The Abdus Salam
International Centre
for Theoretical Physics

2369-21

CIMPA/ICTP Geometric Structures and Theory of Control

1 - 12 October 2012

Lectures on MHD, Turbulence and Application to Solar Wind

William Matthaeus

Department of Physics and Astronomy, Bartol Research Institute, University of Delaware , USA

Lectures on MHD, Turbulence and Application to Solar Wind

W. H. Matthaeus

Department of Physics and Astronomy, Bartol Research Institute, University of Delaware

Models starting from kinetic theory

N.B. can also start from macroscopic conservation laws (c.f. Batchelor. Landau and Lifshitz Fluid Mech.)

- Kinetic theory
  - With collisions (Chapman Enskog, e.g., Huang textbook)
  - Without collisions (unmagnetized, e.g., Montgomery + Tidman; Montgomery texts; magnetized….???)
- MHD
  - Is "always correct" at large scales, low frequencies… but you can add important additional effects
- Plasma physics! Hall effect, FLR, Two fluid, Vlasov, Boltzmann, PIC. Hybrid…

Turbulence: Navier Stokes Equations

Incompressible model: Velocity $\mathbf{v}(\mathbf{x}, t)$, density $\rho = constant$, $\nabla \cdot \mathbf{v} = 0$, pressure p

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v}$$

Fourier representation

$$\mathbf{v}(\mathbf{x}, t) = \int d^3k \; e^{i\mathbf{k}\cdot\mathbf{x}} \; \tilde{\mathbf{v}}(\mathbf{k}, t)$$

Energy spectrum $S(\mathbf{k}) \sim \langle |\mathbf{v}(\mathbf{k})|^2 \rangle$

Large, energy-containing eddies, wavelength $\lambda \sim 1/|\mathbf{k}| \approx \lambda_c$ correlation scale. Small scale fluctuations $|\mathbf{k}| >> 1/\lambda_c$.

Some properties of the NS equation (incompressible)

- $\nabla \cdot \boldsymbol{v} = 0$ preserves constant density
- Pressure is a constraint
- Formal theory of approach to incompressibility exists (Klainerman and Majda 1980)
- With $\nu = 0$ (ideal), and Vn = 0 or periodic BCs, the energy $< |\boldsymbol{v}|^2 > =$ constant
- Ignore nonlinearity → linear viscous decay
- Reynolds number Re = vL/ν is a measure of the strength of nonlinear vs. linear terms

A simple model to show that quadratic nonlinearities seek equipartition**

**(unless there are extra conservation laws)

.3 Few-mode models; Lorenz model, etc.

example: 5-mode model of coupled ODEs

$$\frac{du_i}{dt} = u_{i+1}u_{i+2} + u_{i-1}u_{i-2} - 2\, u_{i+1}u_{i-1} \qquad (i = 1, 5)$$

with $u_{i+5} \equiv u_i$

is of analogous structure to (Euler) inviscid N-S equation.

- quadratic nonlinearity
- energy conserving $\quad \sum_{i=1}^{5} |u_i|^2 = \text{constant in time}$
- Liouville Theorem $\quad \sum_{i=1}^{5} \frac{\partial}{\partial u_i}\left(\frac{du_i}{dt}\right) = 0$

  ⇒ this means you can do a statistical mechanics in the 5D phase space $(u_1, u_2, u_3, u_4, u_5)$
- System is strongly mixing and ergodic — ergodic
- Tendency towards statistical equilibrium with, on average, equipartition in each mode ⇒ $\langle u_1^2\rangle = \langle u_2^2\rangle = \dots \langle u_5^2\rangle$

Stretching of material line elements in isotropic turbulence -1

- A number of quantities (e.g. magnetic field) evolve according to equations that contain terms like

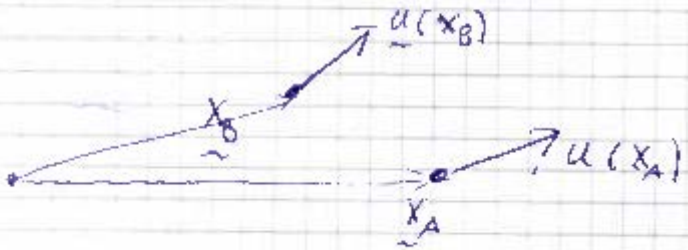
$$\frac{\partial \underset{\sim}{b}}{\partial t} + \cdots = \underset{\sim}{u} \cdot \nabla \underset{\sim}{b}$$

$\underset{\sim}{b}$ quantity transported
$\underset{\sim}{u}$ turbulent velocity

This is often called a "line stretching" term.

- Let us review a proof that line elements that follow fluid elements ("material line elements") are stretched by turbulence, i.e. their length increases in time, on ~~a~~ average.

i) equation obeyed by material line element: points move with fluid at velocity $\underset{\sim}{u}(\underset{\sim}{x})$



$$\underset{\sim}{x}_A' = \underset{\sim}{x}_A + \underset{\sim}{u}(\underset{\sim}{x}_A)\,\Delta t$$

$$\underset{\sim}{x}_B' = \underset{\sim}{x}_B + \underset{\sim}{u}(\underset{\sim}{x}_B)\,\Delta t$$
$$= \underset{\sim}{x}_B + [\underset{\sim}{u}(\underset{\sim}{x}_A) + \underset{\sim}{\ell} \cdot \nabla \underset{\sim}{u}]\,\Delta t$$

line element $\underset{\sim}{\ell}(t) = \underset{\sim}{x}_B - \underset{\sim}{x}_A$ $\qquad$ $\underset{\sim}{\ell}(t+\Delta t) = \underset{\sim}{x}_B' - \underset{\sim}{x}_A'$

Stretching of material line elements in isotropic turbulence -2

$$\Delta \underline{l} = \underline{l}(t+\Delta t) - \underline{l}(t) = \underline{x}_B' - \underline{x}_A' - (\underline{x}_B - \underline{x}_A)$$

$$= \underline{x}_B + \underline{u}(x_A)\Delta t + \underline{l}\cdot\nabla\underline{u}\,\Delta t - \underline{x}_A - \underline{u}(x_A)\Delta t - \underline{x}_B + \underline{x}_A$$

$$= \underline{l}\cdot\nabla\underline{u}\,\Delta t$$

$$\frac{\Delta \underline{l}}{\Delta t} = \underline{l}\cdot\nabla\underline{u} \xrightarrow[\lim \Delta t \to 0]{} \boxed{\frac{d\underline{l}}{dt} = \underline{l}\cdot\nabla\underline{u}}$$

ii) Matrix solution / expansion

$$\frac{d l_a}{dt} = T_{ab}\, l_b \xrightarrow{\text{abbreviate}} \frac{d \underline{l}}{dt} = T\,\underline{l}$$

(T ← matrix; $\underline{l}$ ← column vector)

Stretching of material line elements in isotropic turbulence -3

which can be written as $\ell(t) = U(t)\,\ell(0)$ (U: matrix; $\ell(0)$: column vector)

$$\ell_a(t) = U_{ab}(t)\,\ell_b(0)$$

so

$$\ell^2(t) = \underset{\sim}{\ell}\cdot\underset{\sim}{\ell}(t) = U_{ab}U_{ac}\,\ell_b(0)\,\ell_c(0) \equiv W_{bc}\,\ell_b(0)\,\ell_c(0)$$

Now (for 1st time) ensemble average, and use isotropy of the turbulence, which implies there is no preferred direction and, therefore

$$W_{bc} = \lambda(t)\,\delta_{bc}$$

so

$$\ell^2(t) = \lambda(t)\,\ell^2(0)$$

- Suppose eigenvalues of the symmetric matrix W are $\lambda_1, \lambda_2, \lambda_3$
- The matrix W (no ensemble average) can always be written in its diagonal form

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

and the eigenvectors form an orthogonal triad $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$

Stretching of material line elements in isotropic turbulence -4

$$\epsilon^3 \, \underline{a}_3 \cdot (\underline{a}_1 \times \underline{a}_2) = V$$

after evolution by Δt the volume is:

$$\epsilon^3 \, \underline{a}_3' \cdot (\underline{a}_1' \times \underline{a}_2') = V'$$

clearly $V' = \lambda_1 \lambda_2 \lambda_3 V = \det(W)\, V$

if the flow is incompressible then $V' = V$

and

$$\lambda_1 \lambda_2 \lambda_3 = 1$$

• but

$$\lambda(t) = \left\langle \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} \right\rangle = \frac{1}{3} \langle \mathrm{Tr}\, W \rangle$$

$$\geq \langle (\lambda_1 \lambda_2 \lambda_3)^{1/3} \rangle = 1$$

by Holder's inequality ($\overline{\text{arithmetic}} > \overline{\text{geometric}}$)

$\therefore \quad \lambda(t) \geq 1$, or line elements stretch on average   QED.

turbulence

- Line elements stretch
- Nonlinearities seek equipartition
- Similarity decay of energy (von Karman-Howarth
- Self similarity at high Re (Kol41)
- Intermittency and higher order statistics (Kol62)

Von Karman-Howarth and similarity decay of energy

$$du^2/dt = \alpha \; u^3/L = \alpha u^2/\tau_{nl}$$

$$dL/dt = \beta \; u = \text{[illegible]} \, L/\tau_{nl}$$

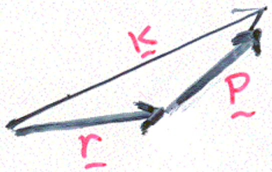
$$\tau_{nl} = L/u$$

Turbulence: nonlinearity and cascade

$$\frac{\partial \underline{V}}{\partial t} \sim \underline{V}\cdot\nabla\underline{V} \xrightarrow{\text{FOURIER}} \frac{\partial V_\alpha(\underline{k})}{\partial t} \sim \sum_{\underline{r}+\underline{p}=\underline{k}} C_{\alpha\beta\gamma}\, V_\beta(\underline{r})\, V_\gamma(\underline{p})$$

NL

"Triad interaction"



CHARACTERISTIC PROCESSES

line stretching, vortex coalescence, reconnection . . .

ORDER → CHAOS

ergodicity, mixing

CHAOS → ORDER

coherent structures

intermittency

Energy decay in turbulence

Wind tunnel measurements of energy vs. distance (time)

Batchelor and Townsend, 1949

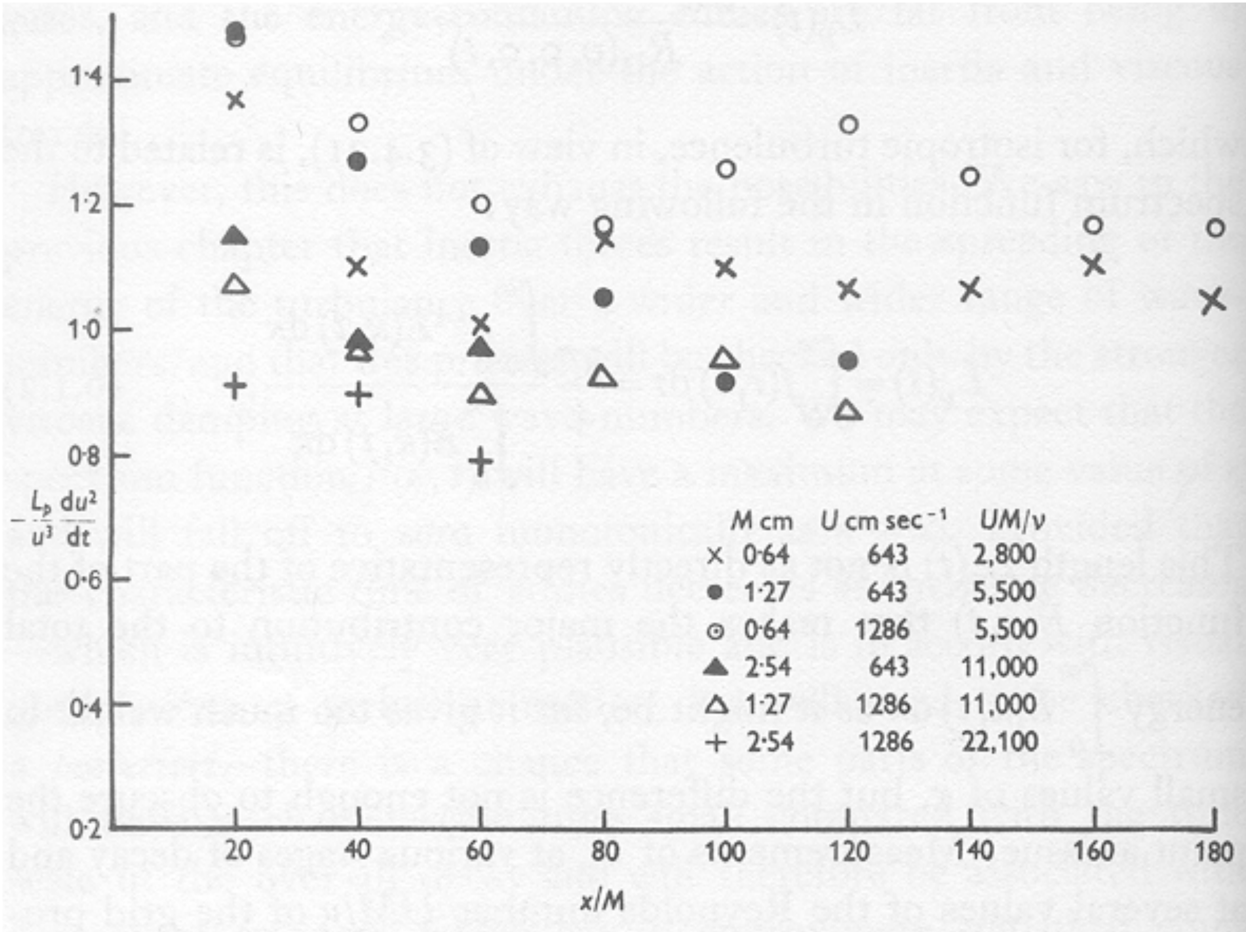
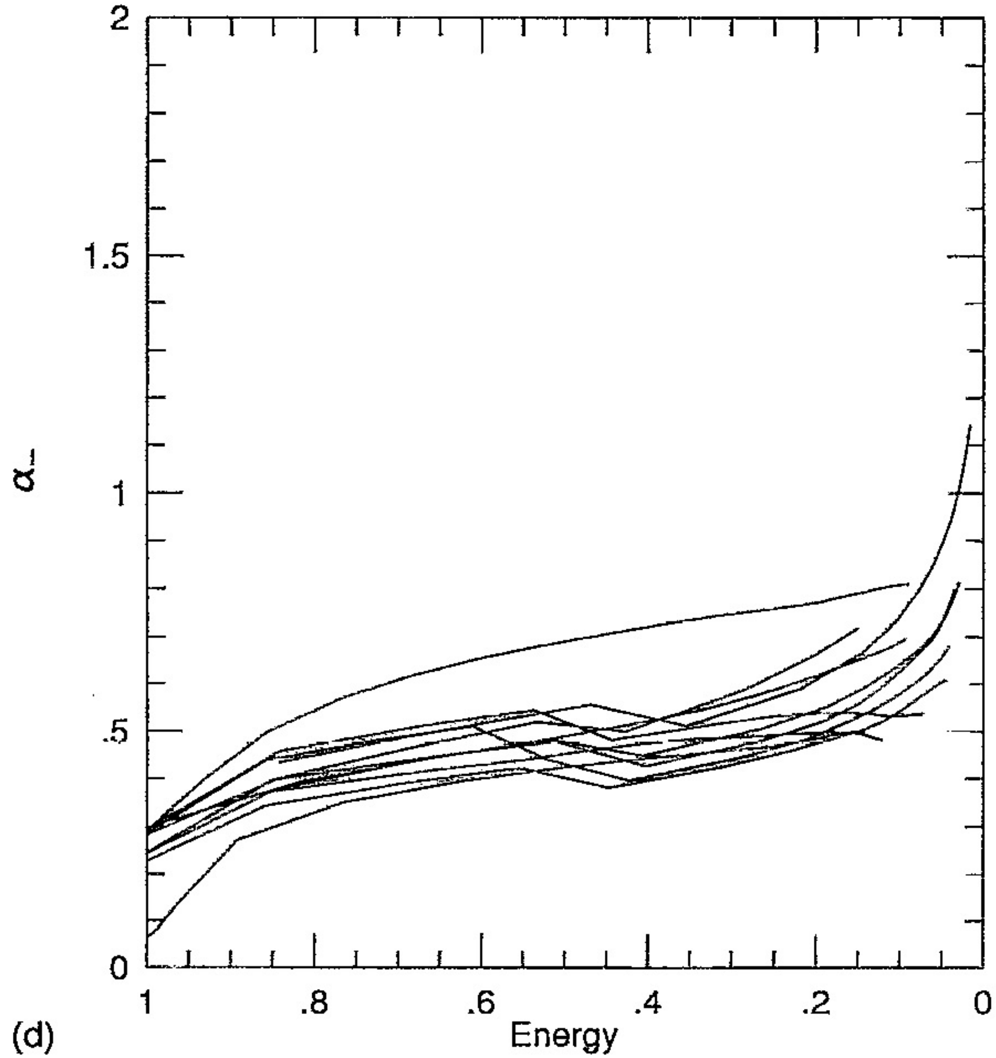
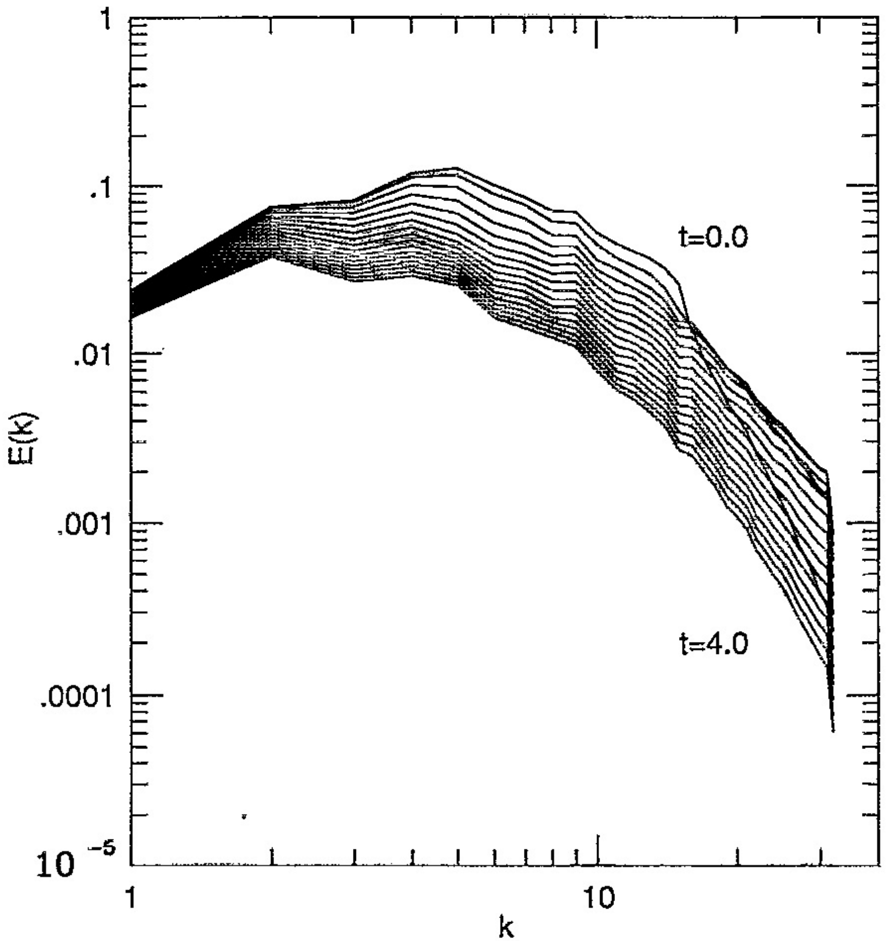


