \partial u^1 = J \nabla u^2 \times \nabla u^3$$

Jacobian is defined by $J \equiv \frac{\partial \boldsymbol{x}}{\partial u^1} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u^2} \times \frac{\partial \boldsymbol{x}}{\partial u^3} \right)$

Take inner product , $\partial \mathbf{x}/\partial u^1 \cdot \nabla u^1 = a_1 (\partial \boldsymbol{x} / \partial u^2) \times (\partial \boldsymbol{x} / \partial u^3)$



In general : $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$

$$\nabla u^i = \frac{1}{J} \left(\frac{\partial \boldsymbol{x}}{\partial u^j} \times \frac{\partial \boldsymbol{x}}{\partial u^k} \right)$$

$$\frac{\partial \boldsymbol{x}}{\partial u^i} = J \nabla u^j \times \nabla u^k$$

A-2: Many formulas comes from orthogonal and dual relations

Covariant form

$$a = \sum_i a^i \frac{\partial x}{\partial u^i} = \sum_i (a \cdot \nabla u^i) \frac{\partial x}{\partial u^i}$$

Contravariant form

$$a = \sum_i a_i \nabla u^i = \sum_i \left(a \cdot \frac{\partial x}{\partial u^i}\right) \nabla u^i$$

Metric

$$g_{ij} = \frac{\partial x}{\partial u^i} \cdot \frac{\partial x}{\partial u^j}, \quad g^{ij} = \nabla u^i \cdot \nabla u^j$$

Differential length

$$ds^2 = dx \cdot dx = \sum_{i,j} \frac{\partial x}{\partial u^i} \cdot \frac{\partial x}{\partial u^j} du^i du^j = \sum_{i,j} g_{ij} du^i du^j$$

Inner product

$$a \cdot b = \sum_i a_i b^i = \sum_i a^i b_i$$

Gradient

$$\nabla f = \sum_i \frac{\partial f}{\partial u^i} \nabla u^i$$

Divergence

Dual relation

$$\nabla \cdot a = \nabla \cdot \sum_i a^i \frac{\partial x}{\partial u^i} = J^{-1} \sum_i \frac{\partial J a^i}{\partial u^i}$$

Magnetic field trajectory

$$dx/ds = b \ (b = B/|B|)$$

$$dx/ds = \sum (\partial x/\partial u^j) du^j/ds$$

Orthogonal relation

$$[\, dx/ds = b \,] \cdot \nabla u^i$$

$$\boxed{du^i/ds = b \cdot \nabla u^i}$$

A-3: Variational principle for field lines

Action integral to give field line is the path integral of vector potential.

$$\delta S = \delta \int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x} = 0$$

Use $\boldsymbol{A}=\phi\nabla\theta-\psi\nabla\zeta+\nabla G$ ⟵ *Note : Gauge term does not contribute*

$$\delta S(\theta,\phi) = \int \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} - \frac{\partial \psi}{\partial \phi} \right) \delta\phi - \left(\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} + \frac{\partial \psi}{\partial \theta} \right) \delta\theta + \frac{\mathrm{d}(\phi\delta\theta)}{\mathrm{d}\zeta} \right] \mathrm{d}\zeta$$

$$\begin{aligned}
\delta(A\cdot dx) &= \delta(\phi d\theta - \psi d\zeta + dG) \\
&= \delta[\phi(d\theta/d\zeta) - \psi + (dG/d\zeta)]d\zeta \\
&= [\,(d\theta/d\zeta)\delta\phi + \phi(d\delta\theta/d\zeta) - (\partial\psi/\partial\phi)\delta\phi - (\partial\psi/\partial\theta)\delta\theta + (d\delta G/d\zeta)]d\zeta \\
&= [\{d\theta/d\zeta - \partial\psi/\partial\phi\}\,\delta\phi - \{d\phi/d\zeta + \partial\psi/\partial\theta\}\,\delta\theta + \{d(\delta G + \phi\delta\theta)/d\zeta\}\,]d\zeta
\end{aligned}$$

A-4: Variational principle to give plasma equilibrium

Action integral :
$$S = \int L \, \mathrm{d}V = \int \left[\frac{B^2}{2\mu_0} + \frac{P}{\gamma - 1} \right] \mathrm{d}V$$

$$\delta L = \vec{B} \cdot \frac{\delta \vec{B}}{\mu_0} + \frac{\delta p}{\gamma - 1}$$

$$\delta P = -\gamma P \nabla \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla P$$
$$\delta \boldsymbol{B} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}) \, .$$

$$\delta p = -\vec{\xi} \cdot \nabla p - \gamma p \nabla \cdot \vec{\xi} = (\gamma - 1)\vec{\xi} \cdot \nabla p - \nabla \cdot (\gamma \vec{\xi} \, p)$$

$$\vec{B} \cdot \frac{\delta \vec{B}}{\mu_0} = \frac{1}{\mu_0} \vec{B} \cdot \nabla \times \left(\vec{\xi} \times \vec{B}\right) = \frac{1}{\mu_0} \left[\left(\vec{\xi} \times \vec{B}\right) \cdot \nabla \times \vec{B} - \nabla \cdot \left(\vec{B} \times \left(\vec{\xi} \times \vec{B}\right)\right)\right]$$

$$\delta L = -\vec{\xi} \cdot \left(\vec{j} \times \vec{B}\right) + \vec{\xi} \cdot \nabla p - \nabla \cdot \underbrace{\left(\frac{1}{\mu_0} \vec{B} \times \left(\vec{\xi} \times \vec{B}\right) + \frac{\gamma}{\gamma - 1} \vec{\xi} \, p\right)}_{\text{Surface integral}}$$

