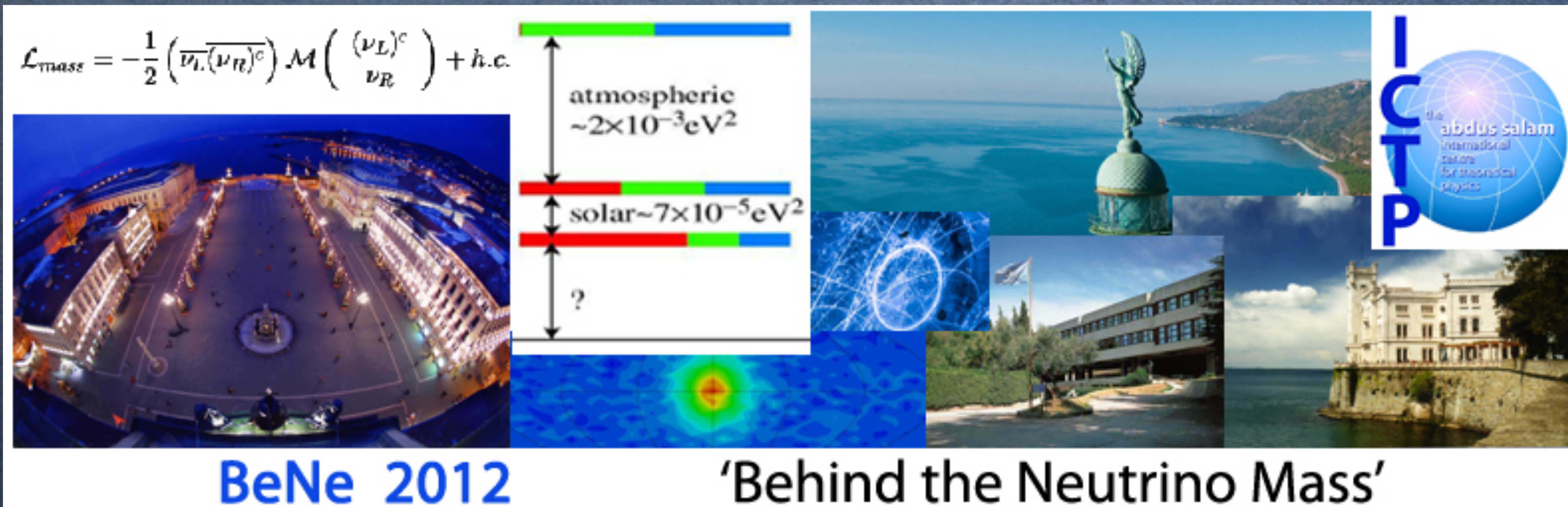
UNIVERSITY OF Southampton
School of Physics and Astronomy

A Grand Δ(96) Model

Steve King

September 21st 2012, Trieste



Simple LO mixing patterns $\theta_{13} = 0 \quad \theta_{23} = 45^o$

- **Bimaximal**

V. Barger, S. Pakvasa, T. Weiler and K. Whisnant

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} P \quad \theta_{12} = 45^o$$

- **Tri-bimaximal**

Harrison, Perkins and Scott

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$\theta_{12} = 35.26^o$$

- **Golden ratio**

Datta, Ling, Ramond; Kajirama, Raidal, Strumia; Everett, Stuart,Ding: Feruglio, Paris

$$U_{GR} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \tan\theta_{12} = \frac{1}{\phi} \qquad \theta_{12} = 31.7^o$$

King 1205.0506

Data prefers Tri-bimaximal-Cabibbo Mixing

Combine TB mixing with $\theta_{13} \approx \frac{\theta_C}{\sqrt{2}} \approx 9.2^o$

$$s_{13} = \frac{\lambda}{\sqrt{2}}, \quad s_{12} = \frac{1}{\sqrt{3}}, \quad s_{23} = \frac{1}{\sqrt{2}} \qquad \lambda = 0.2253 \pm 0.0007$$

$$U_{TBC} \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{3}}(1-\frac{1}{4}\lambda^2) & \frac{1}{\sqrt{2}}\lambda e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\lambda e^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \\ \frac{1}{\sqrt{6}}(1-\lambda e^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}\lambda e^{i\delta}) & \frac{1}{\sqrt{2}}(1-\frac{1}{4}\lambda^2) \end{pmatrix} + \mathcal{O}(\lambda^3)$$

Approximate description of lepton mixing

Hints of a connection with quark mixing

King; Parke; Pakvasa, Rodejohann, Weiler

useful to expand PMNS about TB mixing

$$U_{\rm PMNS} \approx \begin{pmatrix} \frac{2}{\sqrt{6}}(1-\frac{1}{2}s) & \frac{1}{\sqrt{3}}(1+s) & \frac{1}{\sqrt{2}}re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+s-a+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}s-a-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+a) \\ \frac{1}{\sqrt{6}}(1+s+a-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1-\frac{1}{2}s+a+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1-a) \end{pmatrix} P$$

$$\sin\theta_{12} = \frac{1}{\sqrt{3}}(1+s)\ , \qquad \sin\theta_{23} = \frac{1}{\sqrt{2}}(1+a)\ , \qquad \sin\theta_{13} = \frac{r}{\sqrt{2}}$$

Fogli, Lisi, Marrone, Palazzo, Rotunno '12

$$-0.066 \le s \le -0.013, \quad -0.146 \le a \le -0.094, \quad 0.208 \le r \le 0.231,$$

s = solar
below Tri-max

a = atmospheric
below Bi-max

r = reactor
Cabibbo-like

Global fits hint at atmospheric angle in first octant
i.e. a<0

Tri-bimaximal Hydras

- **Tri-bimaximal (s=a=r=0)**

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

Harrison, Perkins, Scott

- **Tri-bimaximal-reactor (s=a=0)**

$$U_{TBR} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}}(1-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

King; Antusch,Boudjemaa,King; Morisi,Patel,Peinado; Luhn, King

- **Tri-maximal 1 (s=0, a=r.cosδ)**

$$U_{\mathrm{TM}_1} = P' \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1-\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+re^{-i\delta}) \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1+\frac{3}{2}re^{i\delta}) & -\frac{1}{\sqrt{2}}(1-re^{-i\delta}) \end{pmatrix} P$$

Lam; Albright,Rodejohann; Antusch, King, Luhn, Spinrath

Haba,Watanabe,Yoshioka; He,Zee; Grimus,Lavoura; Albright,Rodejohann; King, Luhn

- **Tri-maximal 2 (s=0, a=-r/2.cosδ)**

$$U_{\mathrm{TM}_2} = P' \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}(1-\frac{1}{2}re^{-i\delta}) \\ -\frac{1}{\sqrt{6}}(1-\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}(1+\frac{1}{2}re^{-i\delta}) \end{pmatrix} P$$