Fig. 6.1. The empirical relation $\frac{du^2}{dt} = -A\frac{u^3}{l}$.

Von Karman-Taylor constant for a group of MHD simulations (from dZ-/dt equation)



Spectra vs time for MHD simulation



The Kolmogorov 1941 spectrum

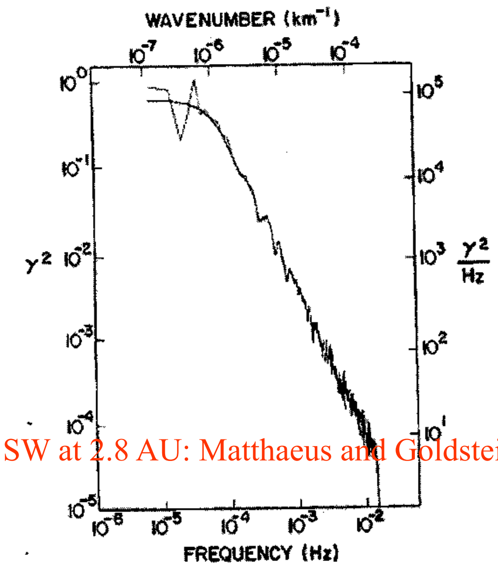
- Well separated in scale from sources and dissipation, the distribution of energy over scale (omnidirectional spectrum) should depend only on
  - ε the rate of driving=supply=transfer through scale
  - k the wavenumber (inverse scale)
- Therefore by dimensional analysis

$$E(k) \sim \text{[illegible]}\alpha\ k\beta$$

→ $$E(k)=Ck\ \varepsilon^{2/3}k^{-5/3}$$

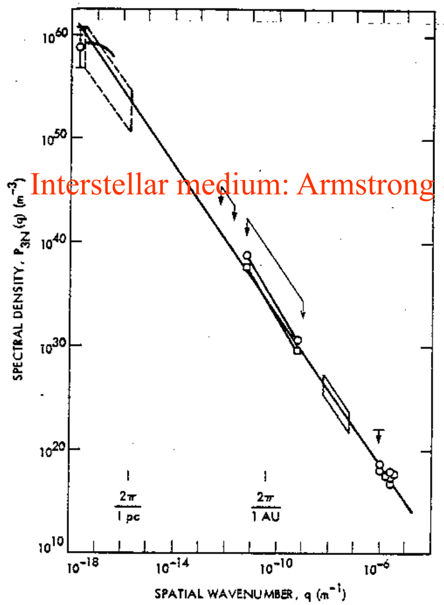
“Powerlaws everywhere”→

Broadband self-similar spectra are a signature of cascade



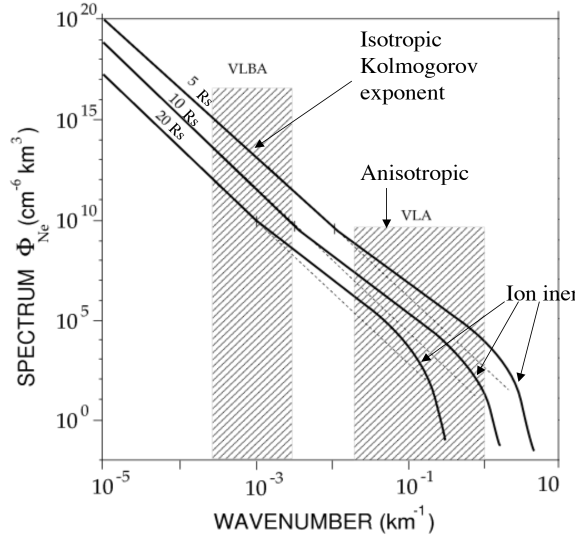
SW at 2.8 AU: Matthaeus and Goldstein

- Solar wind
- Corona
- Diffuse ISM
- Geophysical flows

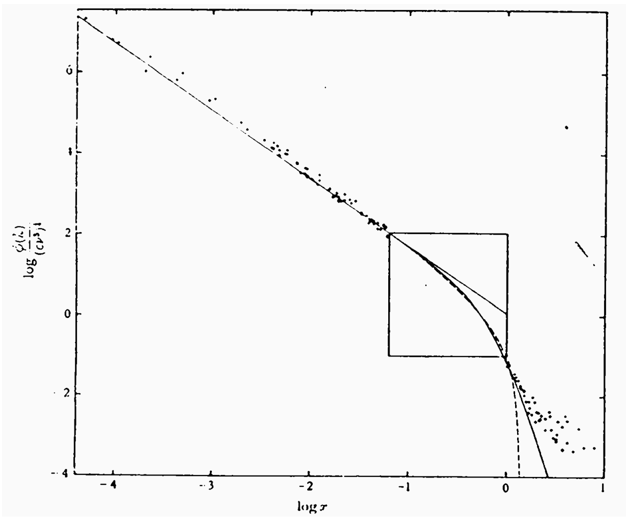


Interstellar medium: Armstrong et al

Solar Wind Density Spectra

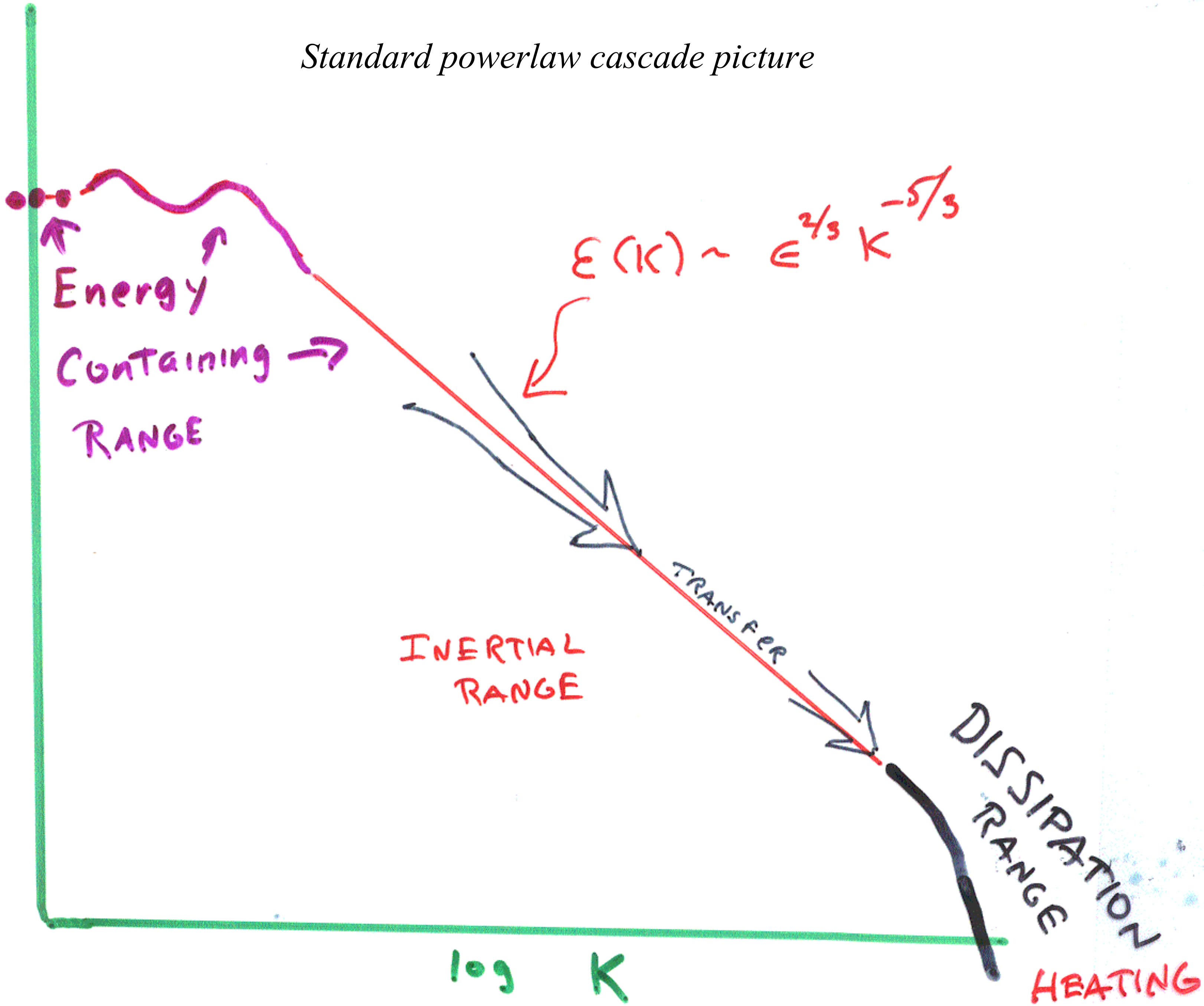


Coronal scintillation results (Harmon and Coles)



Tidal channel: Grant, Stewart and Moilliet

Standard powerlaw cascade picture



Turbulent fluctuations have structure!

Kolmogorov '41 *ε : dissipation rate*

δv_r : *velocity increment*

$\delta v_r \sim (\varepsilon r)^{1/3}$

$\rightarrow \langle \delta v_r^{\,p} \rangle = \text{const.}\ \varepsilon^{p/3}\, r^{p/3}$

But this is NOT observed!

Why? Spatial fluctuations of dissipation are very large – gradients Are not uniformly distributed; the cascade produces ***intermittency***

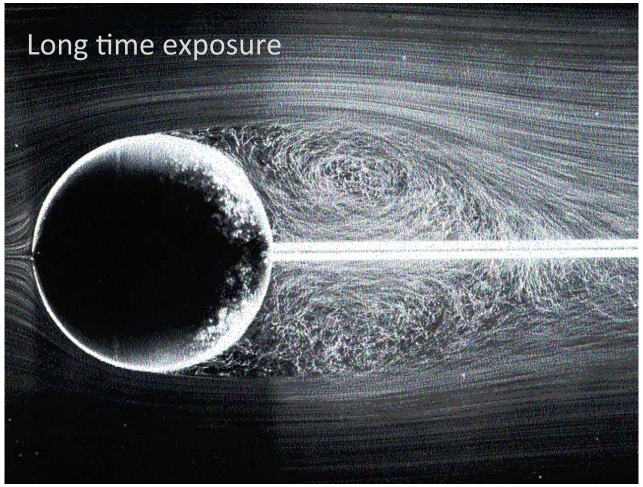
Kolmogorov '62 $\varepsilon_r = r^{-3} \int_r d^3x'\, \varepsilon(x')$

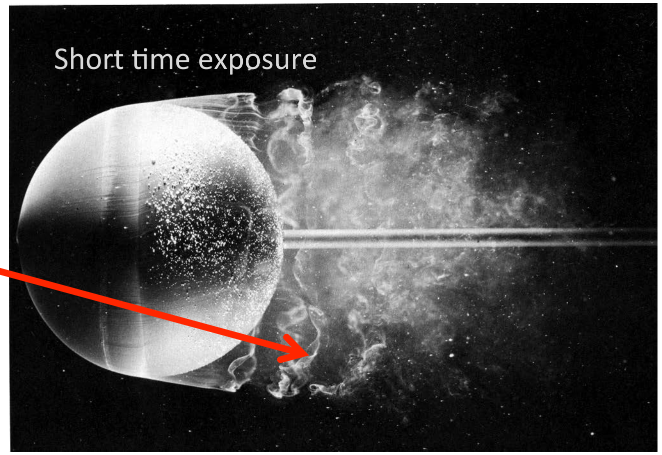
$\delta v_r \sim (\varepsilon_r\, r)^{1/3}$

$\rightarrow \langle \delta v_r^{\,p} \rangle = \text{const.}\ \langle \varepsilon_r^{\,p/3} \rangle\, r^{p/3}$

$= \text{const.}\ \varepsilon^{p/3}\, r^{p/3 + \xi(p)}$

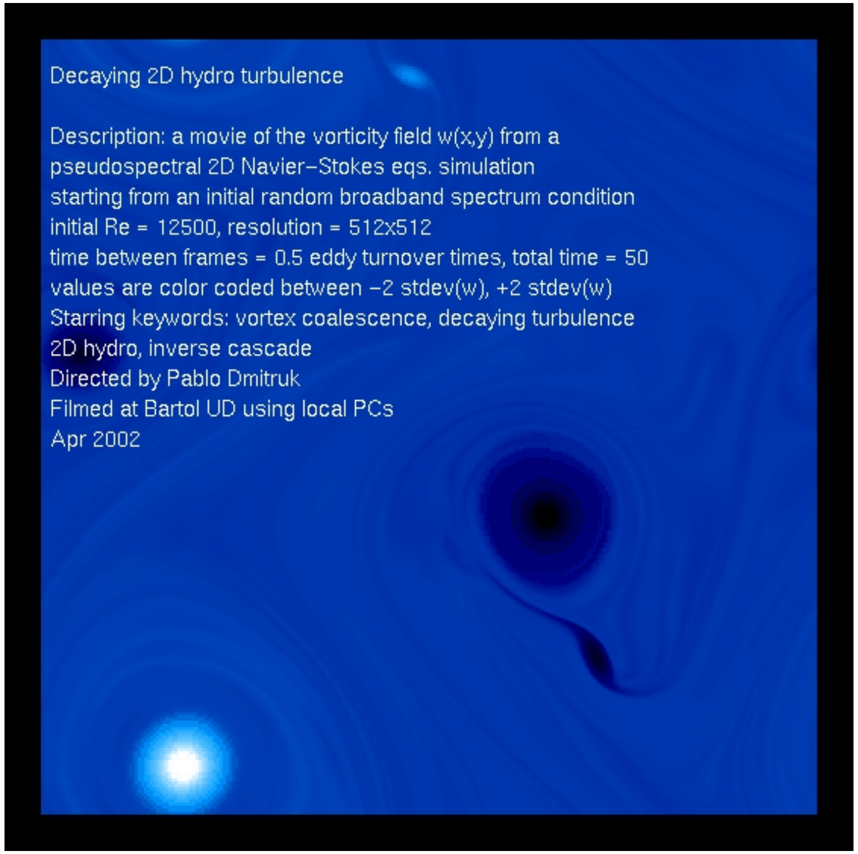
hydrodynamic flow around a sphere:



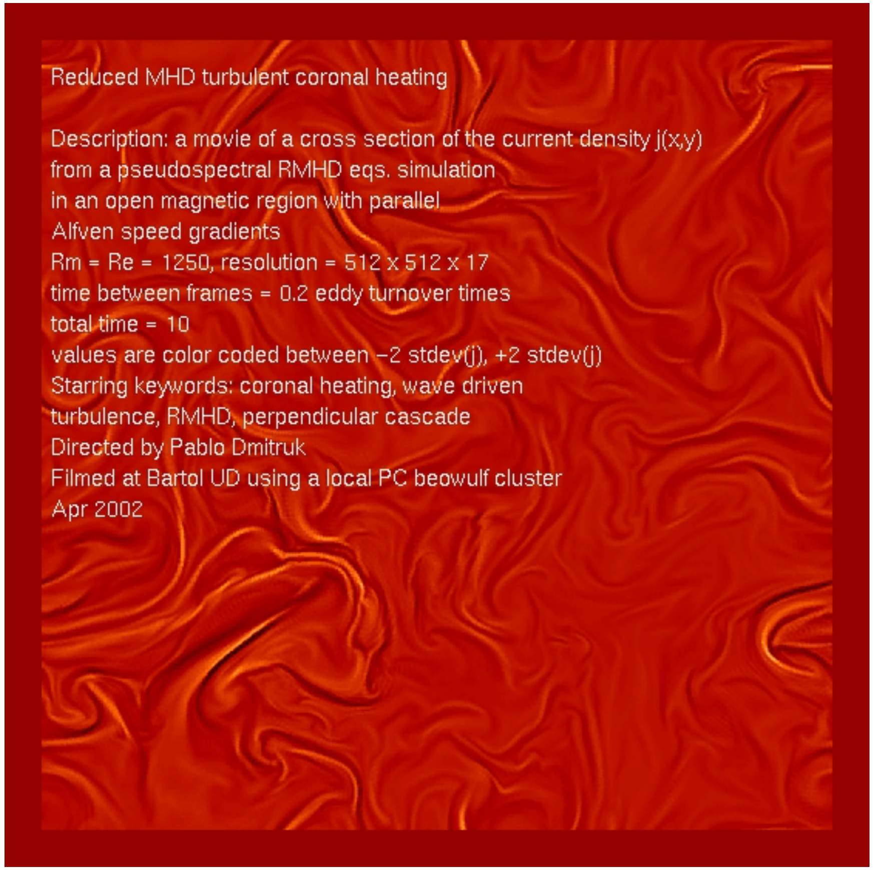


(Oubukhov '62) → multifractal theory etc, comes from this!

