Finite Horizontal Symmetries for Neutrino Mixing

(group theory of mixing)

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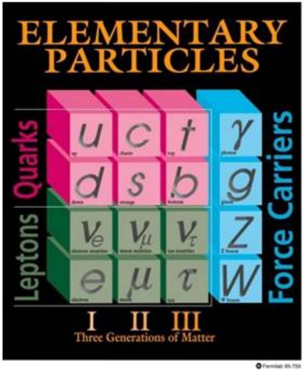
CONTENTS

(arXiv: 1208.5527)

I

Introduction



Is there a finite horizontal symmetry consistent with the neutrino mixing data?

$\theta_{13} = 0, \;\; \theta_{23} = 45^\circ$

S_4

$$\frac{1}{\sqrt{6}} \begin{bmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}$$

A_4

$$\frac{1}{\sqrt{6}} \begin{bmatrix} \times & \sqrt{2} & \times \\ \times & \sqrt{2} & \times \\ \times & \sqrt{2} & \times \end{bmatrix}$$

$\theta_{13} = 0, \; \theta_{23} = 45^\circ$ S_4 $\frac{1}{\sqrt{6}} \begin{bmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}$

(full symmetry/mixing)

A_4 $\frac{1}{\sqrt{6}} \begin{bmatrix} \times & \sqrt{2} & \times \\ \times & \sqrt{2} & \times \\ \times & \sqrt{2} & \times \end{bmatrix}$

(partial symmetry/mixing)

$\theta_{13} = 0, \;\; \theta_{23} = 45^\circ$ $\quad S_4 \quad$ $\frac{1}{\sqrt{6}} \begin{bmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}$

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///

$\theta_{13} \simeq 9^\circ, \;\; \theta_{23} < 45^\circ$

$\theta_{13} = 0, \ \theta_{23} = 45^\circ$ $\quad S_4$ $\quad \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}$

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(partial symmetry/mixing)

//

$\theta_{13} \simeq 9^\circ, \ \theta_{23} < 45^\circ$ $\quad$ none $\quad$ ($SU(3)$ subgp of order < 512 w/ 3dIR)

(full)

(partial)

$\Delta(150)$ $\quad$ 3rd column $\quad \sin^2 2\theta_{13} = 0.11$

$\sin^2 2\theta_{23} = 0.94$

$\theta_{13} = 0,\ \theta_{23} = 45^\circ$ S_4 $\frac{1}{\sqrt{6}}\begin{bmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}$

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$\theta_{13} \simeq 9^\circ,\ \theta_{23} < 45^\circ$ none (full) ($SU(3)$ subgp of order < 512 w/ 3dIR)

(partial)

$\Delta(150)$ $3^{\rm rd}$ column $\sin^2 2\theta_{13} = 0.11$

$\sin^2 2\theta_{23} = 0.94$

S_4 1^{st} or 2^{nd} column

A_4 2^{nd} column

$\cdots$ 1^{st} or 2^{nd} column

Main Assumption

(Group Theory of Mixing)

1. Known: every mixing matrix carries a $[Z_n, Z_2 \times Z_2]$ symmetry.

assume either

2a. $[Z_n, Z_2 \times Z_2]$ is a residual symmetry of the horizontal symmetry (full symmetry/mixing),

or

2b. $[Z_n, Z_2]$ is a residual symmetry of the horizontal symmetry (partial symmetry/mixing).