N.B. Atmospheric sum rules: a=r.cosδ, a=-r/2.cosδ

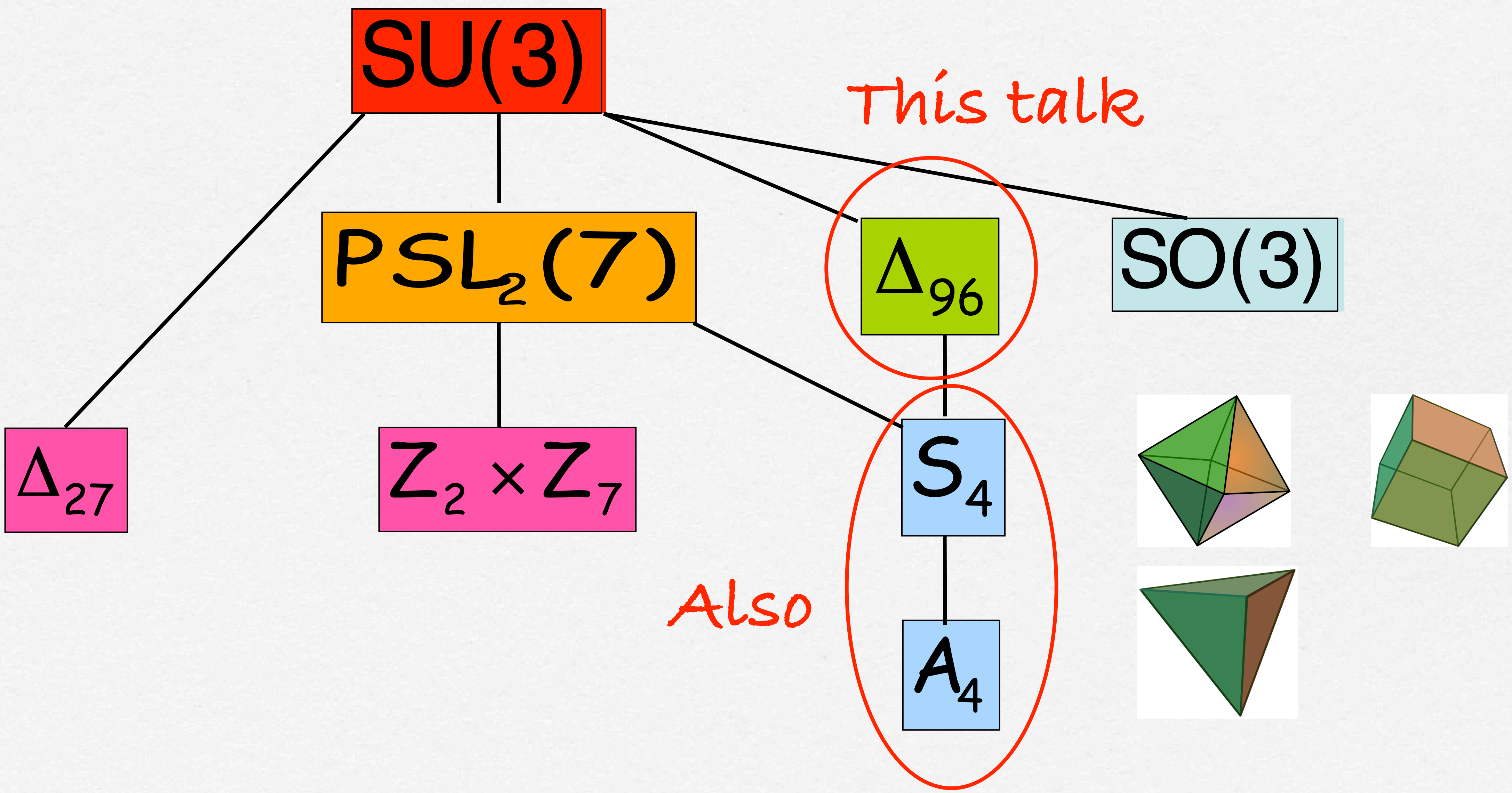
Theory Road Map

Daya Bay/RENO

- → **Family Symmetry**
  - → Indirect
  - → Direct
- → **Anarchy**
  - → Landscape

Family Symmetry

Family Symmetries G_F which contain triplet reps (three families in a triplet)



King, Ross, de Medeiros Varzielas, Antusch, Malinsky,...

The Indirect Approach

Starting point is type I see-saw $A^T = (A_1, A_2, A_3)$ $B^T = (B_1, B_2, B_3)$

$$m_{LR} = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} \quad M_{RR} = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix} \rightarrow m^{\nu} = \frac{AA^T}{M_1} + \frac{BB^T}{M_2} + \frac{CC^T}{M_3}$$

Construct the columns (A,B,C) from flavon fields

G_F yields special vacuum alignments, for example:

- (A,B,C) proportional to columns of PMNS called Form Dominance

  Chen, King('09)

- $AA^T/M_1 \ll BB^T/M_2 \ll CC^T/M_3$ called Sequential Dominance (SD)

  King('98,'02)

- SD with B=b(1,1,-1) and C=c(0,1,1) called CSD gives TB Mixing

  King('05)

Indirect Models after Daya Bay/RENO

King; Antusch,Boudjemaa; King,Luhn

Tri-bimaximal-reactor

$$m^{v} = \frac{AA^T}{M_1} + \frac{BB^T}{M_2} + \frac{CC^T}{M_3}$$

dominant

$\frac{AA^T}{M_1} \to 0$

$$B \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad C \propto \begin{pmatrix} re^{-i\delta} \\ 1 \\ -1 \end{pmatrix}$$

PCSD

$$U_{TBR} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+re^{i\delta}) & \frac{1}{\sqrt{3}}(1-\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}}(1-re^{i\delta}) & -\frac{1}{\sqrt{3}}(1+\frac{1}{2}re^{i\delta}) & \frac{1}{\sqrt{2}} \end{pmatrix} P$$

$$s = a = 0, \; r \neq 0$$

Special case TBC: $r = \lambda$

Antusch,King,Luhn,Spinrath

Trimaximal1

$$m^{v} = \frac{AA^T}{M_1} + \frac{BB^T}{M_2} + \frac{CC^T}{M_3}$$

dominant

$\frac{AA^T}{M_1} \to 0$

$$B \propto \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad C \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

CSD2

$$U_{\mathrm{TM}_1} = P' \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1-\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{2}}(1+re^{-i\delta}) \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}(1+\frac{3}{2}re^{i\delta}) & -\frac{1}{\sqrt{2}}(1-re^{-i\delta}) \end{pmatrix} P$$

$$a = r\cos\delta \qquad s = 0$$

Atmospheric sum rule

Altarelli, Feruglio, Ma, Hagedorn, Merlo, Luhn, ...