Turbulence (in its various forms) produces coherent structures as an integral part of the cascade process



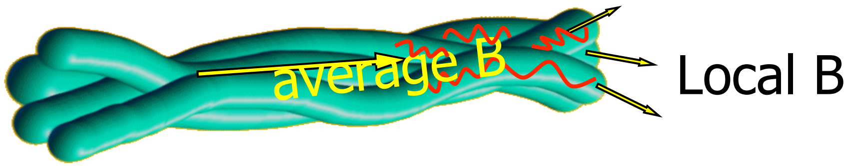
2D hydrodynamics:
Vorticity sheets and maximum entropy "cores"



2D magnetohyrdodynamics:
current sheets and flux tube cores

Alternative views of origin of "flux tubes" and discontinuities/ current sheets in SW

In turbulence: expect structure from outer scale to dissipation range



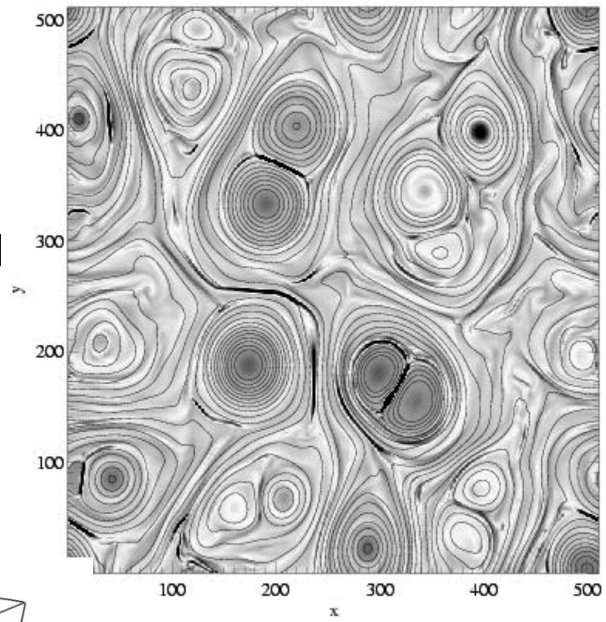
Spaghetti models:

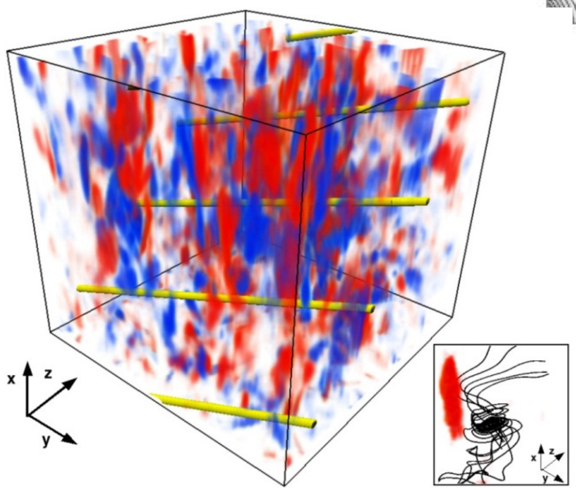
e.g.,. Bruno et al. [2004]

Figure 1. A sketch of the flux tube texture of the solar-wind plasma. Each flux tube contains a different plasma and the flux tubes move independently. A depiction (left) looking at the sides of the tubes indicates that the tubes are tangled about the direction of the Parker spiral. An end view (right) depicts the cross sections of the network of tubes. The scale sizes of the flux tubes correspond to the scale sizes of granules on the solar surface. The median diameter of a flux tube at 1 AU is 5.5×10^5 km.

- Passive flux tubes with boundaries
  -- Borovsky 2008

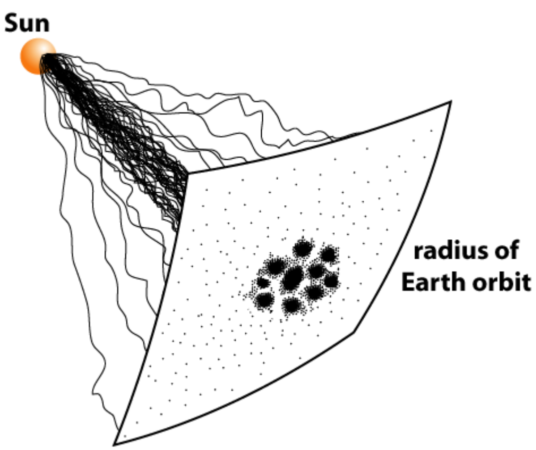
Current and Magnetic field in 2D MHD simulation





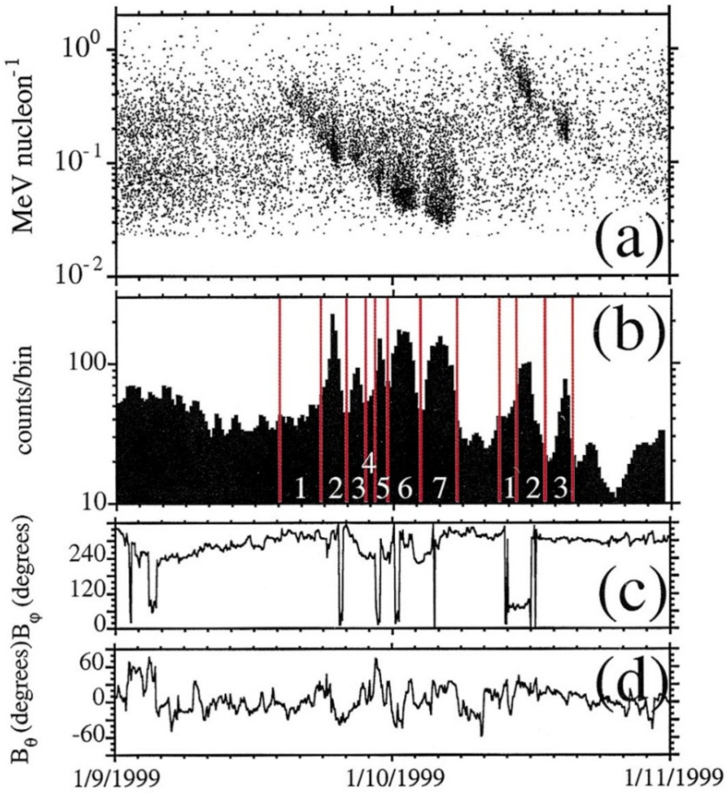
3D *MHD* compressible simulation with mean B_0

Spatially structured turbulence is expected to have transport or "trapping" boundaries



Ruffolo et al.2003

Boundaries are observed: "dropouts" of Solar energetic particles

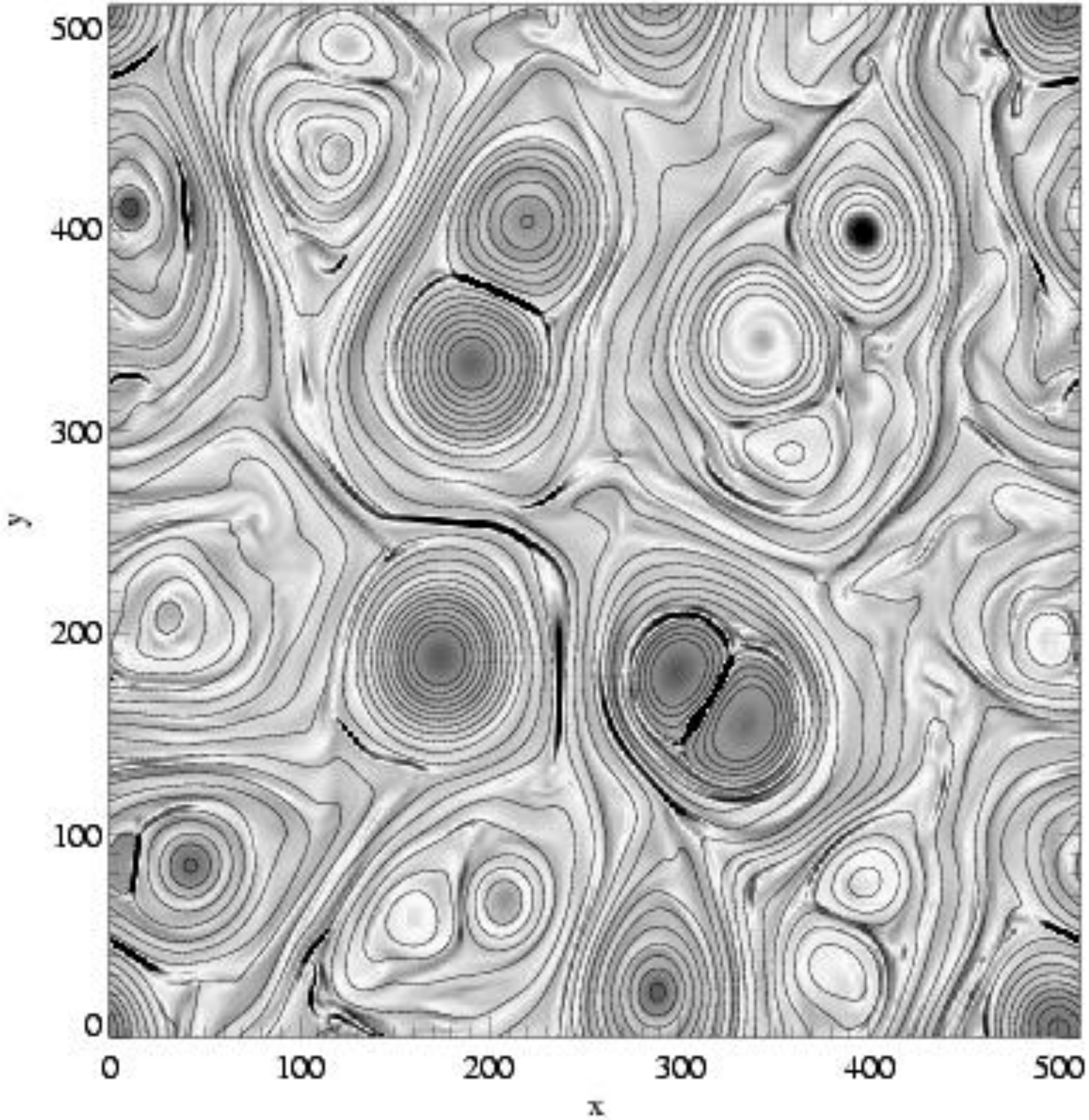


H-FE ions vs arrival time
For 9 Jan 1999 SEP event
From Mazur et al, ApJ (2000)

Temporarily trapped
Particles-- Tooprakai et al, 2007

Two-dimensional turbulence

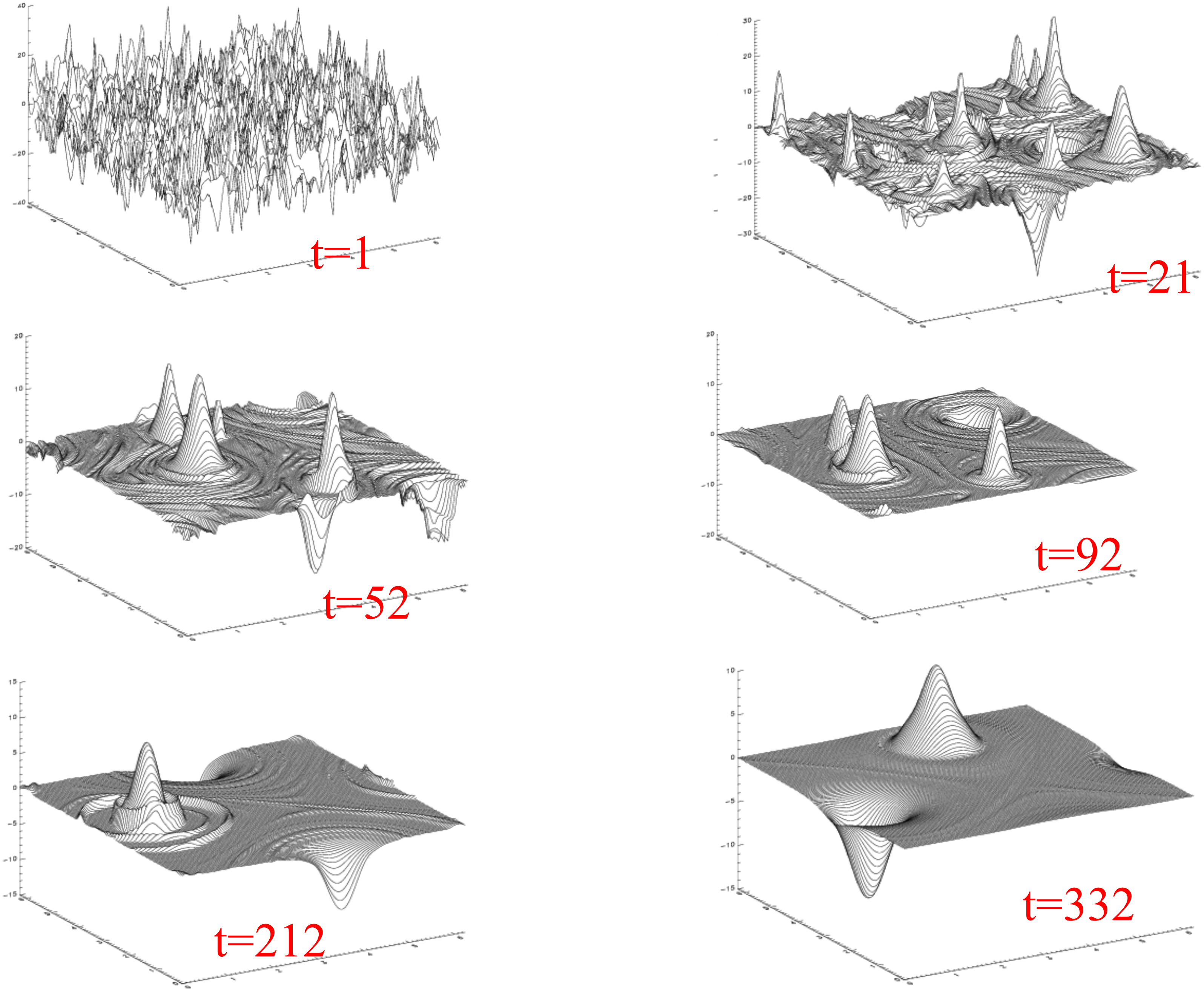
512^2 spectral method simulation



See
Fyfe and Montgomery, 1976
Fyfe et al, 1976, 1977

2D NS turbulence, Re=14,000, visualization of vorticity

Montgomery et al, 1990



Magnetohydrodynamics (MHD), etc

Distinctive effects in MHD

The incompressible MHD model, in terms of the fluid velocity $\mathbf{u}$ and the magnetic field $\mathbf{B}$, involves the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}(\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{u} \quad (1)$$

and the magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \mu \nabla^2 \mathbf{B}. \quad (2)$$

- Two fields and multiplicity

of length scales

- Anistropy/propagation

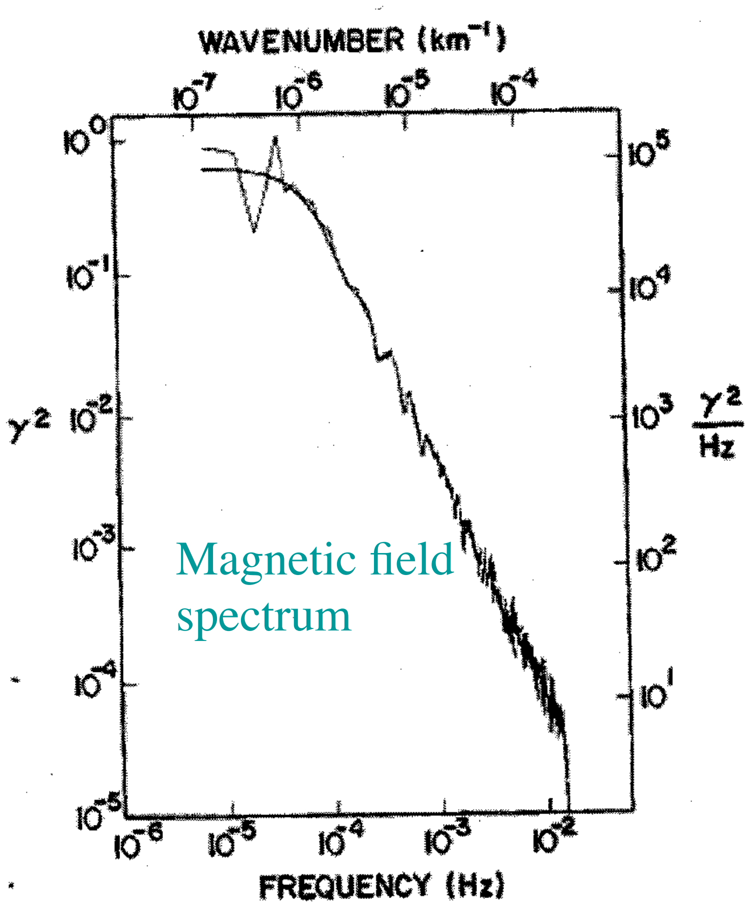
$$\frac{\partial \mathbf{z}_\pm}{\partial t} \mp \mathbf{V_A} \cdot \nabla \mathbf{z}_\pm = -\mathbf{z}_\mp \cdot \nabla \mathbf{z}_\pm - \frac{1}{\rho}\nabla P + \mu \nabla^2 \mathbf{z}_\pm;$$

- Multiple ideal invariants/direct/inverse cascades (see also “quasi-invariants”)
- Dimensionless parameters

There are numerous reasons to doubt that MHD turbulence admits the same sort of “universality” that hydro does.