$[Z_n,\ Z_2 \times Z_2]$ Symmetry of any Mixing Matrix U

Mixing Matrix $U = \begin{bmatrix} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

- $\overline{M}_\nu$ Majorana & symmetrical
- $U^T \overline{M}_\nu U =$ diagonal
- $\overline{M}_e = M_e^\dagger M_e$ diagonal
- L-leptons only

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- $$G_1 = +u_1 u_1^\dagger - u_2 u_2^\dagger - u_3 u_3^\dagger$$ $$G_2 = -u_1 u_1^\dagger + u_2 u_2^\dagger - u_3 u_3^\dagger$$ $$G_3 = -u_1 u_1^\dagger - u_2 u_2^\dagger + u_3 u_3^\dagger$$
- $G^T \overline{M}_\nu G = \overline{M}_\nu$

  $GG^\dagger = 1$

  u_i: invariant eigenvector of G_i

  ('mixing vectors')

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- $$\begin{aligned} G_1 &= +u_1 u_1^\dagger - u_2 u_2^\dagger - u_3 u_3^\dagger \\ G_2 &= -u_1 u_1^\dagger + u_2 u_2^\dagger - u_3 u_3^\dagger \\ G_3 &= -u_1 u_1^\dagger - u_2 u_2^\dagger + u_3 u_3^\dagger \end{aligned}$$

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- $G_i^2 = 1, \quad G_i G_j = G_k$

  $Z_2 \times Z_2$ Symmetry

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- $G_1 = +u_1 u_1^\dagger - u_2 u_2^\dagger - u_3 u_3^\dagger$
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- $G^T \overline{M}_\nu G = \overline{M}_\nu$
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  u_i: invariant eigenvector of G_i
  ('mixing vectors')

- $G_i^2 = 1, \quad G_i G_j = G_k$
  $Z_2 \times Z_2$ Symmetry

- $F^\dagger \overline{M}_e F = \overline{M}_e, \quad FF^\dagger = 1$
- F diagonal $\quad Z_n$ Symmetry

Unbroken Symmetry

$$\boxed{\mathcal{G} \supset \quad \langle F, G_i \rangle}$$

one i: Partial $\quad (|\mathcal{G}| = 2n)$

two (three) i: Full $\quad (|\mathcal{G}| = 4n)$

Unbroken Symmetry

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TBM Illustration

$$U = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}$$

$$G_1 \quad G_2 \quad G_3$$

$$F = \mathrm{diag}(1, \omega_3, \omega_3^2), \quad (\omega_3 = e^{2\pi i/3})$$

$$S_4 = \langle F, (G_1), G_2, G_3 \rangle = \langle F, G_1 \rangle$$

Full mixing group (fits 3 columns)

$$A_4 = \langle F, G_2 \rangle$$

Partial mixing group (fits one column)

(the other 2 columns fitted)

Unbroken Symmetry

$$\boxed{\mathcal{G} \supset \quad \langle F, G_i \rangle}$$

one i: Partial $\quad (|\mathcal{G}| = 2n)$

two (three) i: Full $\quad (|\mathcal{G}| = 4n)$

- 'Unbroken' symmetry group $\mathcal{G}$
- G_i: order-2 element
- F: order > 2, non-degenerate
- 3-dim unitary representation
- u_i: invariant eigenvector of G_i in F-diagonal basis
- $U = \begin{bmatrix} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$
- GAP

TBM Illustration

$$U = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & \sqrt{3} \\ -1 & \sqrt{2} & -\sqrt{3} \end{bmatrix}$$

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(the other 2 columns fitted)

Group Theory and Dynamical Theory

Group Theory

- no need for valons/alignment
  nor Clebsch-Gordan coefficients
- no need to know R-fermions
- Similarity and difference with the texture-zero approach

Group Theory and Dynamical Theory

Group Theory

- no need for valons/alignment nor Clebsch-Gordan coefficients
- no need to know R-fermions
- Similarity and difference with the texture-zero approach

Dynamical Theory

(spontaeously broken)

- to be consistent with Group Theory, the valon alignments $\langle \phi \rangle$ must be an invariant eigenvector

  of F in the e-sector

  of G in the ν-sector

- each of these invariant eigenvectors is naturally a solution of a $\mathcal{G}$-invariant valon potential.
  details

- when F, G are both involved:
  - like spin-spin coupling. To avoid that, arrange
  - additional symmetry to forbid it
  - or, assume weak coupling