The Direct Approach

Family Symmetry G_F broken in special way

Subgroups preserved in neutrino/charged lepton sectors

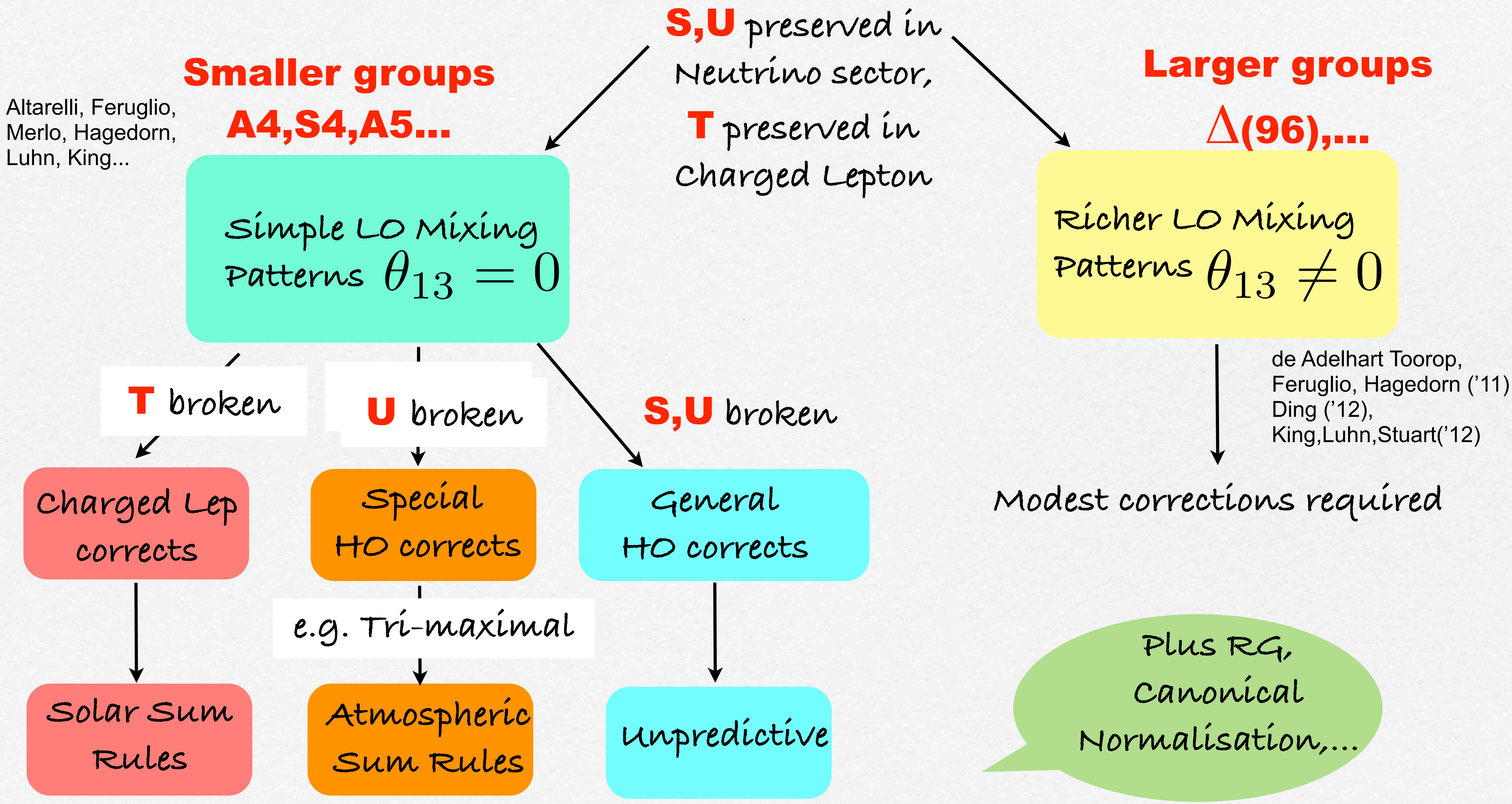
$G_F \supset S,T,U$ ← Group generators

"Klein symmetry" **S,U** only preserved

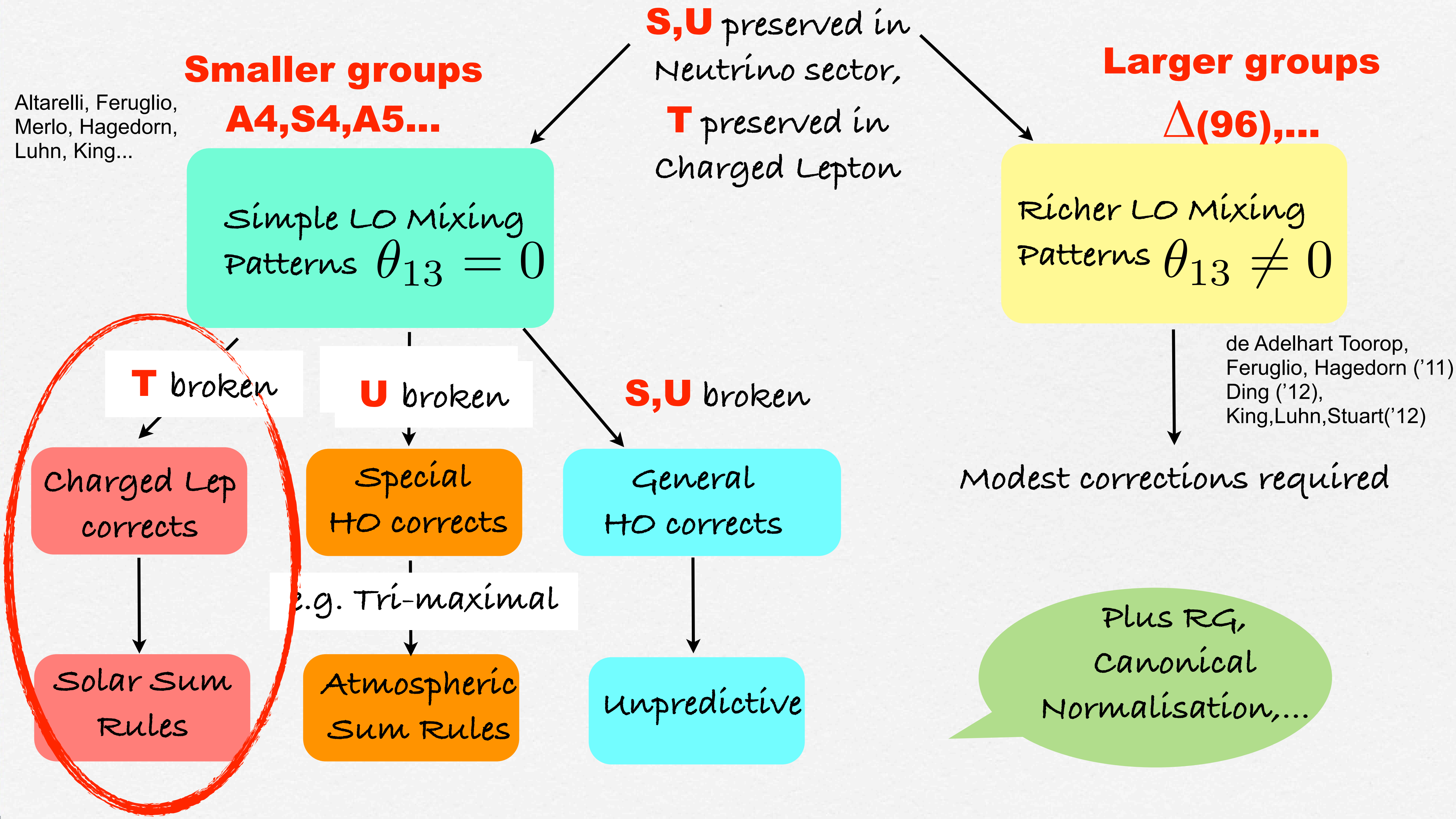
T only preserved

Neutrino Mass Matrix ↔ U_{PMNS} ↔ **Charged lepton Mass Matrix**

Direct Models after Daya Bay and RENO



Direct Models after Daya Bay and RENO



Solar Sum Rule s=r.cosδ

King ('05); Masina ('05); Antusch, King ('05) Antusch, Maurer ('11) Mazocca, Petcov, Romanino, Spinrath ('11) King 1205.0506 Antusch,Gross, Maurer, Sluka 1205.1051;

$$U_{PMNS} = V^e V^{\nu\dagger} = \begin{pmatrix} 1 & \lambda e^{-i\delta} & 0 \\ -\lambda e^{i\delta} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Cabibbo-like