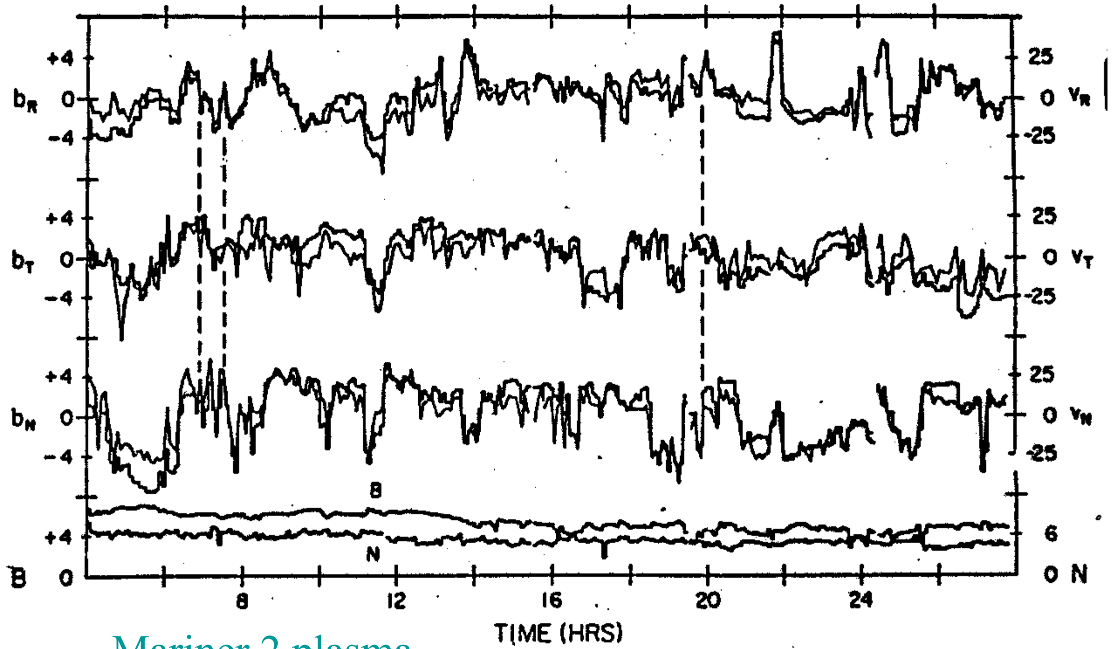
Solar wind: indications of both turbulence and wave-like properties:

- Powerlaws
- "Alfvenic fluctuations"



Magnetic field spectrum

SW at 2.8 AU: Matthaeus and Goldstein



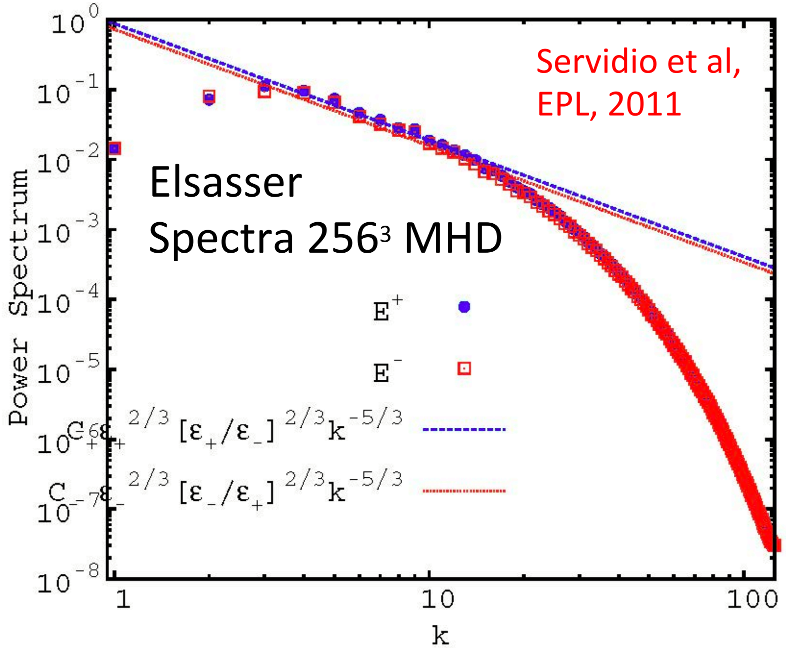
Mariner 2 plasma and magnetic field data

Belcher and Davis, JGR, 1972

- Theory

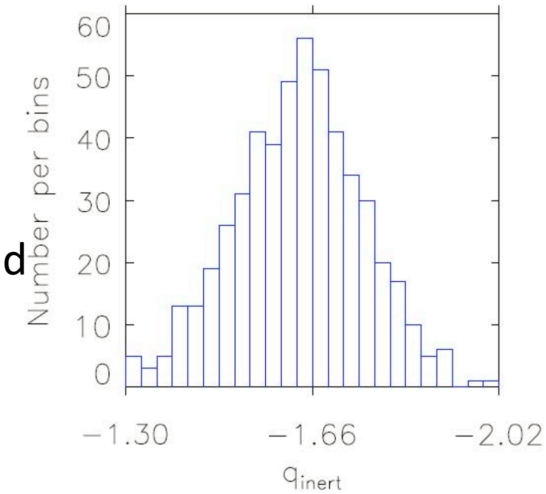
  K41 & I-Kr65

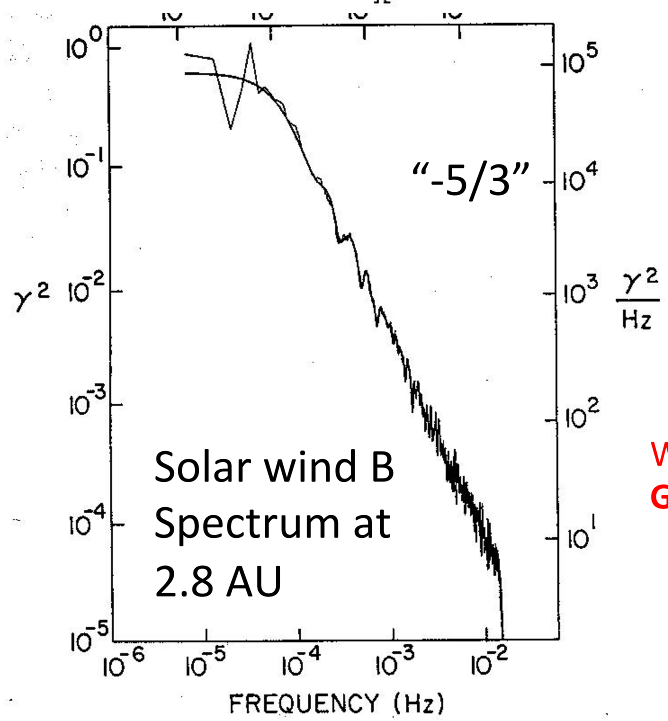
- Simulation →



- Solar wind observation

But if you look at many intervals, there's a broad distribution of powerlaws
(Vasquez et al, 07



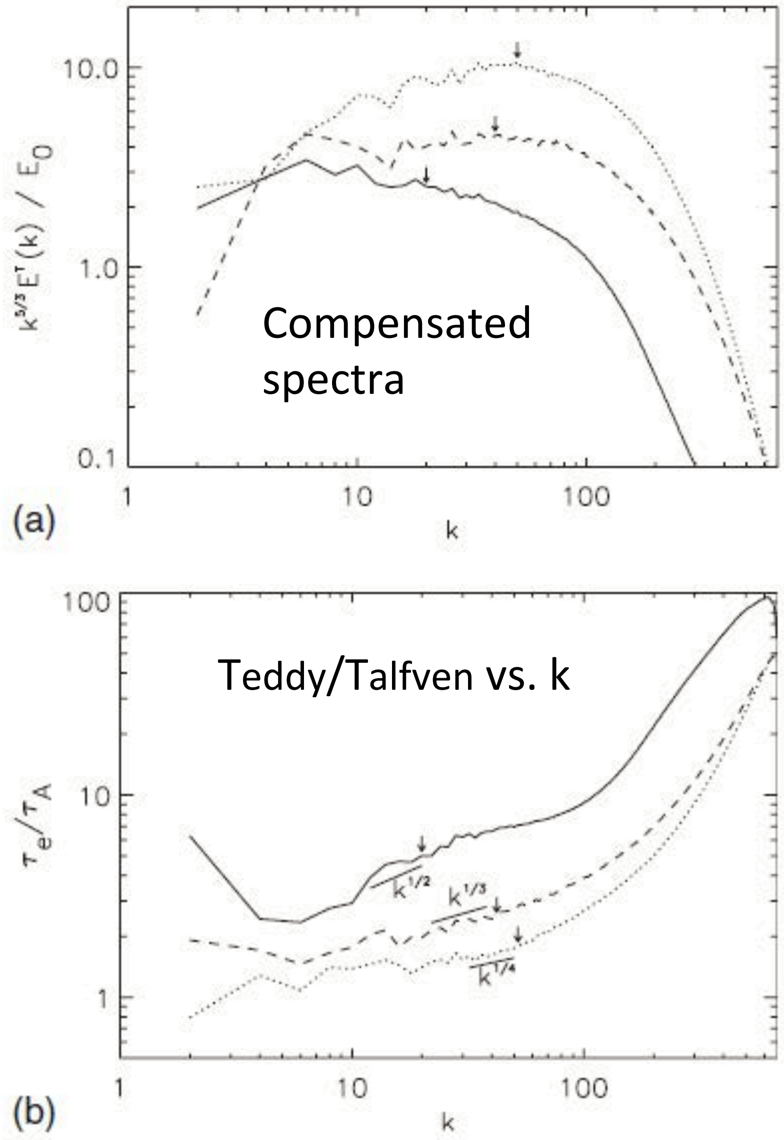


WHM & **Gol**dstein, 1982

MHD spectra are highly variable!

In simulations
AND solar wind

- Lee at al (PRE, 2010) showed that initial conditions with same E, Hc and Hm, and same initial spectra can *evolve very differently* →
- The reason for this is not agreed upon
- May point to role of the structure of 4th order moments (Wan et al JFM 2011, submitted) ; if these are nonuniversal then MHD is not universal.
- NB: B0 and 4th order correlations do not enter third order law, so anisotropy at 3rd and 4th order may be very important



Nonlinear (incompressible) MHD

Velocity fluctuation v $\quad \nabla \bullet v = 0$

Magnetic fluctuation b $\quad \nabla \bullet B = 0$

Mean magnetic fld B_0 $\quad B = B_0 + b$

Induction

$\partial \mathbf{b} / \partial t = □ - \mathbf{v}\, □ \nabla \mathbf{b} + \mathbf{b}\, □ \nabla \mathbf{v} + B_0$

Momentum

$\partial \mathbf{v} / \partial t \; □ - v\, □ \nabla v + J \times B + \nu \nabla^2 v$

nonlinear

linear

$\boldsymbol{b}(\boldsymbol{x}) = □\, d3k\, \boldsymbol{b}(\boldsymbol{k})\, e\, ik □\, x$

$\mathbf{b}(\mathbf{k},t) \sim |\mathbf{b}(k,0)|\, e^{i\omega t}$

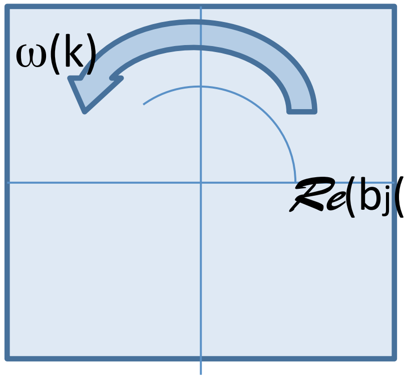
$\partial \boldsymbol{b}(k)/\partial t \sim \sum k = r + p \uparrow$ [illegible] $\boldsymbol{b}(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{p})$

Cross scale couplings: chaos
Random walk in phase space
→ "turbulence"

eigenmodes (independent)
systematic rotation in
phase space → "waves"

Trajectory in time of a Fourier mode in strong turbulence

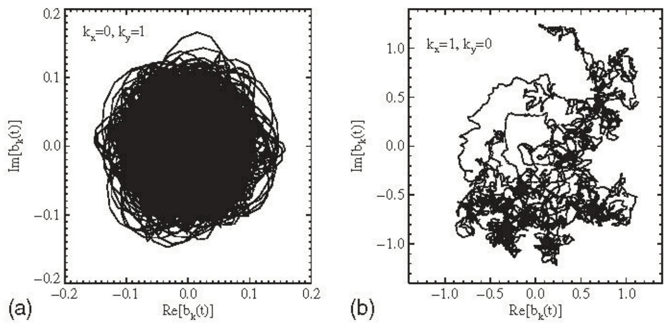
pure wave Fourier mode
ω(k): dispersion relation



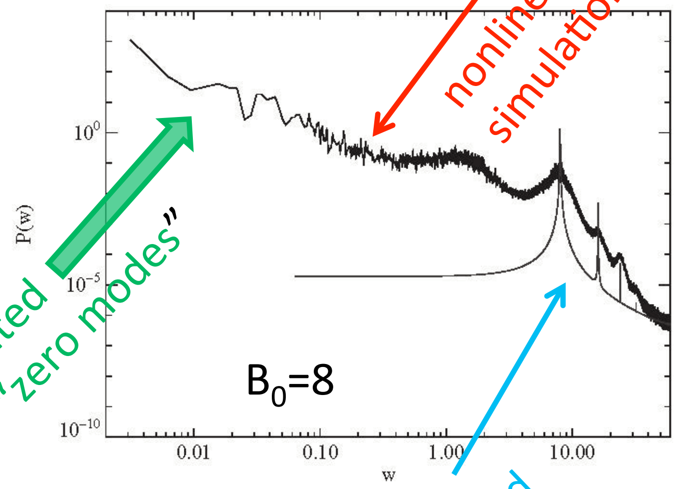
Numerical experiments on MHD Turbulence with mean field:

maximum of ~13% of energy in linear modes – that occurs when dB/B0 ~ 1/2

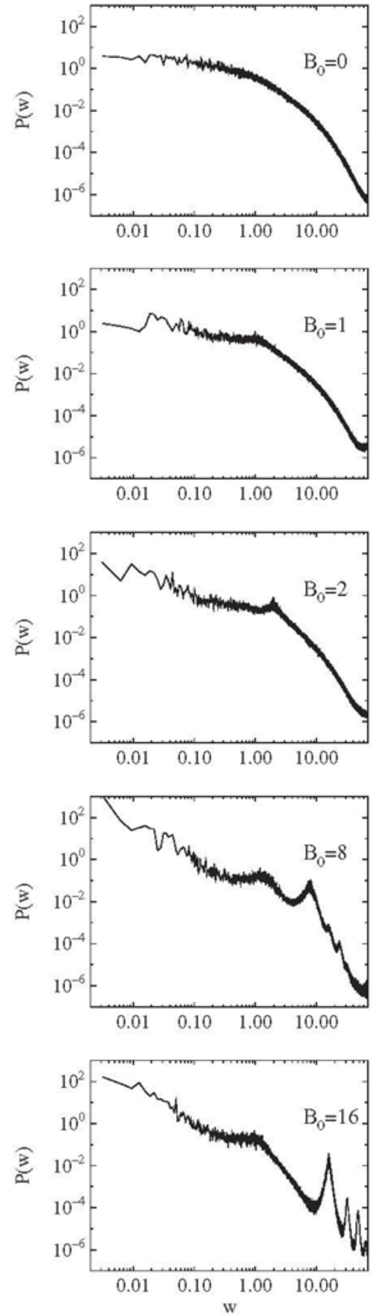
behavior of a Fourier mode In time, from simulation



Eulerian frequency spectrum: transform of one point two time Correlation fn.