II

Experimental Data on Mixing

θ_{13}

- Daya Bay:
  $\sin^2 2\theta_{13} = 0.089 \pm 0.010 \pm 0.005$
- Double Chooz:
  $\sin^2 2\theta_{13} = 0.109 \pm 0.030 \pm 0.025$
- RENO:
  $\sin^2 2\theta_{13} = 0.113 \pm 0.013 \pm 0.019$
- T2K: (N)/(I, Prelim)
  $\sin^2 2\theta_{13} = 0.104 + 0.060 - 0.045$
  $\sin^2 2\theta_{13} = 0.128 + 0.070 - 0.055$

θ_{23}

- MINOS: $(\nu)/(\overline{\nu})$
  $\sin^2 2\theta_{23} = 0.96 \pm 0.04$
  $\sin^2 2\theta_{23} = 0.97 \pm 0.03/0.08$

.

PMNS Mixing Matrix U

$$
\begin{array}{cccc}
 & 211 & 111 & 011 \\
|U_{TBM}| = & \begin{bmatrix} .816 & .577 & .000 \\ .408 & .577 & .707 \\ .408 & .577 & .707 \end{bmatrix}
\end{array}
$$

$$|U_{aN}| = \begin{bmatrix} .814 \pm .010 & .558 \pm .014 & .161 \pm .011 \\ .327 \pm^{.160}_{.036} & .645 \pm^{.113}_{.035} & .691 \pm .046 \\ .480 \pm^{.160}_{.026} & .522 \pm^{.118}_{.044} & .705 \pm .045 \end{bmatrix}$$

$$|U_{aI}| = \begin{bmatrix} .813 \pm .010 & .558 \pm .014 & .164 \pm .011 \\ .485 \pm .022 & .500 \pm .041 & .718 \pm .041 \\ .322 \pm .032 & .663 \pm .031 & .676 \pm .043 \end{bmatrix}$$

$$|U_{bN}| = \begin{bmatrix} .822 \pm .010 & .547 \pm .015 & .157 \pm .010 \\ .354 \pm^{.098}_{.019} & .698 \pm^{.060}_{.015} & .623 \pm .022 \\ .446 \pm^{.099}_{.015} & .462 \pm^{.080}_{.022} & .766 \pm .018 \end{bmatrix}$$

$$|U_{bI}| = \begin{bmatrix} .822 \pm .010 & .547 \pm .015 & .157 \pm .010 \\ .348 \pm^{.096}_{.020} & .694 \pm^{.058}_{.017} & .631 \pm .025 \\ .451 \pm^{.093}_{.016} & .469 \pm^{.078}_{.025} & .760 \pm .021 \end{bmatrix}$$

- a=Forero, Tortola, & Valle, arXiv 1205.4018
- b=Fogli, Lisi, Marrone, Montanino, Palazzo, & Rotunni, arXiv1205.5254
- N=normal hierarchy; I=inverse hierarchy

χ^2 of Fits of TBM

$$\chi_j^2 \doteq \sum_{i=1}^{3}(|U_{\mathcal{G}}|_{i,j} - |U_{global}|_{i,j})^2/2\sigma_{i,j}^2 \ , \quad (j = 1, 2, 3)$$

$$|U_{TBM}| = \begin{bmatrix} .816 & .577 & .000 \\ .408 & .577 & .707 \\ .408 & .577 & .707 \end{bmatrix}$$

		aN	aI	bN	bI
1	$(2,1,1)^T$	4.12	9.82	3.65	4.00
2	$(1,1,1)^T$	2.85	6.62	34.7	26.1
3	$(0,1,1)^T$	110	119	126	122

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● $\mathcal{G} \supset S_4$ ○ $\mathcal{G} \supset A_4$