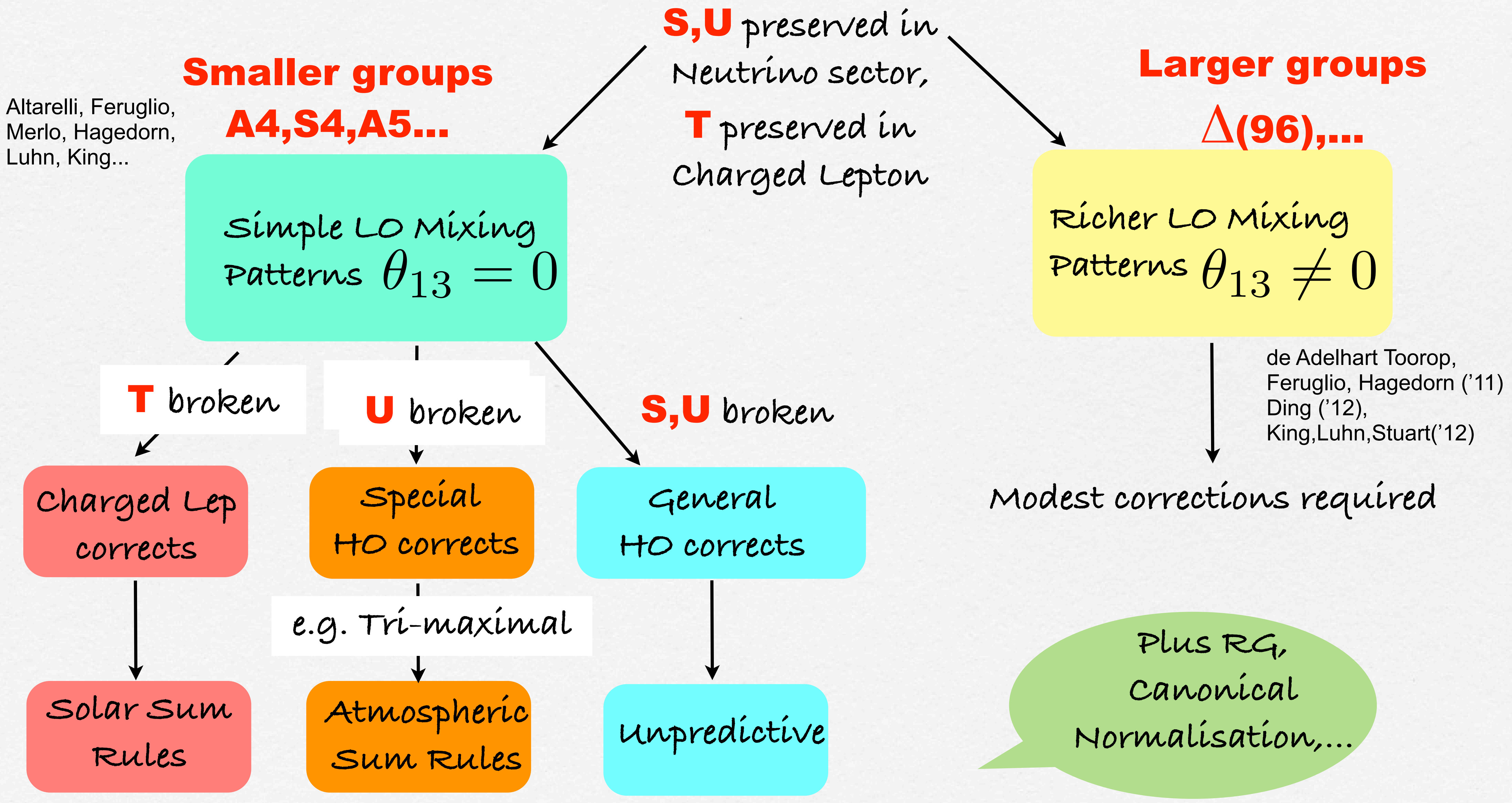
Tri-bimaximal

$$\lambda = \text{Wolfenstein} \qquad \sin\theta_{13} = \frac{\lambda}{\sqrt{2}}$$

s r

$$U_{PMNS} \approx \begin{pmatrix} \sqrt{\frac{2}{3}}(1-(\lambda/2)\cos\delta) & \frac{1}{\sqrt{3}}(1+\lambda\cos\delta) & \frac{1}{\sqrt{2}}\lambda e^{-i\delta} \\ -\frac{1}{\sqrt{6}}(1+(2\lambda)\cos\delta) & \frac{1}{\sqrt{3}}(1-\lambda\cos\delta) & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Direct Models after Daya Bay and RENO



Direct Models after Daya Bay and RENO

S,U preserved in Neutrino sector, **T** preserved in Charged Lepton

Smaller groups

A4,S4,A5...

Altarelli, Feruglio, Merlo, Hagedorn, Luhn, King...

Simple LO Mixing Patterns $\theta_{13} = 0$

T broken

Charged Lep corrects

Solar Sum Rules

U broken

Special HO corrects

e.g. Tri-maximal

Atmospheric Sum Rules

S,U broken

General HO corrects

Unpredictive

Larger groups

Δ(96),...

Richer LO Mixing Patterns $\theta_{13} \neq 0$

de Adelhart Toorop, Feruglio, Hagedorn ('11)
Ding ('12),
King,Luhn,Stuart('12)

Modest corrections required

Plus RG, Canonical Normalisation,...

Trimaximal2 from ~~U~~

$$\frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{pmatrix}\quad \begin{pmatrix}1 & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & \omega\end{pmatrix} \mp \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$S, T, U \in S_4$

- TB mixing respects Klein symmetry

$$SM^\nu S = M^\nu \quad TM^E T = M^E \quad UM^\nu U = M^\nu$$

- TM2 mixing respects discrete sym $S, T \in A_4$

$$SM^\nu S = M^\nu \quad TM^E T = M^E$$

U is broken

$$s \approx 0\ , \qquad a \approx -\frac{1}{2} r\cos\delta$$

$$U_{TB} = \begin{pmatrix}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{pmatrix} P, \qquad U_{\mathrm{TM}_2} = P'\begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} re^{-i\delta}\\ -\frac{1}{\sqrt{6}}(1+\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}(1-\frac{1}{2}re^{-i\delta})\\ -\frac{1}{\sqrt{6}}(1-\frac{3}{2}re^{i\delta}) & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}(1+\frac{1}{2}re^{-i\delta})\end{pmatrix} P$$

This suggests that TM2 mixing can arise from A4 or S4 broken to A4 (breaking U)

King, Luhn

□ A4 Model

L=3, N^c=3, Hu=1

Cooper, King, Luhn;
Shimizu, Tanimoto, Watanabe

Altarelli, Ferguglio,...

$$W^{\nu}_{A_4} = yLH_uN^c + (y_1\underset{\mathbf{3}}{\varphi_S} + y_2\underset{\mathbf{1}}{\xi} + y_3'\underset{\mathbf{1'}}{\xi'} + y_3''\underset{\mathbf{1''}}{\xi''})N^cN^c$$

$$m_D = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} y\,v_u \qquad \langle\varphi_S\rangle = v_S\begin{pmatrix}1\\1\\1\end{pmatrix}$$

TB violation $\Delta = \frac{1}{2}(\gamma'' - \gamma')$

$$M_R = \left[\alpha\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix} + \beta\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + \gamma'\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} + \gamma''\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\right]$$

$$\alpha = y_1v_S,\ \beta = y_2\langle\xi\rangle,\ \gamma' = y_3'\langle\xi'\rangle,\ \gamma'' = y_3''\langle\xi''\rangle$$

without ξ', ξ'' m_D, M_R respect S and U invariance → TB mixing

with ξ', ξ'' M_R respects S but not U invariance → TM2 mixing

S4 Model

Hagedorn, King, Luhn

3′ 1 2 1′ 2

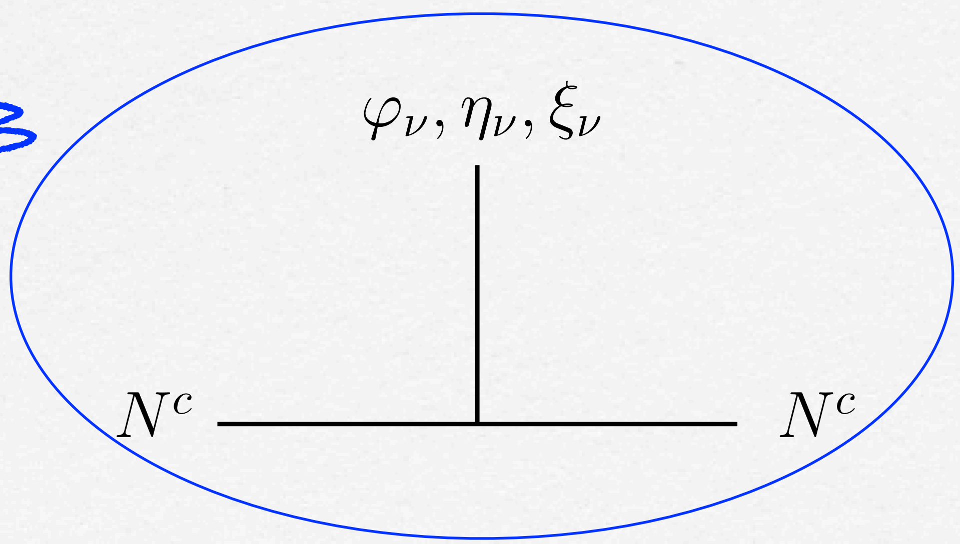
$$W_{S_4}^{\nu,\text{eff}} \sim LH_u N^c + (\varphi_\nu + \xi_\nu + \eta_\nu) N^c N^c + \frac{\zeta_\nu}{M_\chi} \eta_\nu N^c N^c$$