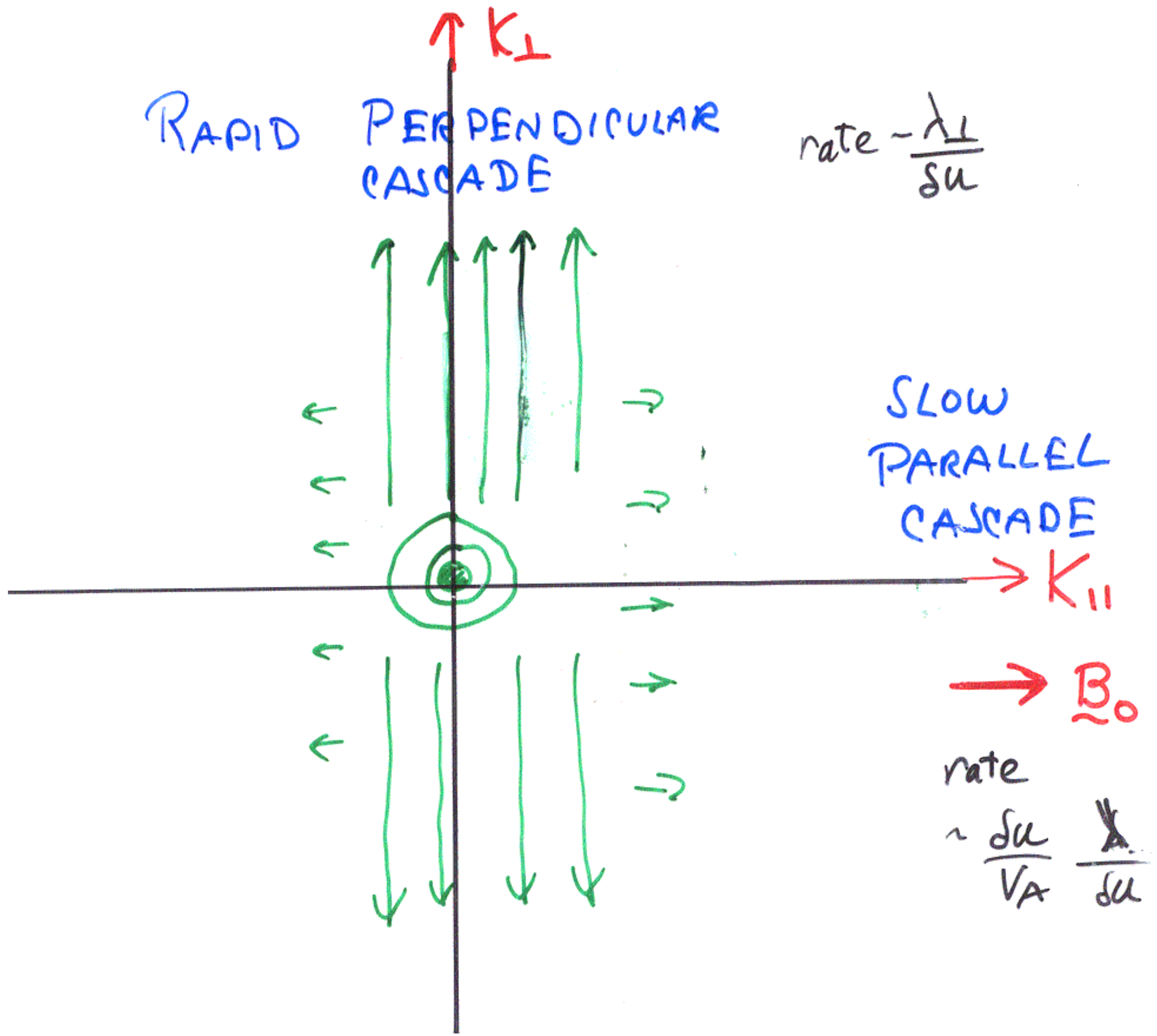


Eulerian frequency spectra



Increasing B_0

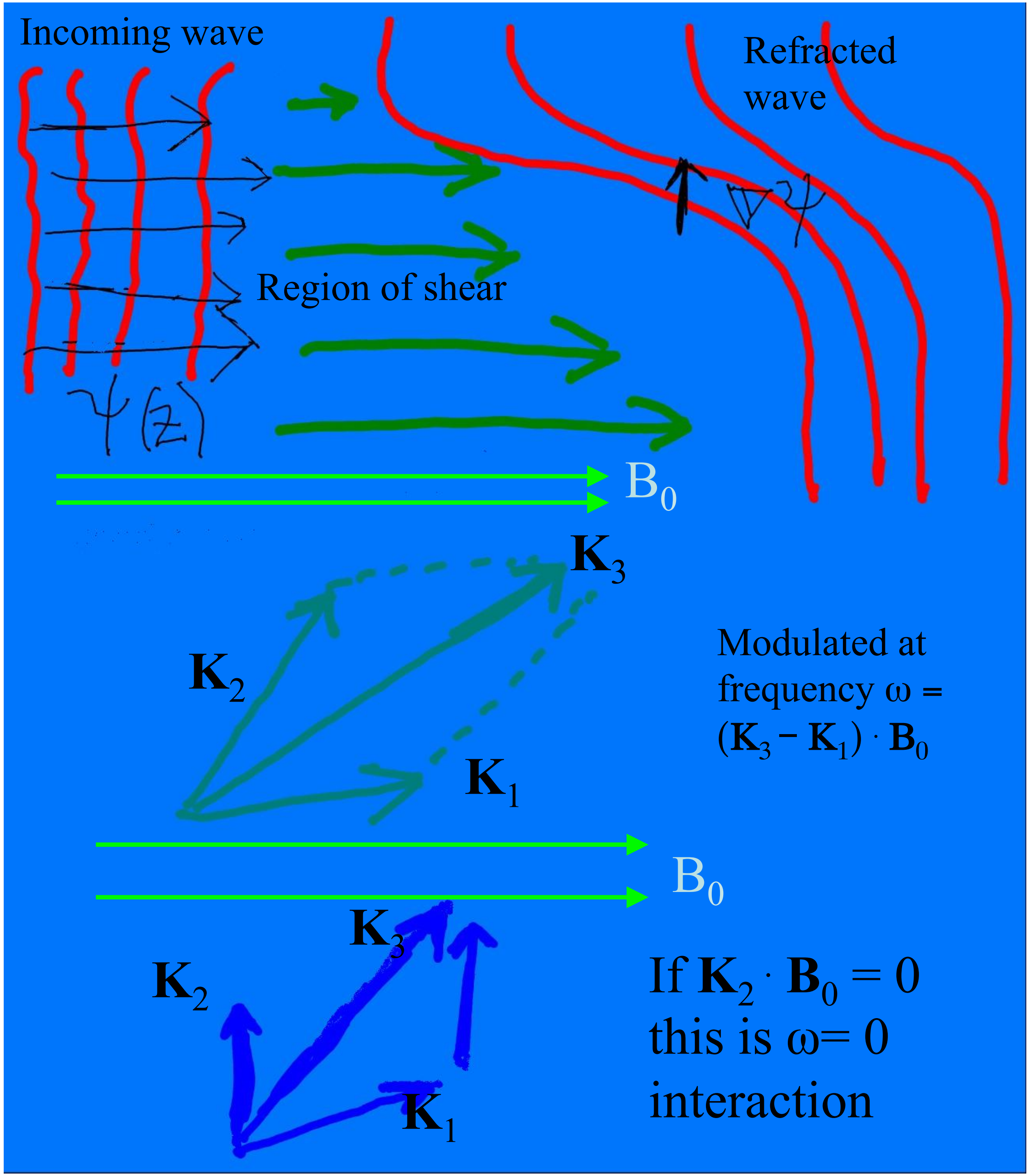
Dmitruk & Matthaeus, 2009


$K_\perp$
RAPID PERPENDICULAR CASCADE
rate $\sim \frac{\lambda_\perp}{\delta u}$
SLOW PARALLEL CASCADE
$K_\parallel$
B_0
rate
$\sim \frac{\delta u}{V_A} \frac{\lambda}{\delta u}$

Cross sections ♎B/B_0 = 1/10

Jz and Bz in an x-z plane

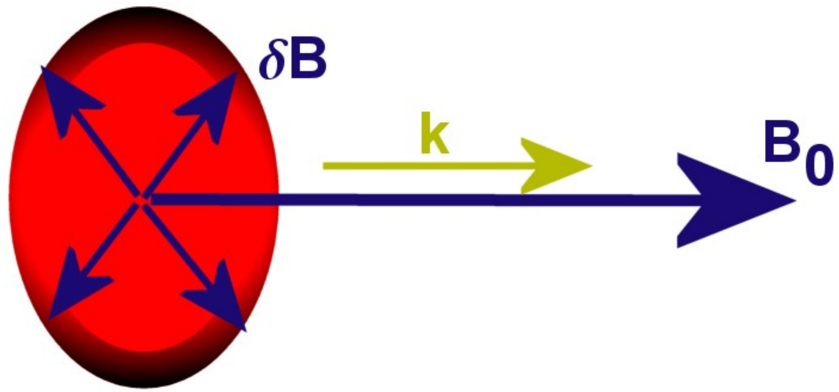
Jz and Bx, By in an x-y plane


Incoming wave
Refracted wave
$\nabla\psi$
$\psi(z)$
Region of shear
B_0
$\mathbf{K}_3$
$\mathbf{K}_2$
$\mathbf{K}_1$
Modulated at frequency $\omega = (\mathbf{K}_3 - \mathbf{K}_1) \cdot \mathbf{B}_0$
B_0
$\mathbf{K}_3$
$\mathbf{K}_2$
$\mathbf{K}_1$
If $\mathbf{K}_2 \cdot \mathbf{B}_0 = 0$ this is $\omega = 0$ interaction

Turbulence Geometry

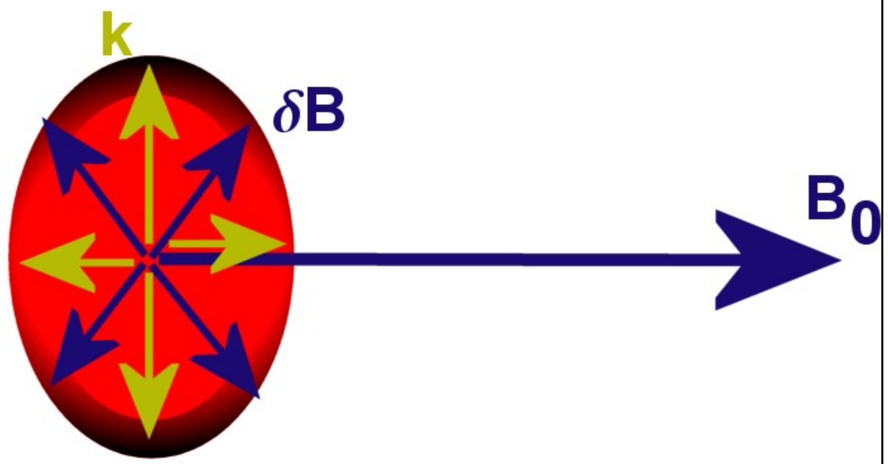
Slab Geometry: Wavevectors k parallel to mean field B_0. Fluctuating field δB perpendicular to B_0.

Motivations: Parallel propagating Alfvén waves. Computational simplicity.

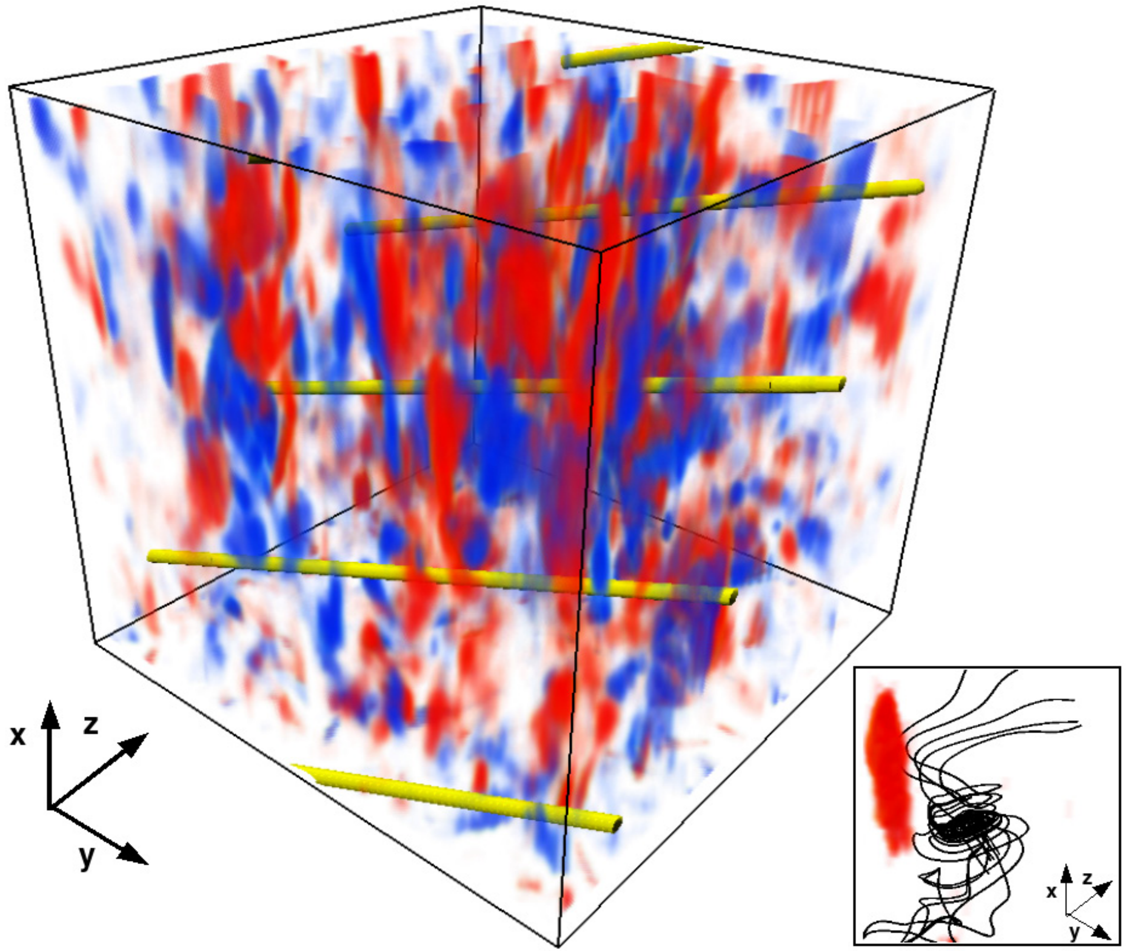


2D Geometry: k and δB both perpendicular to B_0.

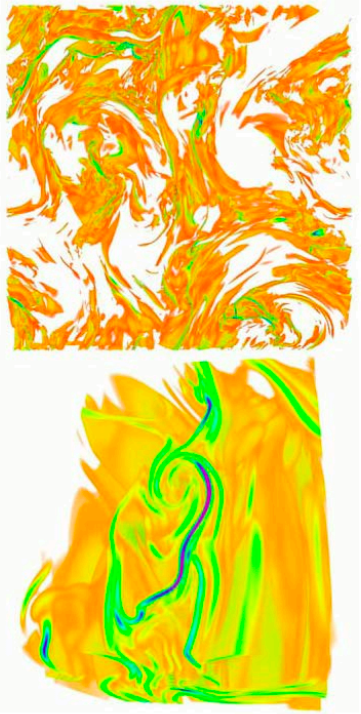
Motivations: “Structures.” Turbulence theory. Laboratory experiments.