III

Finite Subgroups of $U(3)$

A list of Finite Subgroups of $U(3)$

P.O. Ludl, arXiv:1006.1479

- Order($\mathcal{G}$) < 512, with 3-dim irreducible representation, direct product groups excluded
- $SU(3)$ subgroups: 59 $\Longleftarrow$
- Central extensions not included, but they produce nothing new

  e.g., $T' = 2.A_4$, $O' = 2.S_4$, $I' = 2.A_5$

A	B	C
$[12,3]$	A_4, T	$\circ$
$[21,1]$	T_7	$\times$
$[24,12]$	$S_4, O, \Delta(24)$	$\bullet$
$[27,3]$	$\Delta(27)$	$\times$
$[39,1]$	T_{13}	$\times$
$[48,3]$	$\Delta(48)$	$\circ$
$[54,8]$	$\Delta(54)$	p
$[57,1]$	T_{19}	$\times$
$[60,5]$	$A_5, I, \Sigma(60)$	$\circ$
$[75,2]$	$\Delta(75)$	$\times$
$[81,9]$		$\times$
$[84,11]$		$\circ$
$[93,1]$	T_{31}	$\times$
$[96,64]$	$\Delta(96)$	$\bullet$
$[108,15]$	$\Sigma(36\varphi)$	p
$[108,22]$	$\Delta(108)$	$\circ$
$[111,1]$	T_{37}	$\times$
$[129,1]$	T_{43}	$\times$
$[147,1]$	T_{49}	$\times$
$[147,5]$	$\Delta(147)$	$\times$

$\circ \supset A_4$ $\bullet \supset S_4$ $\times$ *odd order* p *only partial sym* — *no fit*

A	B	C	mix vec	χ^2	fit	col
[12, 3]	A_4, T	∘	[.577, .577, .577]	2.85	aN	2
[21, 1]	T_7	×				
[24, 12]	S_4, O, $\Delta(24)$	•	[.816, .408, .408]	3.65	bN	1
[27, 3]	$\Delta(27)$	×				
[39, 1]	T_{13}	×				
[48, 3]	$\Delta(48)$	∘				
[54, 8]	$\Delta(54)$	p	–	–	–	–
[57, 1]	T_{19}	×				
[60, 5]	A_5, I, $\Sigma(60)$	∘				
[75, 2]	$\Delta(75)$	×				
[81, 9]		×				
[84, 11]		∘				
[93, 1]	T_{31}	×				
[96, 64]	$\Delta(96)$	•				
[108, 15]	$\Sigma(36\varphi)$	p	–	–	–	–
[108, 22]	$\Delta(108)$	∘				
[111, 1]	T_{37}	×				
[129, 1]	T_{43}	×				
[147, 1]	T_{49}	×				
[147, 5]	$\Delta(147)$	×				

$\circ \supset A_4$ $\quad \bullet \supset S_4$ $\quad \times$ *odd order* $\quad p$ *only partial sym* $\quad$ – *no fit*

A	B	C	mix vec	χ^2	fit	col
[150, 5]	$\Delta(150)$	p	[.812, .332, .480]	.018	aN	1
			[.812, .480, .332]	.086	al	1
			[.170, .607, .777]	1.25	bN	3
[156, 14]		$\circ$				
[162, 14]		p	—	—	—	—
[168, 42]	$\Sigma(168)$, $PSL(3,2)$	$\bullet$	[.815, .363, .452]	.267	bN	1
			[.815, .452, .363]	.269	bl	1
[183, 1]	T_{61}	$\times$				
[189, 8]		$\times$				
[192, 3]	$\Delta(192)$	$\circ$				
[201, 1]	T_{67}	$\times$				
[216, 88]	$\Sigma(72\varphi)$	p	—	—	—	—
[216, 95]	$\Delta(216)$	$\bullet$				
[219, 1]	T_{73}	$\times$				
[228, 11]		$\circ$				
[237, 1]	T_{79}	$\times$				
[243, 26]	$\Delta(243)$	$\times$				
[273, 3]	T_{91}	$\times$				
[273, 4]	T'_{91}	$\times$				
[291, 1]	T_{97}	$\times$				
[294, 7]	$\Delta(294)$	p	[.814, 460, .351]	1.16	al	1
			[.814, .351, .460]	.312	bl	1
			[.707, .500, .500]	4.95	bl	2
			[.122, .638, .760]	5.80	bl	3