1′

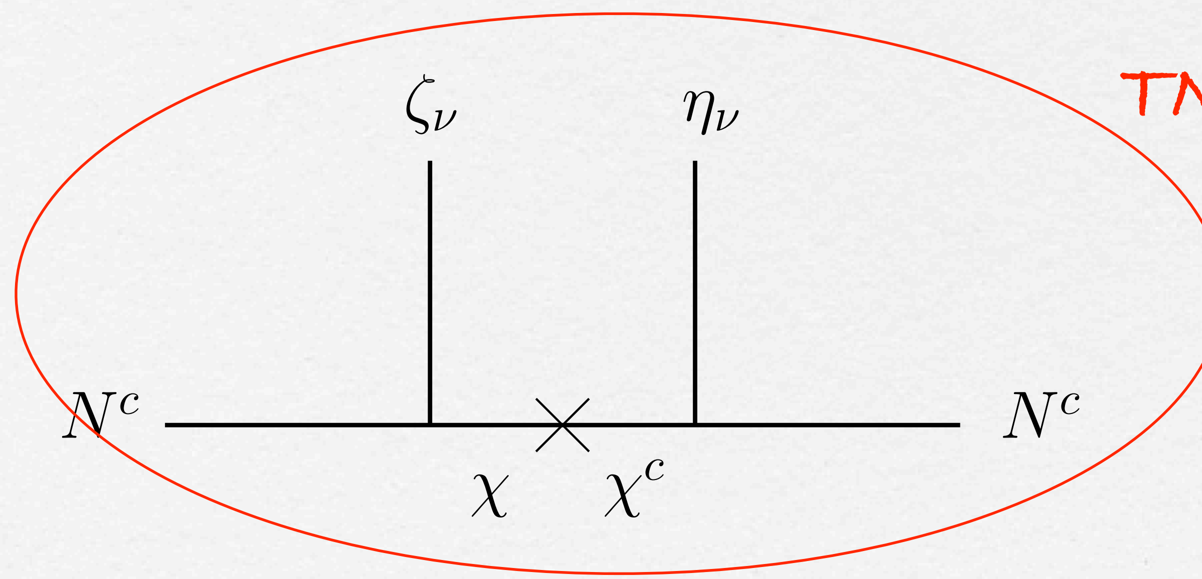
unifies ξ′, ξ″ into a doublet η, restores TB