3D MHD compressible simulation with mean B_0



current structures can form complex boundaries



Slice & blowup

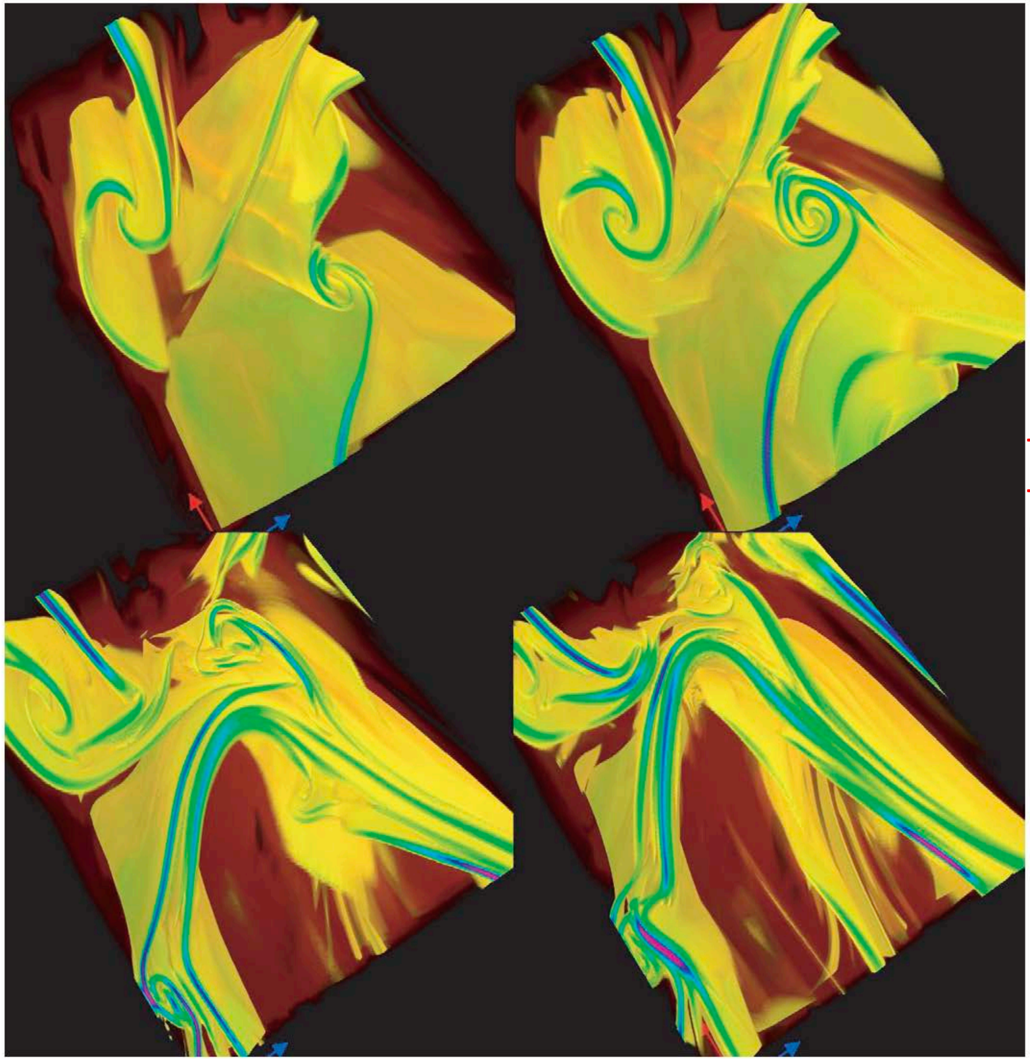


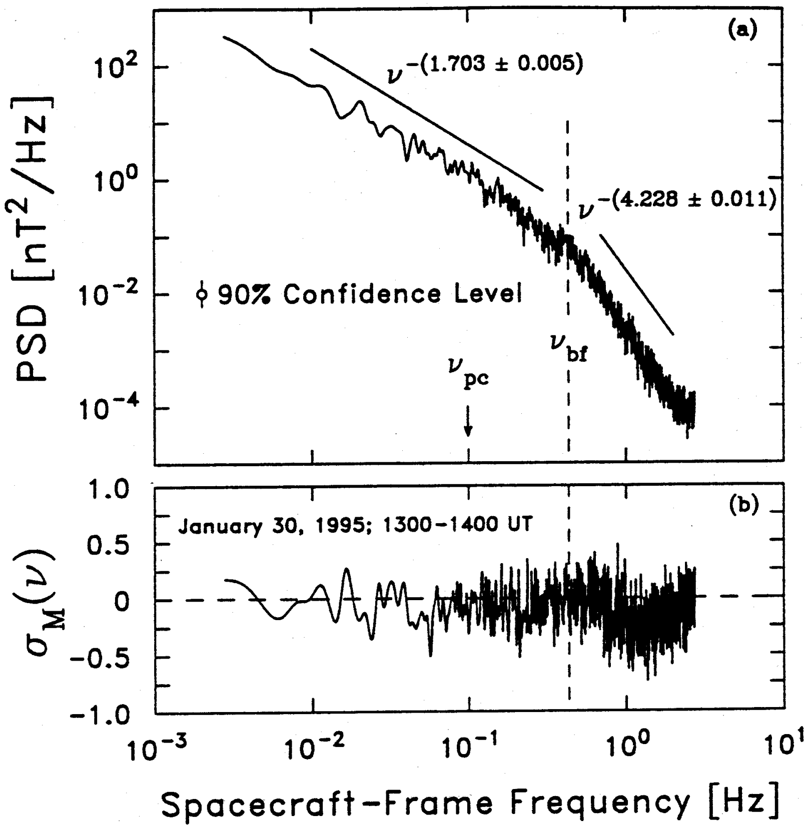
Figure 10. Same as in figure 9, but showing only the current intensity. The associated movie (available from stacks.iop.org/NJP/10/125007/mmedia) shows the temporal evolution.

1536^3 3D MHD

- Mininni et al NJP
- 2008

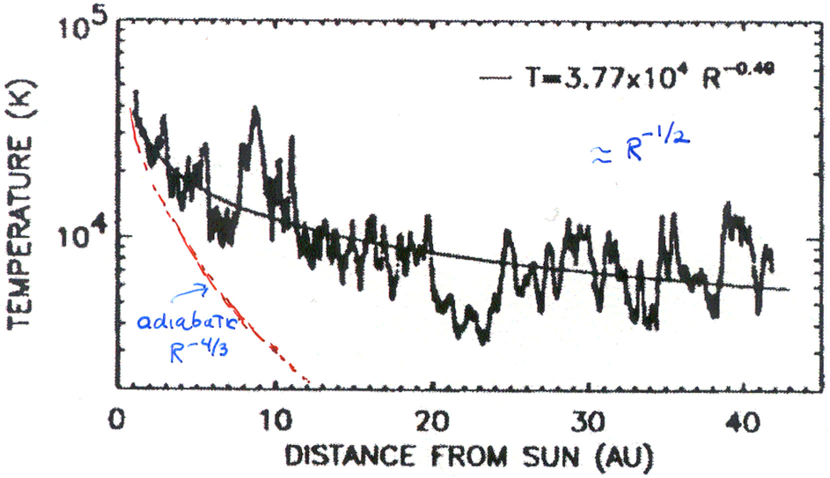
Solar Wind Dissipation

- steepening near 1 Hz (at 1 AU) -- breakpoint scales best with ion inertial scale
- Helicity signature → proton gyroresonant contributions ~50%
- Appears inconsistent with solely parallel resonances
- K_{par} and K_{perp} (to B_0) are both involved
- Consistent with dissipation in oblique current sheets



Leamon et al, 1998, 1999, 2000

Observed heating in the solar wind: can fluid turbulence and kinetic response explain this?

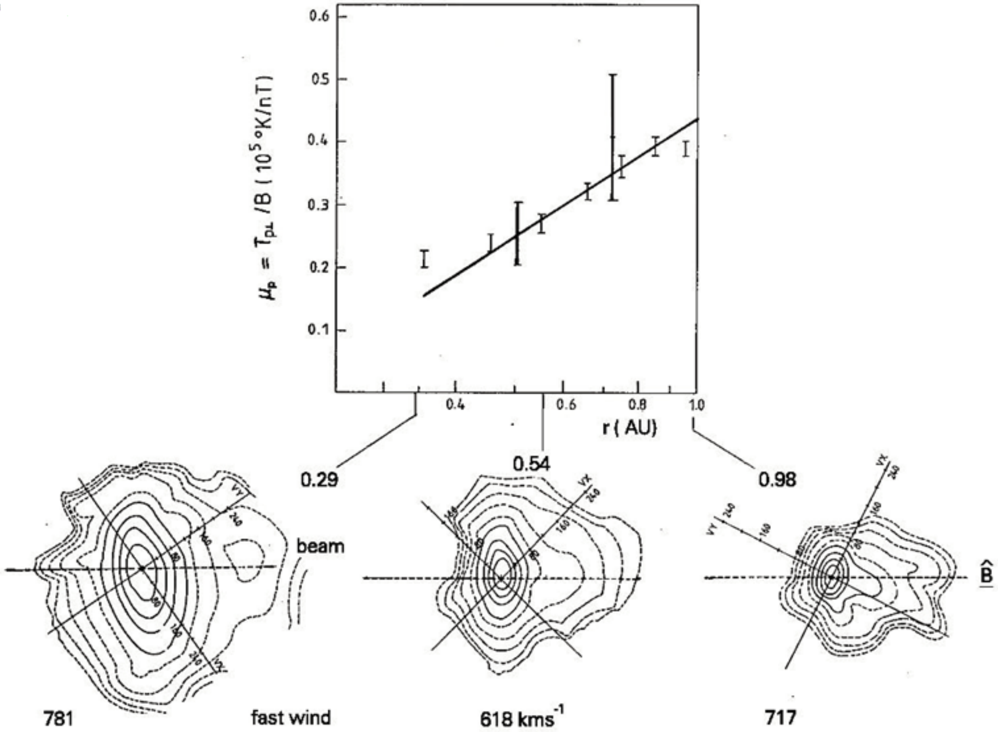


Voyager proton temperatures

Richardson et al, GRL, 1995

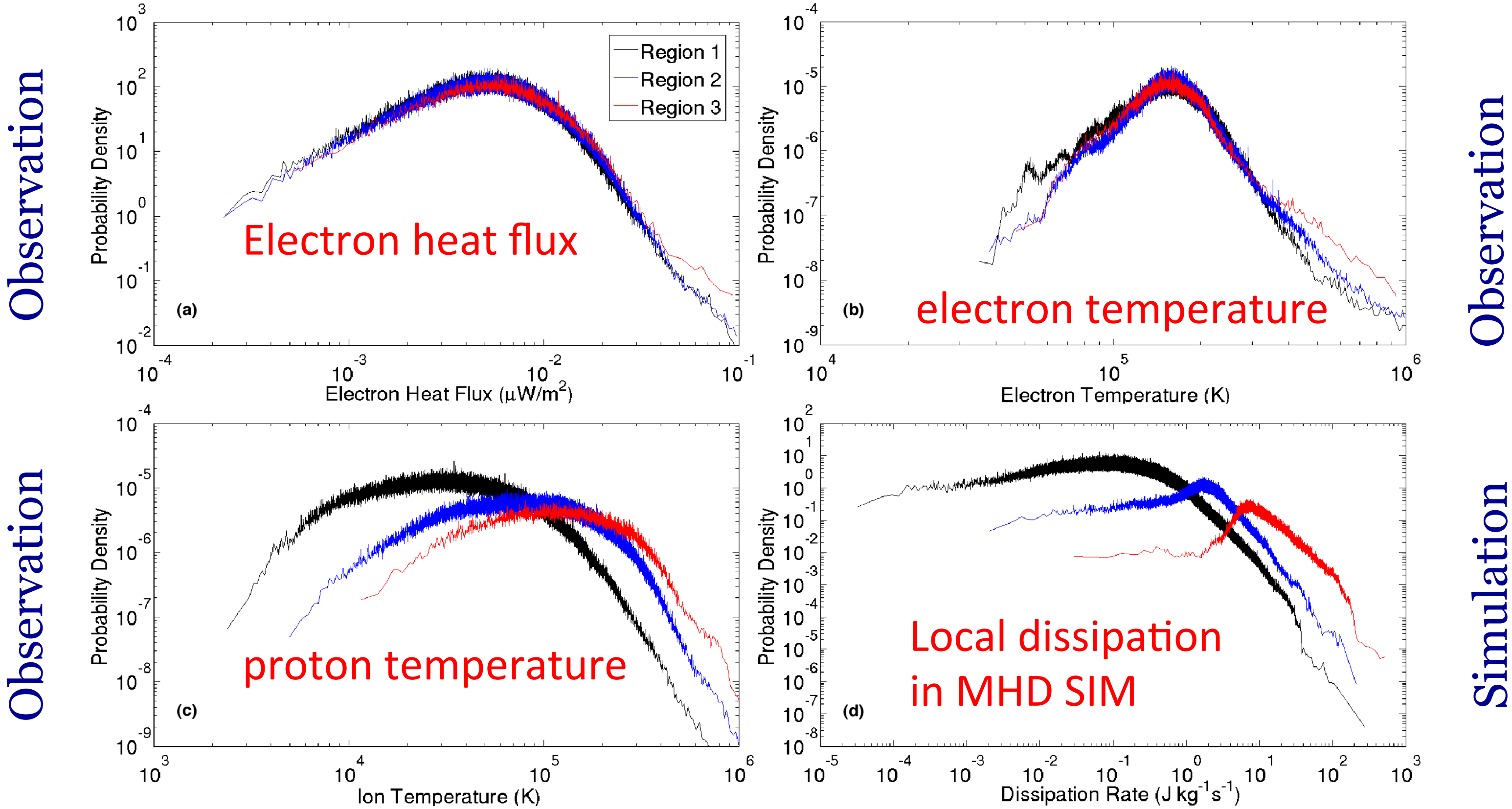
Helios protons:
magnetic moment increase
and distinctive vdfs

Marsch, 1991



Coherent structures are associated with enhanced heating

Each of the heating diagnostics is conditionally sampled so that the values associated with each of the three regions can be plotted as a separate PDF.



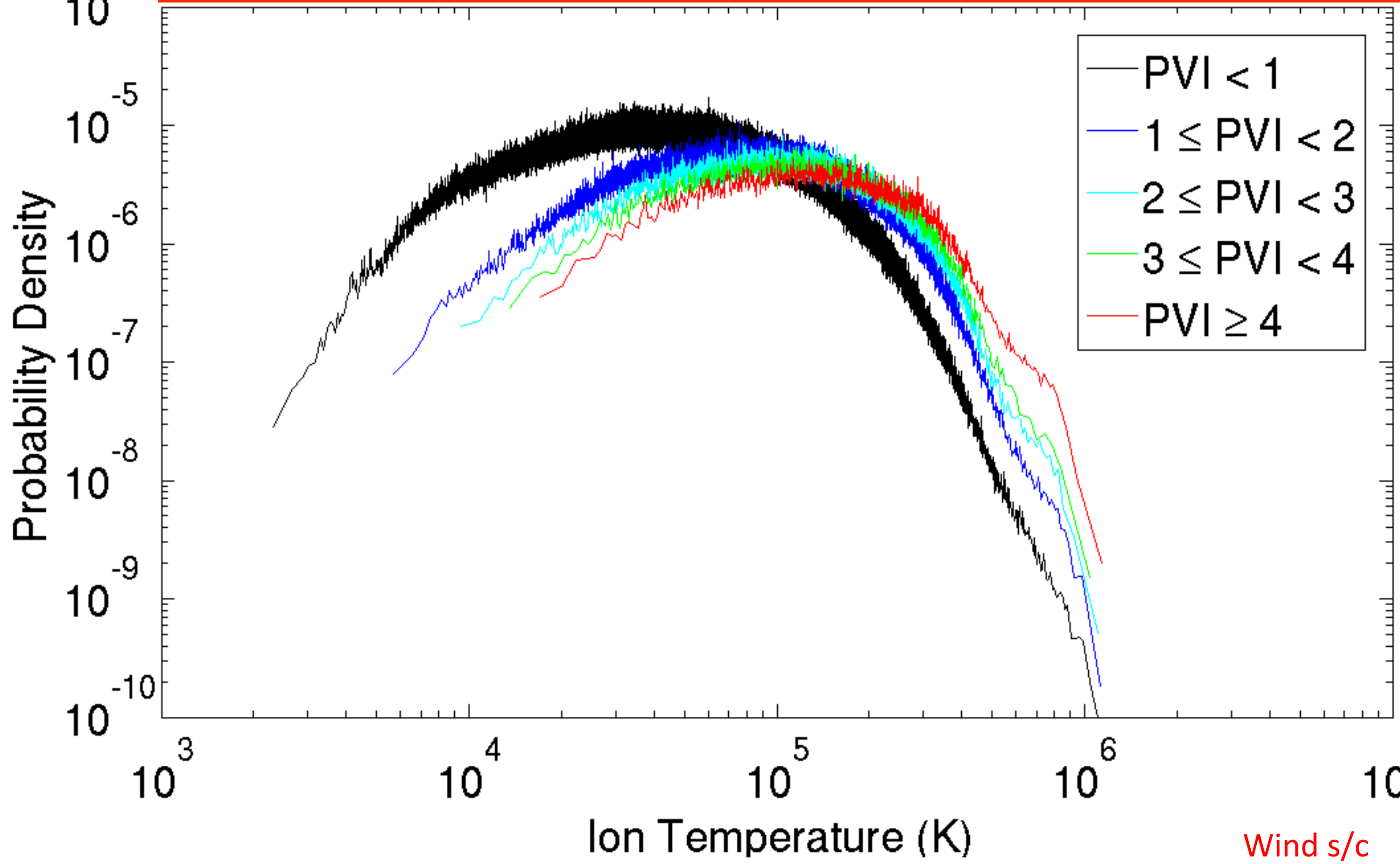
Red pdfs have PVI ≥4 ←→ Region 3

$$PVI = |\Delta \boldsymbol{B}(x, s)| / \langle |\Delta \boldsymbol{B}(x, s)|^2 \rangle^{1/2}$$

Evidence that coherent structure are sites of enhanced heating:
Solar wind proton temperature distribution conditioned on

$$PVI = |\Delta \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{s})| / \langle|\Delta \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{s})|^2\rangle^{1/2}$$