δS=0 $\longrightarrow$ $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$

A-5. Hamada Coordinates:

Dr. S. Hamada (1931-2001) was a professor of Nihon University who invented Hamada coordinates in 1962, which is a special flux coordinates.
In Hamada coordinates, both **B** and **J** becomes straight lines with Jacobian J=1.
Since div**J**=0, we can apply same discussion to **J** to find stream function.

[$\boldsymbol{a}=c_2\partial\mathbf{x}/\partial\theta+c_3\partial\mathbf{x}/\partial\zeta$]

$$\boldsymbol{a} = a_2\nabla\zeta \times \nabla u + a_3\nabla u \times \nabla\theta \quad (\boldsymbol{a} = \boldsymbol{B}, \boldsymbol{J})$$

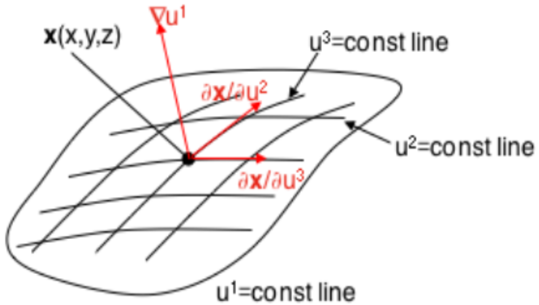
Since $\boldsymbol{a}$ is divergence free $\nabla\cdot\boldsymbol{a} = 0\ (\boldsymbol{a} = \boldsymbol{B}, \boldsymbol{J})$,we obtain $\partial a_2/\partial\theta + \partial a_3/\partial\zeta = 0$

Therefore $\boldsymbol{a}$ has stream function. $\boldsymbol{a}=\nabla u \times \nabla h$ $\quad a_2 = \dfrac{\partial h}{\partial\zeta}, \quad a_3 = \dfrac{\partial h}{\partial\theta}$

Periodicity of a_2 and a_3 in θ and ζ leads to following form of stream function for ***B*** *(b)* and ***J****(j)*

$\boldsymbol{B}=\nabla u \times \nabla b$ $\qquad b(u,\theta,\zeta) = b_2(u)\theta + b_3(u)\zeta + \tilde{b}(u,\theta,\zeta)$

$\boldsymbol{J}=\nabla u \times \nabla j$ $\qquad j(u,\theta,\zeta) = j_2(u)\theta + j_3(u)\zeta + \tilde{j}(u,\theta,\zeta)$



Hamada considered coordinate transformation to eliminate $\tilde{b}(u,\theta,\phi),\ \tilde{j}(u,\theta,\phi)$.

$$\theta_h = \theta + \theta_1 \text{ and } \zeta_h = \zeta + \zeta_1$$

A-5: Hamada Coordinates (cont.)

Coordinate transformation to eliminate $\tilde{b}(u,\theta,\phi)$, $\tilde{j}(u,\theta,\phi)$ is easy.

$$\theta_h = \theta + \theta_1 \text{ and } \zeta_h = \zeta + \zeta_1 \qquad \theta_1 = \frac{\tilde{b} j_3 - \tilde{j} b_3}{b_2 j_3 - b_3 j_2}, \quad \zeta_1 = \frac{-\tilde{b} j_2 + \tilde{j} b_2}{b_2 j_3 - b_3 j_2}$$

Coefficients b_2, b_3, j_2, j_3 of stream functions are related to magnetic flux and current flux.

$$\psi'(u) = -b_3(u),\ \phi'(u) = b_2(u), \qquad g'(u) = j_3(u),\ f'(u) = j_2(u)$$

$$2\pi\phi(u) = \int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{a}_\zeta \qquad 2\pi f(u) = \int \boldsymbol{J}\cdot \mathrm{d}\boldsymbol{a}_\zeta$$

$$2\pi\psi(u) = -\int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{a}_\theta \qquad 2\pi g(u) = \int \boldsymbol{J}\cdot \mathrm{d}\boldsymbol{a}_\theta$$

B and ***J*** are expressed as follows.

$$\boldsymbol{B} = \nabla u \times \nabla b \quad\longrightarrow\quad \boldsymbol{B} = \nabla\phi \times \nabla(\theta_h - \zeta_h/q) = \nabla\phi\times\nabla\alpha$$

$$\boldsymbol{J} = \nabla u \times \nabla j \quad\longrightarrow\quad \boldsymbol{J} = \nabla f \times \nabla(\theta_h - \zeta_h/q_J) = \nabla f\times\nabla\alpha_J$$

$$q = \mathrm{d}\phi/\mathrm{d}\psi(u) \text{ and } q_J = -\mathrm{d}f(u)/\mathrm{d}g(u)$$