< ζ$_\nu$ > breaks S4→A4 and TB→TM2

TB



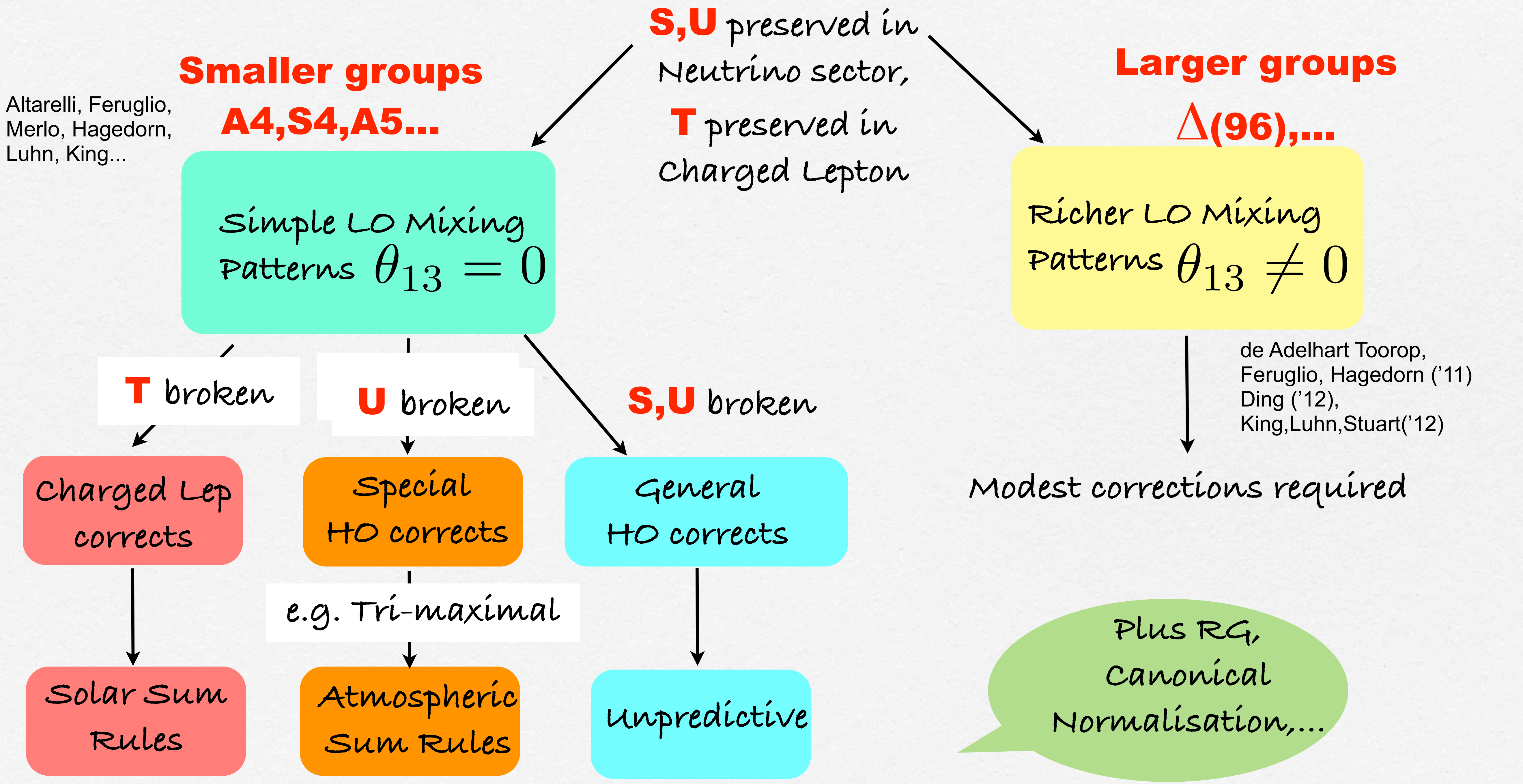
TM2



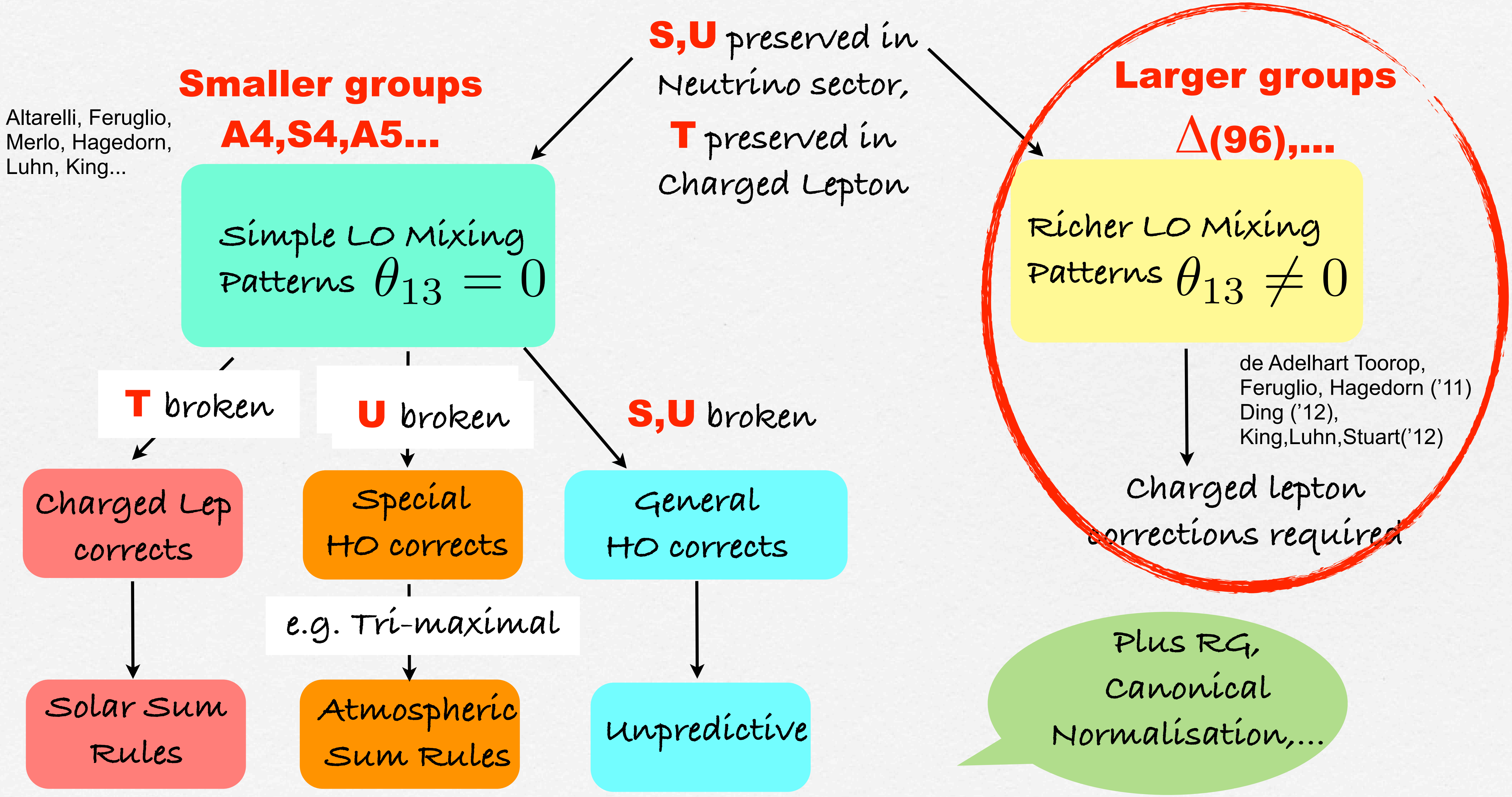
S4 model has TB at LO with TM2 corrections at HO

Explains why $\theta_{13} < \theta_{12}, \theta_{23}$

Direct Models after Daya Bay and RENO



Direct Models after Daya Bay and RENO



$\Delta(96)$

$\Delta(6n^2)$ group with $n = 4$

$\Delta(96) \cong (Z_4 \times Z_4) \rtimes S_3$

$$\mathbf{1} \otimes \mathbf{x} = \mathbf{x} \text{ with } \mathbf{x} \text{ any } \Delta(96) \text{ irrep}$$
$$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}$$
$$\mathbf{1}' \otimes \mathbf{2} = \mathbf{2}$$
$$\mathbf{1}' \otimes \mathbf{r} = \mathbf{r}' \text{ when } \mathbf{r} = \mathbf{3},\ \widetilde{\mathbf{3}},\text{ or } \overline{\mathbf{3}}$$
$$\mathbf{1}' \otimes \mathbf{r}' = \mathbf{r} \text{ when } \mathbf{r} = \mathbf{3},\ \widetilde{\mathbf{3}},\text{ or } \overline{\mathbf{3}}$$
$$\mathbf{1}' \otimes \mathbf{6} = \mathbf{6}$$
$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}$$
$$\mathbf{2} \otimes \mathbf{r}^m = \mathbf{r} \oplus \mathbf{r}' \text{ when } \mathbf{r} = \mathbf{3},\ \widetilde{\mathbf{3}},\text{ or } \overline{\mathbf{3}}$$
$$\mathbf{2} \otimes \mathbf{6} = \mathbf{6} \oplus \mathbf{6}$$
$$\mathbf{3}^m \otimes \mathbf{3}^n = \widetilde{\mathbf{3}}^p \oplus \overline{\mathbf{3}}' \oplus \overline{\mathbf{3}}$$
$$\mathbf{3}^m \otimes \widetilde{\mathbf{3}}^n = \overline{\mathbf{3}}^p \oplus \mathbf{6}$$
$$\mathbf{3}^m \otimes \overline{\mathbf{3}}^n = \mathbf{1}^q \oplus \mathbf{2} \oplus \mathbf{6}$$
$$\widetilde{\mathbf{3}}^m \otimes \widetilde{\mathbf{3}}^n = \mathbf{1}^q \oplus \mathbf{2} \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}'$$
$$\widetilde{\mathbf{3}}^m \otimes \overline{\mathbf{3}}^n = \mathbf{3}^p \oplus \mathbf{6}$$
$$\overline{\mathbf{3}}^m \otimes \overline{\mathbf{3}}^n = \mathbf{3} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}}^p$$
$$\mathbf{3}^m \otimes \mathbf{6} = \mathbf{3} \oplus \widetilde{\mathbf{3}} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}}' \oplus \mathbf{6}$$
$$\widetilde{\mathbf{3}}^m \otimes \mathbf{6} = \mathbf{3} \oplus \overline{\mathbf{3}} \oplus \mathbf{3}' \oplus \overline{\mathbf{3}}' \oplus \mathbf{6}$$
$$\overline{\mathbf{3}}^m \otimes \mathbf{6} = \widetilde{\mathbf{3}} \oplus \overline{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \overline{\mathbf{3}}' \oplus \mathbf{6}$$
$$\mathbf{6} \otimes \mathbf{6} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}' \oplus \mathbf{6} \oplus \mathbf{6}$$

King, Luhn, Stuart arXiv:1207.5741

	S	T	U
$\mathbf{1}$:	1	1	1
$\mathbf{1}'$:	1	1	-1
$\mathbf{2}$:	$\mathcal{I}_{2\times 2}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{3}$:	s_3	t_3	u_3
$\overline{\mathbf{3}}$:	s_3	t_3^*	u_3
$\mathbf{3}'$:	s_3	t_3	$-u_3$
$\overline{\mathbf{3}}'$:	s_3	t_3^*	$-u_3$
$\widetilde{\mathbf{3}}$:	$\mathcal{I}_{3\times 3}$	t_3	$v s_3$
$\widetilde{\mathbf{3}}'$:	$\mathcal{I}_{3\times 3}$	t_3	$-v s_3$
$\mathbf{6}$:	$\begin{pmatrix} s_3 & 0 \\ 0 & s_3 \end{pmatrix}$	$\begin{pmatrix} t_3 & 0 \\ 0 & t_3 \end{pmatrix}$	$\begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix}$

$$s_3 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad u_3 = \frac{1}{3}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & -1 \\ -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ -1 & -1+\sqrt{3} & -1-\sqrt{3} \end{pmatrix},$$

$$s_3 u_3 = \frac{1}{3}\begin{pmatrix} -1-\sqrt{3} & \sqrt{3}-1 & -1 \\ \sqrt{3}-1 & -1 & -1-\sqrt{3} \\ -1 & -1-\sqrt{3} & \sqrt{3}-1 \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

de Adelhart Toorop, Hagedorn, Feruglio; Ding

Bi-Trimaximal neutrino mixing

Klein Symmetry in $\Delta(96)$:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix},\ U = \frac{1}{3}\begin{pmatrix} -1+\sqrt{3} & -1-\sqrt{3} & -1 \\ -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ -1 & -1+\sqrt{3} & -1-\sqrt{3} \end{pmatrix} \quad T = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

$$SM^\nu S = M^\nu \quad TM^E T = M^E \quad UM^\nu U = M^\nu$$

St. George's Cross

$$U_{\rm BT} = \begin{pmatrix} a_+ & \frac{1}{\sqrt{3}} & a_- \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P, \qquad a_\pm = (1 \pm \tfrac{1}{\sqrt{3}})/2$$