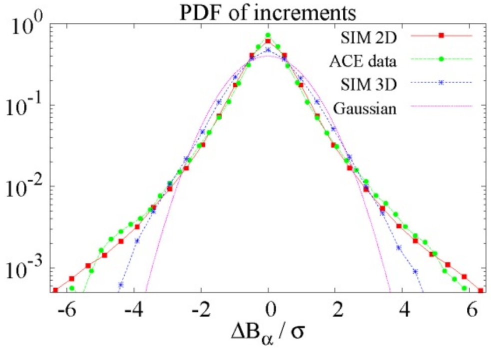
Most intense current sheets associated with largest heating

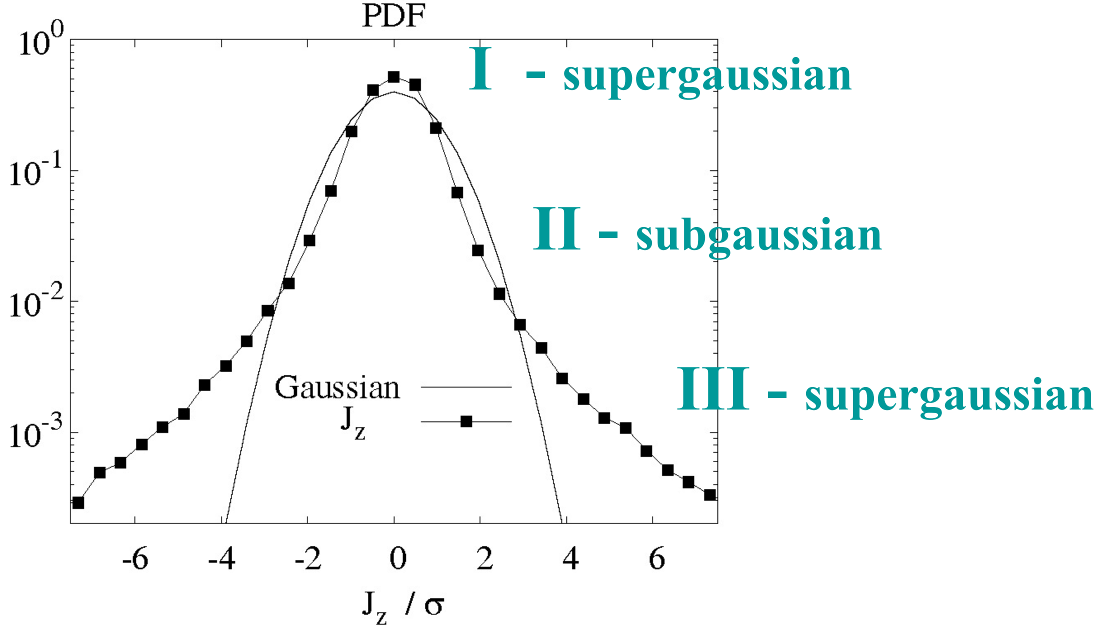


Similar effects in:
- electron Temp
- heat flux
- Points nearby structures

Wind s/c
Data: Osman et al, 2011

Intermittency and the spatial organization of current

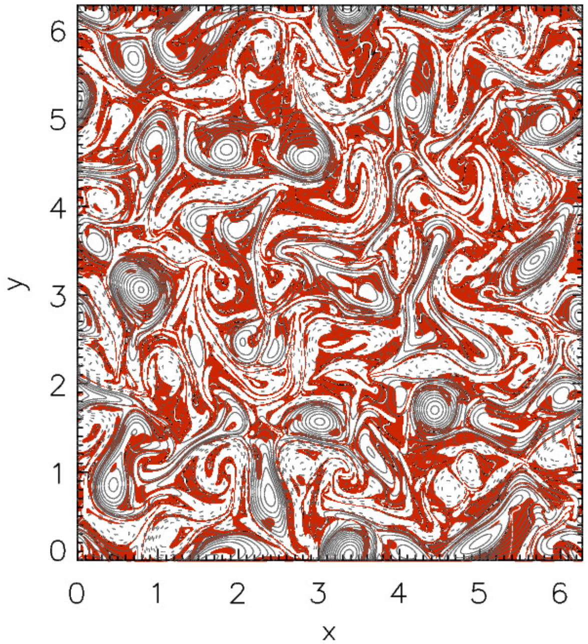


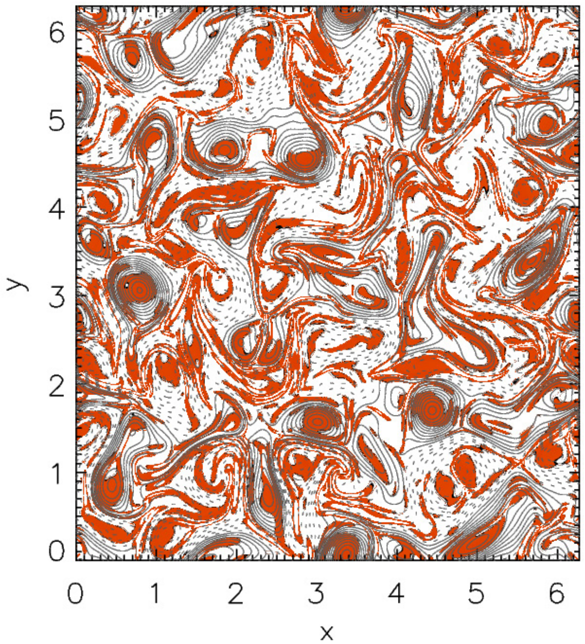


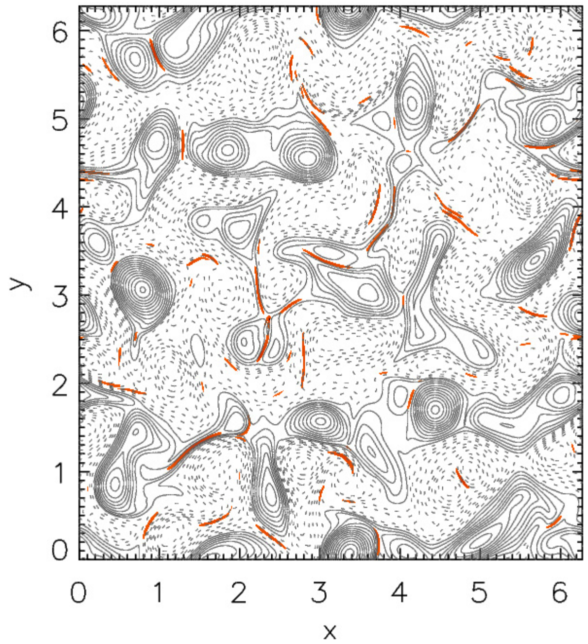
I - weak, supergaussian current lanes

II – subgaussian flux tube cores

III – supergaussian current sheets







Possible extensions to the Kolmogorov Refined Similarity Hypothesis?

- KRSH is the basis of multifractal scalings, etc, but has not been fully discussed in the context of MHD – and not for plasmas (see Merrifield et al, PoP 2005)

For plasma, we do NOT know ε, the local dissipation rate !

Elsasser increments
$\mathbf{Z}_{\pm} = \mathbf{u} \pm \mathbf{b}$
$\delta\mathbf{Z}_{\pm(s)} = \mathbf{Z}_{\pm}(x+s) - \mathbf{Z}_{\pm}(x)$
δZs = magnitude

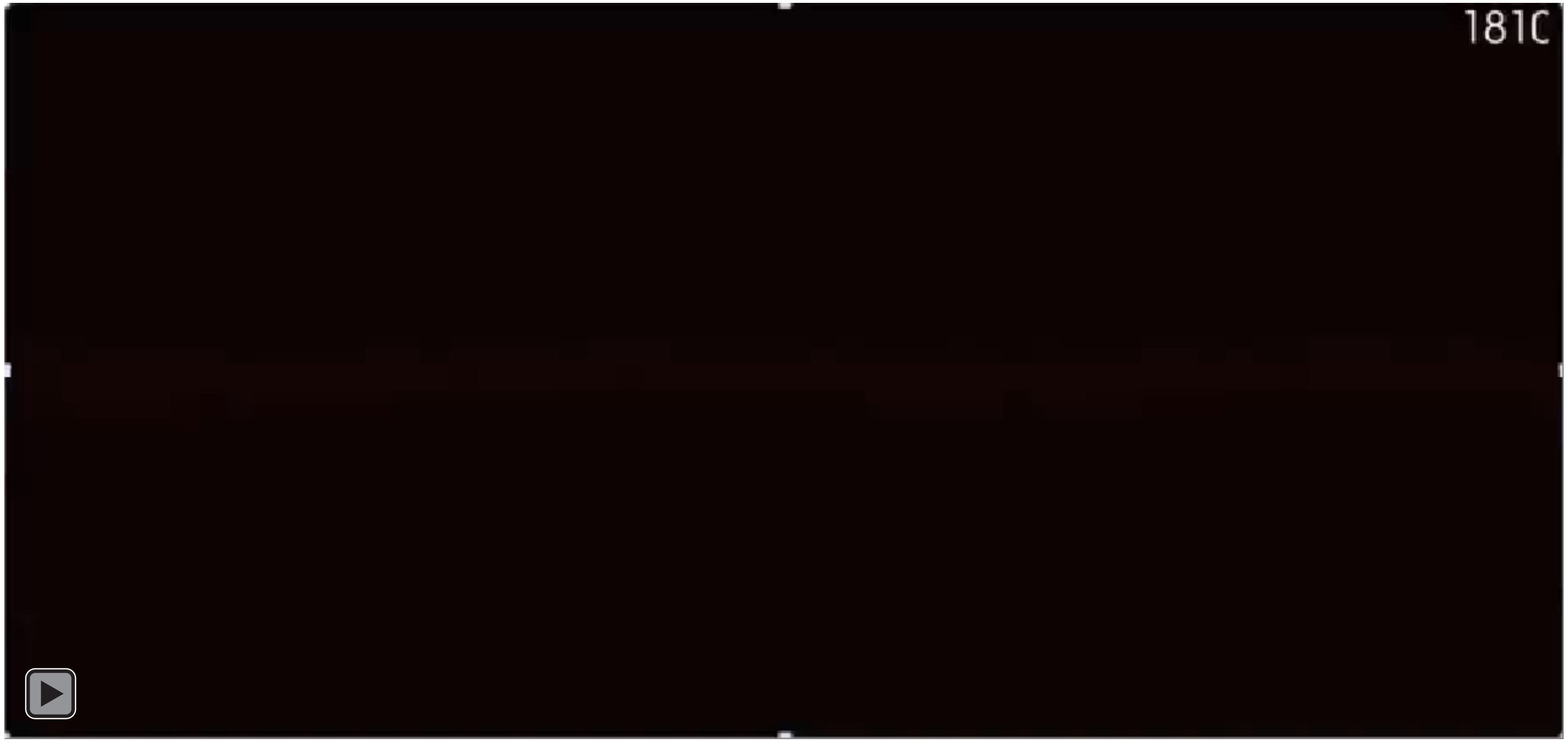
$$\frac{\delta Z_s^3}{s} \sim \epsilon_s \rightarrow T_s$$

Need Reynolds number (or a proxy) to define the conditional statistics

Proper KRSH may contain Mixed increments (+/- Elsasser Fields)

In low density space Plasma, averages of temperatures may be related to heat deposition

Strength of electric current density in shear-driven kinetic plasma (PIC) simulation (V. Roytershtein, H. Karimabadi, et al)



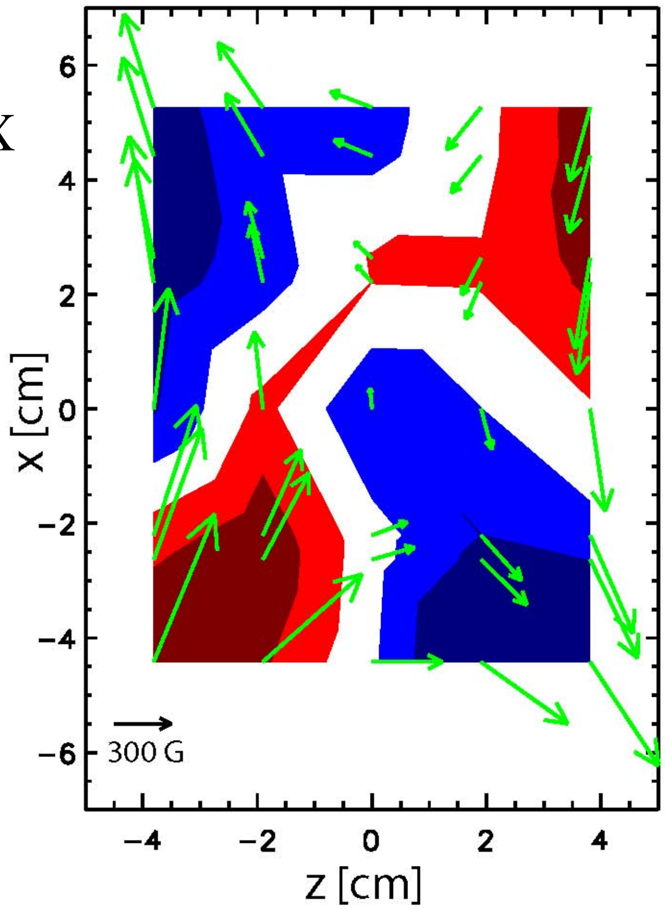
Thinnest sheets seen are comparable to electron inertial length. Sheets are clustered
At about the ion inertial length → heirarchy of coherent, dissipative structures at kinetic scales

Hall magnetic field near X-point: SSX experiment (Swarthmore, M. Brown, PI)

Magnetic field in SSX near in plane X-point (arrows)

An out of plane Quadrupole (colors)

Ion inertial scale ~ 2cm



From Matthaeus et al. GRL 2005

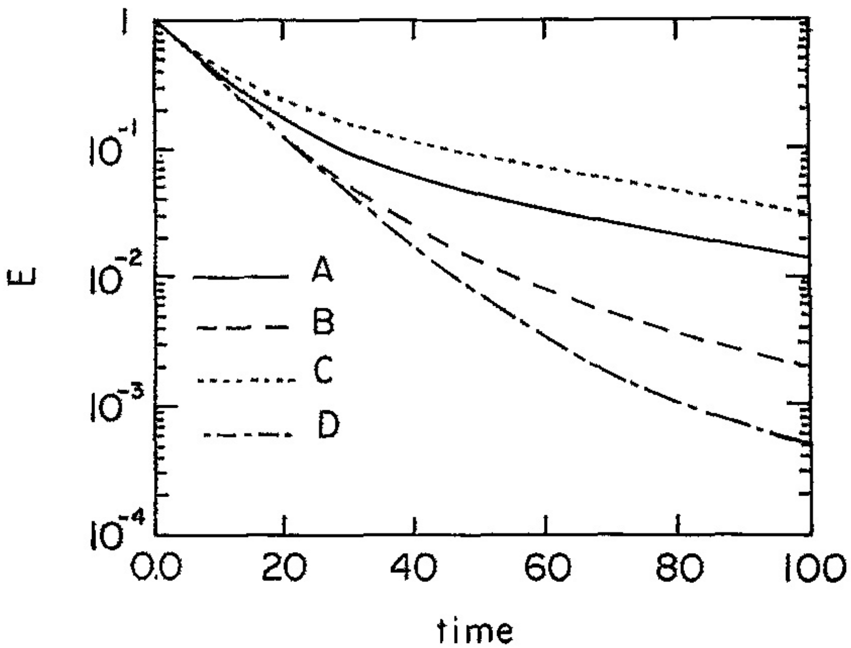
See also Ren et al, PRL 2005 (MRX Princeton)

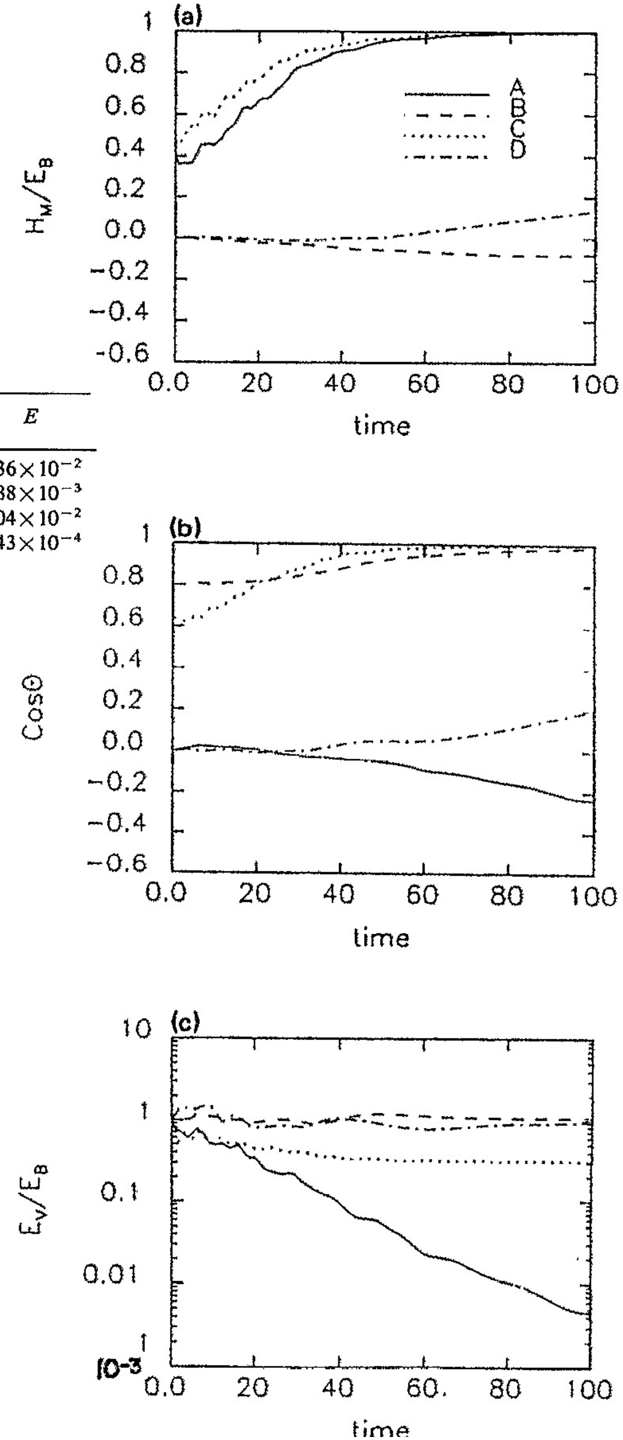
cascades

System	Ideal invariants	Direct cascade	Inverse cascade	Large structure	Small structure
3D hydro	E, (H_v)	E	none	backscatter	$\vec{\omega}$-sheets/threads
2D hydro	E, Ω	Ω	E	MAX-S eddies	ω cores/sheets
2D GCP	E, Ω_n	Ω_n	E	MAX-S potential	ρ cores/sheets
3D MHD	E, H_c, H_m	E, H_c	H_m $[H_c]$	helical **B**	**J**, $\vec{\omega}$ (???)
2D MHD	E , A , H_c	E, H_c	A	magnetic islands	j/ω-cores/sheets
3D Hall MHD	E, H_m, H_g	E, H_g	H_m, $[H_g]$	helical structures	**J**, $\vec{\omega}$, **v** (???)