Jacobian of Hamada Coordinates:

$\boldsymbol{B}=\nabla u \times \nabla(\phi'(u)\theta_h - \psi'(u)\zeta_h)$ $\boldsymbol{J}=\nabla u \times \nabla(f'(u)\theta_h + g'(u)\zeta_h)$

Using above expression and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$, we obtain

$$\boldsymbol{J} \times \boldsymbol{B} = -[(f'\psi' + g'\phi')\nabla\theta_h \times \nabla\zeta_h \cdot \nabla u]\nabla u = [(f'\psi' + g\phi')/J]\nabla u$$

Since $\nabla P = (dP/du)\nabla u$, equilibrium relation $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$ reads

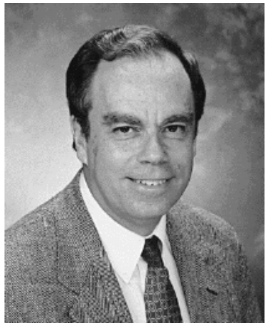
$$f'\psi' + g'\phi' = -JP'(u)$$

So, Jacobian in Hamada coordinates is flux function. i.e. $J=J(u)$.
By definition, Jacobian is related to volume enclosed by a flux surface.

$V = \int J du d\theta d\zeta$ Therefore, in this case $dV/du=(2\pi)^2 J$.

If we choose $u=(2\pi)^{-2}V$, Jacobian $J=1$. Hamada called this coordinates as “natural coordinates”.

A-6: Boozer Coordinates



Dr. A. Boozer is a professor of Columbia University who invented Boozer coordinates in 1981, which is a special flux coordinates.
For the analysis of charged particle motion, it is very important for **B** to express in both **gradient vector** and **tangent vector** to use orthogonal relation. Boozer coordinates is specially designed to meet such an objective.

Since $div\boldsymbol{J}=0$, we have stream function for equilibrium $\boldsymbol{J}$.

$$\boldsymbol{J}=\nabla u \times \nabla j \qquad j(u,\theta,\zeta) = j_2(u)\theta + j_3(u)\zeta + \tilde{j}(u,\theta,\zeta)$$

Added to simplify

$$\underline{\mu_0}\boldsymbol{J}=\nabla\phi \times \nabla[f'(\phi)\theta_m+g'(\phi)\zeta+\nu(\phi,\theta_m,\zeta)] \qquad \tilde{j} = \nu$$

Zero in Hamada

Form of $\boldsymbol{B}$ consistent with $\mu_0\boldsymbol{J}=\nabla\times\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \underline{g(\phi)\nabla\zeta + f(\phi)\nabla\theta_m} - \nu(\phi,\theta_m,\zeta)\nabla\phi + \underline{\nabla F(\phi,\theta_m,\zeta)}$$

Coefficients are flux function

This could be eliminated by coordinate transformation

$$\boldsymbol{B}=\nabla\phi \times \nabla\alpha$$ (This is essentially tangent vector presentation)

A-6: Boozer Coordinates

$$\boldsymbol{B} = g(\phi)\nabla\zeta + f(\phi)\nabla\theta_{\mathrm{m}} - \nu(\phi,\theta_{\mathrm{m}},\zeta)\nabla\phi + \underline{\nabla F(\phi,\theta_{\mathrm{m}},\zeta)}$$

Find way this be 0

$(\theta_b, \zeta_b) = (\theta_{\mathrm{m}} + \eta, \zeta + q(\phi)\eta)$ Conserves form of $\boldsymbol{B}$.

$$\boldsymbol{B} = \nabla\phi \times \nabla\alpha \qquad (\alpha = \theta_b - \zeta_b/q)$$

Since $\alpha=\theta_b-\zeta_b/q=(\theta_m+\eta)-(\zeta+q\eta)/q=\theta_m-\zeta/q$

$$\eta(\phi,\theta_{\mathrm{m}},\zeta) = \frac{F(\phi,\theta_{\mathrm{m}},\zeta)}{g(\phi)q(\phi) + f(\phi)}$$

$$\boldsymbol{B} = g(\phi)\nabla\zeta_b + f(\phi)\nabla\theta_b + \beta_*\nabla\phi$$

$$\beta_* = \eta(\phi,\theta_{\mathrm{m}},\zeta)(q(\phi)g'(\phi) + f'(\phi)) - \nu(\phi,\theta_{\mathrm{m}},\zeta)$$

Boozer coordinates$(\phi, \theta_b, \zeta_b)$

$\boldsymbol{B}=g\nabla(\zeta_b-q\eta)+f\nabla(\theta_b-\eta)-\nu\nabla\phi+\nabla F=g\nabla\zeta_b+f\nabla\theta_b+[\eta(g'q+f')-\nu]\nabla\phi+\nabla(F-(gq+f)\eta)$