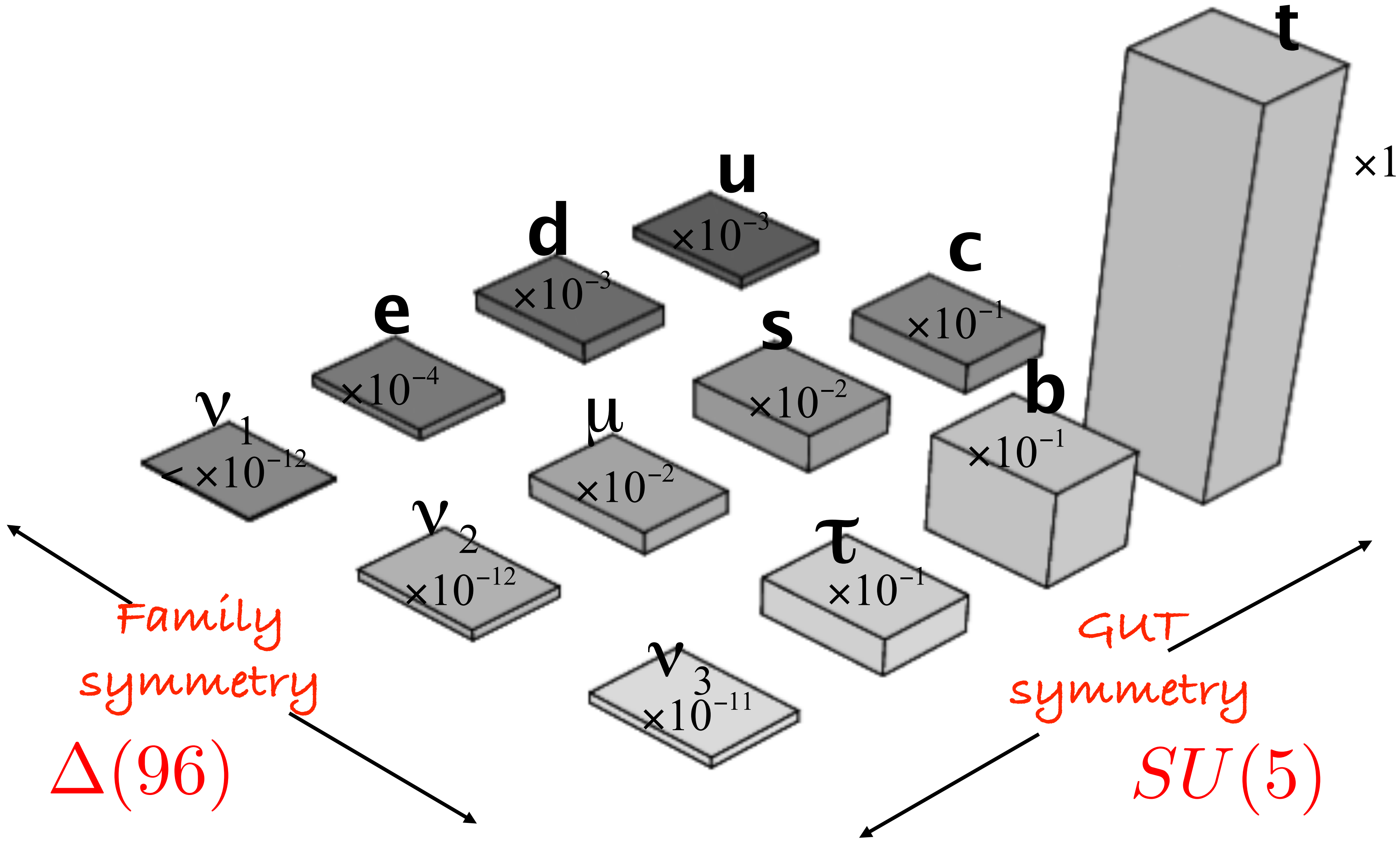
$$\sin\theta_{12} = \sin\theta_{23} = \sqrt{\frac{8-2\sqrt{3}}{13}} \approx 0.591 \qquad (\theta_{12} = \theta_{23} \approx 36.2^\circ)$$
$$\sin\theta_{13} = a_- \approx 0.211 \qquad (\theta_{13} \approx 12.2^\circ).$$

$$s \approx 0.023, \quad a \approx -0.165, \quad r \approx 0.299,$$

Disagrees with data

$$-0.066 \le s \le -0.013, \quad -0.146 \le a \le -0.094, \quad 0.208 \le r \le 0.231,$$

Family Symmetry × GUTs



$\Delta(96)\times SU(5)$

King, Luhn, Stuart arXiv:1207.5741

Flavons

Field	T_3	T	F	N	$H_{5,\overline{5}}$	$H_{\overline{45}}$	Φ^u_2	$\bar{\Phi}^u_2$	$\Phi^d_{\overline{3}}$	$\bar{\Phi}^d_{\overline{3}}$	Φ^d_2	$\Phi^\nu_{\overline{3}'}$	$\Phi^\nu_{\widetilde{3}'}$	$\Phi^\nu_{\widetilde{3}}$
$SU(5)$	$\mathbf{10}$	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{5},\overline{\mathbf{5}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\Delta(96)$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\overline{\mathbf{3}}$	$\mathbf{3}$	$\mathbf{2}$	$\overline{\mathbf{3}}'$	$\widetilde{\mathbf{3}}'$	$\widetilde{\mathbf{3}}$
$U(1)$	0	x	y	$-y$	0	z	$-2x$	0	$-y$	$-x-y-2z$	z	$2y$	$2y$	w
Z_3	1	1	ω^2	ω	$1,\omega$	ω	1	1	1	1	1	ω	ω	ω

Yukawa Operators

$$y_u T_3 T_3 H_5 + y'_u \frac{1}{M} T T \Phi^u_2 H_5 + y''_u \frac{1}{M^2} T T \Phi^u_2 \bar{\Phi}^u_2 H_5,$$

Up

Down

$$y_d \frac{1}{M} F T_3 \Phi^d_{\overline{3}} H_{\overline{5}} + y'_d \frac{1}{M^2} (F \bar{\Phi}^d_{\overline{3}})_1 (T \Phi^d_2)_1 H_{\overline{45}} + y''_d \frac{1}{M^3} (F \Phi^d_2 \Phi^d_2)_3 (T \bar{\Phi}^d_{\overline{3}})_{\overline{3}} H_{\overline{5}},$$

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and Charged Lepton

$$y_D F N H_5 + \overline{y}_M N N \Phi^\nu_{\overline{3}'} + \widetilde{y}_M N N \Phi^\nu_{\widetilde{3}'},$$

Neutrino

Δ(96)×SU(5)