A	B	C	$mix\ vec$	χ^2	fit	col
[150, 5]	$\Delta(150)$	p	[.812, .332, .480]	.018	aN	1
			[.812, .480, .332]	.086	aI	1
			[.170, .607, .777]	1.25	bN	3
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[183, 1]	T_{61}	$\times$				
[189, 8]		$\times$				
[192, 3]	$\Delta(192)$	$\circ$				
[201, 1]	T_{67}	$\times$				
[216, 88]	$\Sigma(72\varphi)$	p	—	—	—	—
[216, 95]	$\Delta(216)$	$\bullet$				
[219, 1]	T_{73}	$\times$				
[228, 11]		$\circ$				
[237, 1]	T_{79}	$\times$				
[243, 26]	$\Delta(243)$	$\times$				
[273, 3]	T_{91}	$\times$				
[273, 4]	T'_{91}	$\times$				
[291, 1]	T_{97}	$\times$				
[294, 7]	$\Delta(294)$	p	[.814, 460, .351]	1.16	aI	1
			[.814, .351, .460]	.312	bI	1
			[.707, .500, .500]	4.95	bI	2
			[.122, .638, .760]	5.80	bI	3

$\sin^2 2\theta_{13} = 0.11$

$\sin^2 2\theta_{23} = 0.94$

$\sin^2 2\theta_{13} = 0.06$

$\sin^2 2\theta_{23} = 0.97$

Neutrino 2012 (June, Kyoto)

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θ_{23}

- MINOS: $(\nu)/(\overline{\nu})$

  $\sin^2 2\theta_{23} = 0.96 \pm 0.04$
  $\sin^2 2\theta_{23} = 0.97 \pm 0.03/0.08$

.

A	B	C	$mix\ vec$	χ^2	fit	col
[300, 43]	$\Delta(300)$	○				
[309, 1]	T_{103}	×				
[324, 50]		○				
[327, 1]	T_{109}	×				
[336, 67]		○				
[351, 8]		×				
[363, 2]	$\Delta(363)$	×				
[372, 11]		○				
[381, 1]	T_{127}	×				
[384, 567]	$\Delta(384)$	●	[.810, .312, .497]	.188	aN	1
			[.810, .497, .312]	.287	aI	1
[399, 3]	T_{133}	×				
[399, 4]	T'_{193}	×				
[417, 1]	T_{139}	×				
[432, 103]	$\Delta(432)$	○				
[444, 14]		○				
[453, 1]	T_{151}	×				
[471, 1]		×				
[486, 61]	$\Delta(486)$	p	[.804, 279, .525]	1.41	aN	1
[489, 1]	T_{163}	×				
[507, 1]	T_{169}	×				
[507, 5]	$\Delta(507)$	×				

Mass Matrices of $\Delta(150)$

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} = F^\dagger \overline{M_e} F \qquad \overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = G^T \overline{M_\nu} G$$

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \overline{M_e} = \begin{bmatrix} \alpha & \beta & \beta^* \\ \beta^* & \alpha & \beta \\ \beta & \beta^* & \alpha \end{bmatrix}$$

$$G = -\begin{bmatrix} 0 & \omega_5^3 & 0 \\ \omega_5^2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \overline{M_\nu} = \begin{bmatrix} a & b & e\omega_5^2 \\ b & a\omega_5 & e \\ e\omega_5^2 & e & f \end{bmatrix}$$

Summary

- No full-symmetry group found.

- Several partial-symmetry groups fit the first two columns approximately, but these fits are tricky and uncertain because CP phase is unknown.

- Only the group $\Delta(150) = [150, 5]$ fits the third column, predicting

$$\sin^2 2\theta_{13} = 0.11 \quad \text{and} \quad \sin^2 2\theta_{23} = 0.94$$

  but the 3-dim irreducible representation is complex.

- Incidentally, none contain D_{14}.