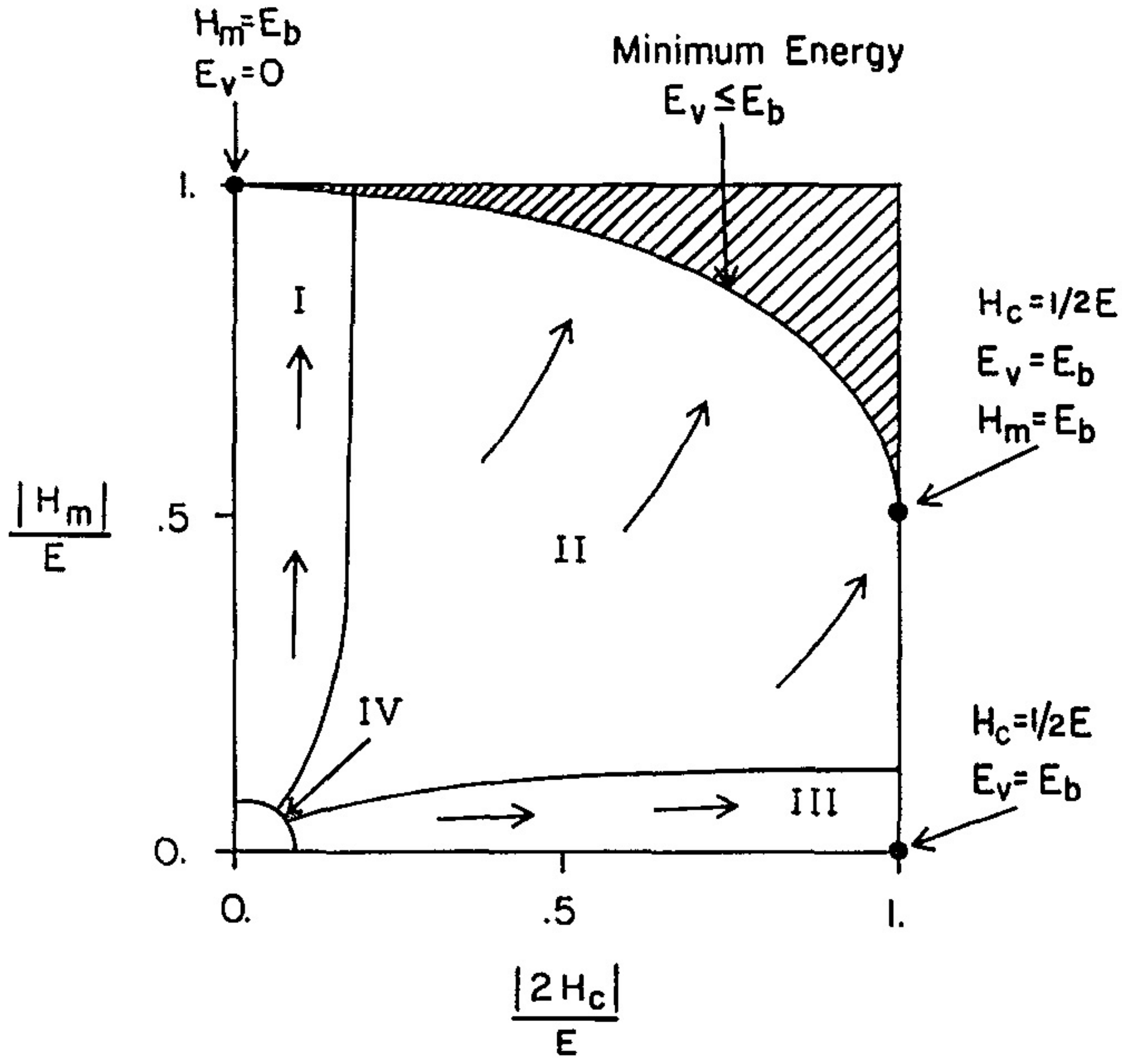
Sample 3D MHD relaxation solutions

Run	Initial value E_v/E_b	Initial value $2H_c/E$	Initial value H_m/E	t_{final}	Value at t_{final} E_v/E_b	Value at t_{final} $2H_c/E$	Value at t_{final} H_m/E	Value at t_{final} E
A	1.0	0.0	0.2	100.0	4.54×10^{-3}	-3.17×10^{-2}	0.992	1.36×10^{-2}
B	1.0	0.8	0.0	100.0	1.07	0.983	3.42×10^{-2}	1.88×10^{-3}
C	0.5	0.6	0.2	100.0	0.317	0.852	0.757	3.04×10^{-2}
D	1.0	0.0	0.0	100.0	0.945	0.2	7.16×10^{-2}	4.43×10^{-4}





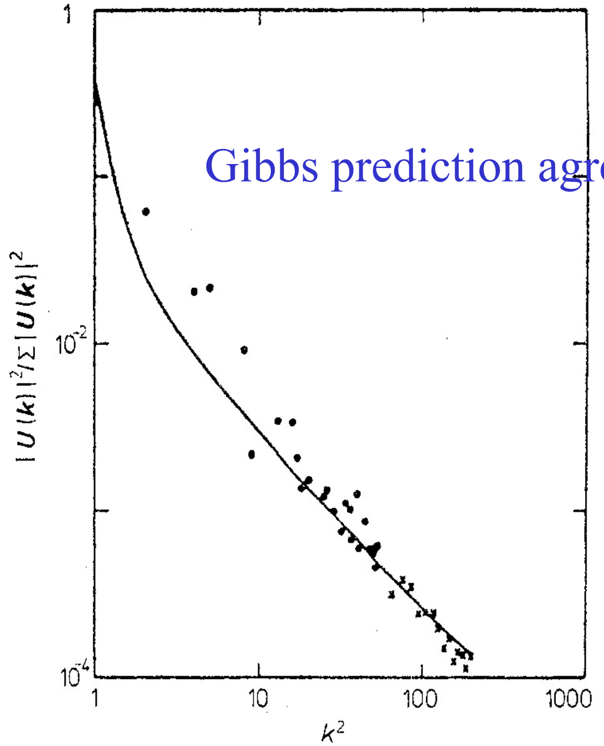
Cartoon of trajectories: 3D MHD



Gibbs statistical mechanics for Galerkin approximations of turbulence

2D hydro Gibbs ensemble

From Seyler, 1975

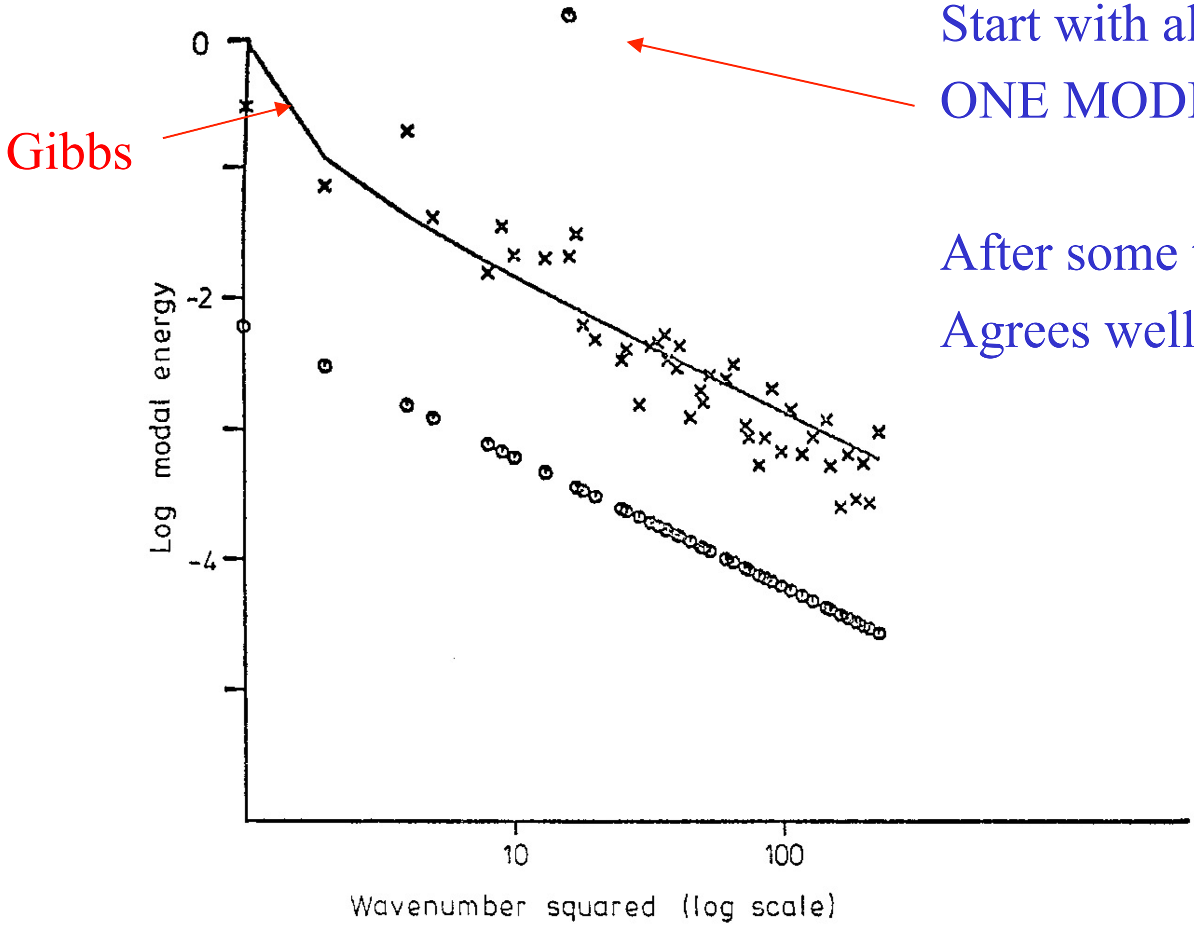


Gibbs prediction agrees well with time averaged spectrum

Streamlines show “self-organization”,
that is maximization of entropy
gives rise to large scale spatial structure

Time →

2D hydro Gibbs is a robust result



Start with almost all the energy in ONE MODE, plus a little broad band noise.

After some time, the averaged spectrum Agrees well with the Gibbs prediction

Gibbs ensemble average spectra for 3D incompressible (Galerkin) ideal MHD

$$\langle E_v(\mathbf{k})\rangle = \frac{2}{\beta}\left(1+\frac{\gamma^2}{4}\frac{(\beta^2-\frac{1}{4}\gamma^2)k^2}{d(k)}\right),$$

$$\langle E_b(\mathbf{k})\rangle = 2\beta(\beta^2-\tfrac{1}{4}\gamma^2)k^2/d(k),$$

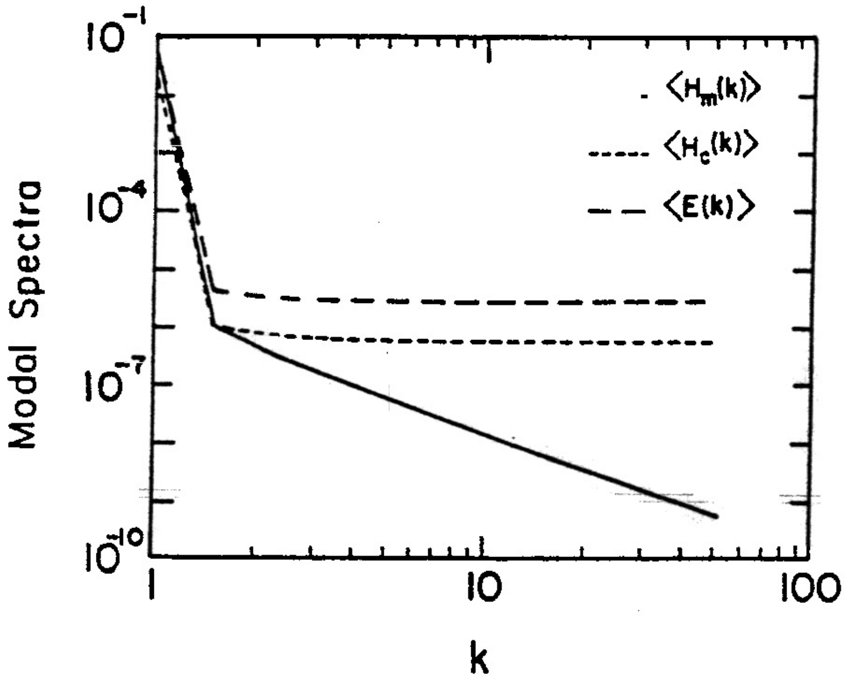
$$\langle E(\mathbf{k})\rangle = 2\beta\,[2(\beta^2-\tfrac{1}{4}\gamma^2)k^2-\theta^2]/d(k),$$

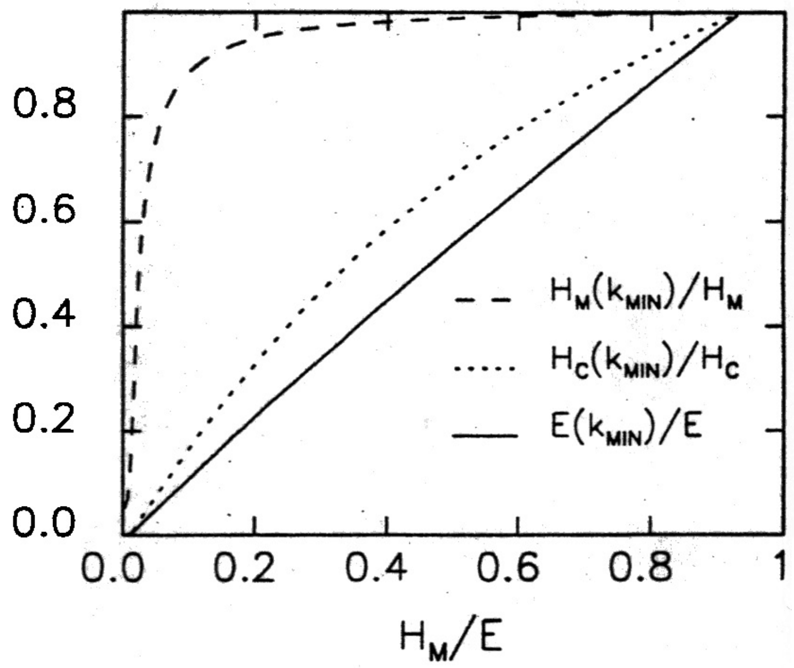
$$\langle H_c(\mathbf{k})\rangle = -\gamma(\beta^2-\tfrac{1}{4}\gamma^2)k^2/d(k),$$

$$\langle H_m(\mathbf{k})\rangle = -2\theta\beta^2/d(k),$$

$$d(k) = (\beta^2-\tfrac{1}{4}\gamma^2)^2k^2-\theta^2\beta^2.$$

Condensation in 3D MHD Gibbs ensemble





Sample 3D MHD relaxation solutions

Run	Initial value E_v/E_b	Initial value $2H_c/E$	Initial value H_m/E	$t_{\rm final}$	Value at $t_{\rm final}$ E_v/E_b	Value at $t_{\rm final}$ $2H_c/E$	Value at $t_{\rm final}$ H_m/E	Value at $t_{\rm final}$ E
A	1.0	0.0	0.2	100.0	4.54×10^{-3}	-3.17×10^{-2}	0.992	1.36×10^{-2}
B	1.0	0.8	0.0	100.0	1.07	0.983	3.42×10^{-2}	1.88×10^{-3}
C	0.5	0.6	0.2	100.0	0.317	0.852	0.757	3.04×10^{-2}
D	1.0	0.0	0.0	100.0	0.945	0.2	7.16×10^{-2}	4.43×10^{-4}