Mass Matrices of

A_4, S_4, $\Delta(150)$

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} = F^\dagger \overline{M_e} F \qquad \overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = G^T \overline{M_\nu} G$$

$$S_4 = \langle F, G_2, G_3 \rangle = \langle F, G_1 \rangle \qquad A_4 = \langle F, G_2, G_3' \rangle$$

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} = F^\dagger \overline{M_e} F \qquad \overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = G^T \overline{M_\nu} G$$

$$S_4 = \langle F, G_2, G_3 \rangle = \langle F, G_1 \rangle \qquad A_4 = \langle F, G_2, G_3' \rangle$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}$$

$$G_1 = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & -2 & 1 \\ -2 & 1 & -2 \end{bmatrix}$$

$$G_2 = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$G_3 = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} = F^\dagger \overline{M_e} F \qquad \overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = G^T \overline{M_\nu G}$$

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$$G_3 = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

F : $\beta = 0, \quad \gamma = 0, \quad \epsilon = 0$

G_1 : $d = a + 3b/2 - c/2 - e$
$f = a - b/2 + 3c/2 - e$ (211)

magic G_2 : $d = a + c - e, \quad f = a + b - e$ (111)

G_3' : $d = c + (a - e)\omega_3^2,$
$f = b + (a - e)\omega_3$ $(1, \omega_3, \omega_3^2)$

2-3 G_3 : $c = b, \; f = d$ (011)

S_4^{Full} : $\overline{M_\nu} = \begin{bmatrix} a & c & c \\ c & d & e \\ c & e & d \end{bmatrix}$ (TBM)

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} = F^\dagger \overline{M_e} F$$

$$\overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = G^T \overline{M_\nu} G$$

$$S_4 = \langle F, G_2, G_3 \rangle = \langle F, G_1 \rangle$$

$$A_4 = \langle F, G_2, G_3' \rangle$$

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} = F^\dagger \overline{M_e} F \qquad \overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = G^T \overline{M_\nu G}$$

$$S_4 = \langle F, G_2, G_3 \rangle = \langle F, G_1 \rangle \qquad A_4 = \langle F, G_2, G_3' \rangle$$

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_3 = -\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} = F^\dagger \overline{M_e} F \qquad \overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = G^T \overline{M_\nu G}$$

$$S_4 = \langle F, G_2, G_3 \rangle = \langle F, G_1 \rangle \qquad A_4 = \langle F, G_2, G_3' \rangle$$

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_3 = -\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$F: \quad \gamma = \beta^*, \quad \delta = \alpha, \quad \phi = \alpha$

$G_1: \quad d = a, \quad e = -c$ (211)

$G_2: \quad c = 0, \quad e = 0$ (111)

$G_3: \quad d = a, \quad e = c$ (011)

$$S_4^{Full}: \quad \overline{M_\nu} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & f \end{bmatrix} \ (TBM)$$

$$\overline{M_e} = \begin{bmatrix} \alpha & \beta & \gamma \\ \beta^* & \delta & \epsilon \\ \gamma^* & \epsilon^* & \phi \end{bmatrix} \qquad \overline{M_\nu} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \overline{M_e} = \begin{bmatrix} \alpha & \beta & \beta^* \\ \beta^* & \alpha & \beta \\ \beta & \beta^* & \alpha \end{bmatrix}$$

$$G = -\begin{bmatrix} 0 & \omega_5^3 & 0 \\ \omega_5^2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \overline{M_\nu} = \begin{bmatrix} a & b & e\omega_5^2 \\ b & a\omega_5 & e \\ e\omega_5^2 & e & f \end{bmatrix}$$

$\Delta(150)$

	$((Z_5 \times$	$Z_5) \rtimes$	$Z_3) \rtimes$	Z_2
	f_4	f_3	f_2	f_1
$f_2 \to$	$f_4 f_3$	$f_4^2 f_3^3$		
$f_1 \to$	$(f_4 f_3)^4$	f_3	f_2^2	

$$\boxed{f_4^d \; f_3^c \; f_2^b \; f_1^a}$$