Driving Fields

Field	X_1^ν	X_2^ν	X_6^ν	X_1^d	Y_1^d	Z_1^d	X_1^u	X_1^{ud}	$X_{1'}^{\nu d}$	X_2^{du}
$\Delta(96)$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{6}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1'}$	$\mathbf{2}$
$U(1)$	$-6y$	$-4y$	$-2y-w$	$4y$	$-2z$	$x+3y+z$	$2x$	$2x+4y$	$x+2y+2z-w$	$2x-z$
Z_3	1	ω	ω	1	1	1	1	1	ω^2	1

Flavon Superpotential

$$X_6^\nu \Phi_{\bar{3}}^\nu \Phi_{\bar{3}'}^\nu \qquad X_2^\nu \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu$$

$$\frac{1}{M} X_1^\nu \left[g_0 \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu + g_1 \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu + g_2 \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu \right]$$

$$Y_1^d \Phi_2^d \Phi_2^d \qquad X_2^{du} \Phi_2^d \Phi_2^u \qquad X_1^u \Phi_2^u \bar{\Phi}_2^u$$

$$\frac{1}{M^2} X_1^d \Phi_{\bar{3}}^d \Phi_{\bar{3}}^d \Phi_{\bar{3}}^d \Phi_{\bar{3}}^d + \frac{1}{M^3} X_1^{ud} \Phi_2^u \Phi_{\bar{3}}^d \Phi_{\bar{3}}^d \Phi_{\bar{3}}^d \Phi_{\bar{3}}^d$$

$$\frac{1}{M} X_{1'}^{\nu d} \Phi_{\bar{3}}^\nu \Phi_{\bar{3}}^d \bar{\Phi}_{\bar{3}}^d + \frac{1}{M^2} Z_1^d \Phi_2^d \Phi_{\bar{3}}^d \Phi_{\bar{3}}^d \bar{\Phi}_{\bar{3}}^d$$

Flavon Alignments

$$\langle \Phi_{\bar{3}'}^\nu \rangle = \varphi_{\bar{3}'}^\nu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \langle \Phi_{\bar{3}'}^\nu \rangle = \varphi_{\bar{3}'}^\nu \begin{pmatrix} v_1 \\ \frac{1}{2}(v_1+v_3) \\ v_3 \end{pmatrix}$$

$$\langle \Phi_2^u \rangle = \varphi_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \langle \bar{\Phi}_2^u \rangle = \bar{\varphi}_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle \Phi_2^d \rangle = \varphi_2^d \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle \Phi_{\bar{3}}^d \rangle = \varphi_{\bar{3}}^d \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \langle \bar{\Phi}_{\bar{3}}^d \rangle = \bar{\varphi}_{\bar{3}}^d \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Quark and Lepton Mass Matrices

$$\varphi_2^u/M \approx \lambda^4 \quad \bar{\varphi}_2^u/M \approx \lambda^4 \quad \varphi_2^d/M \approx \lambda\,, \quad \varphi_{\bar{3}}^d/M \approx \lambda^{1+k}\,, \quad \bar{\varphi}_{\bar{3}}^d/M \approx \lambda^{2+k}$$

$$k = 1$$

Up

$$M_u \approx v_u \begin{pmatrix} y_u'' \bar{\varphi}_2^u \varphi_2^u/M^2 & 0 & 0 \\ 0 & y_u' \varphi_2^u/M & 0 \\ 0 & 0 & y_u \end{pmatrix} \quad M_u \sim v_u \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$m_e/m_d = 1/3, \quad m_\mu/m_s = 3, \quad m_\tau/m_b = 1\,,$$

$$m_d : m_s : m_b \approx \lambda^4 : \lambda^2 : 1,$$

$$\theta_{12}^d \approx \sqrt{m_d/m_s}$$

Down

$$M_d \approx v_d \begin{pmatrix} 0 & y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 & -y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 \\ y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 & y_d' \varphi_2^d \bar{\varphi}_{\bar{3}}^d/M^2 - y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 & -y_d' \varphi_2^d \bar{\varphi}_{\bar{3}}^d/M^2 \\ 0 & 0 & y_d \varphi_{\bar{3}}^d/M \end{pmatrix} \quad M_d \sim v_d \begin{pmatrix} 0 & \lambda^5 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ 0 & 0 & \lambda^2 \end{pmatrix}$$

$$M_e \approx v_d \begin{pmatrix} 0 & y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 & 0 \\ y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 & -3y_d' \varphi_2^d \bar{\varphi}_{\bar{3}}^d/M^2 - y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 & 0 \\ -y_d''(\varphi_2^d)^2 \bar{\varphi}_{\bar{3}}^d/M^3 & 3y_d' \varphi_2^d \bar{\varphi}_{\bar{3}}^d/M^2 & y_d \varphi_{\bar{3}}^d/M \end{pmatrix} \quad M_e \sim v_d \begin{pmatrix} 0 & \lambda^5 & 0 \\ \lambda^5 & 3\lambda^4 & 0 \\ \lambda^5 & 3\lambda^4 & \lambda^2 \end{pmatrix}$$

zero 1-3 and 2-3

Charged Lepton (violates T)

Neutrino (respects S,U)

$$M_D = y_D v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_{Maj} = \overline{y}_M \varphi_{3'}^\nu \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} + \widetilde{y}_M \varphi_{3'}^\nu \begin{pmatrix} v_3 & v_1 & \frac{1}{2}(v_1+v_3) \\ v_1 & \frac{1}{2}(v_1+v_3) & v_3 \\ \frac{1}{2}(v_1+v_3) & v_3 & v_1 \end{pmatrix}$$

King, Luhn, Stuart arXiv:1207.5741

Bi-Trimaximal neutrino mixing with charged lepton corrections ~~T~~

zero 1-3 and 2-3

St. George's Cross

$$U_{\rm BT} = \begin{pmatrix} a_+ & \frac{1}{\sqrt{3}} & a_- \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P, \qquad V_e \approx P' \begin{pmatrix} c^e_{12} & -s^e_{12} e^{-i\delta^e_{12}} & 0 \\ s^e_{12} e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_\pm = (1 \pm \tfrac{1}{\sqrt{3}})/2$$

$$U_{\rm PMNS} = V_{e_{\rm L}} V^\dagger_{\nu_{\rm L}}$$

$$U_{\rm PMNS} \approx P'' \begin{pmatrix} a_+ c^e_{12} + \frac{1}{\sqrt{3}} s^e_{12} e^{-i\delta^e_{12}} & \frac{1}{\sqrt{3}} c^e_{12} - \frac{1}{\sqrt{3}} s^e_{12} e^{-i\delta^e_{12}} & a_- c^e_{12} - \frac{1}{\sqrt{3}} s^e_{12} e^{-i\delta^e_{12}} \\ a_+ s^e_{12} e^{i\delta^e_{12}} - \frac{1}{\sqrt{3}} c^e_{12} & \frac{1}{\sqrt{3}} s^e_{12} e^{i\delta^e_{12}} + \frac{1}{\sqrt{3}} c^e_{12} & a_- s^e_{12} e^{i\delta^e_{12}} + \frac{1}{\sqrt{3}} c^e_{12} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P$$

$\theta^e_{12} \approx \lambda/3 \quad \delta^e_{12} \approx 0$

Predictions: **OK with data**

$$\tan\theta_{23} \approx \frac{\frac{1}{\sqrt{3}} c^e_{12} + a_- s^e_{12}}{a_+} \approx 0.750 \qquad \theta_{23} \approx 36.9^\circ$$

$$\sin\theta_{13} \approx a_- - \frac{1}{\sqrt{3}} \theta^e_{12} \cos\delta^e_{12} \qquad \theta_{13} \approx 9.6^\circ$$

$$\tan\theta_{12} \approx \frac{\frac{1}{\sqrt{3}} c^e_{12} - \frac{1}{\sqrt{3}} s^e_{12}}{a_+ c^e_{12} + \frac{1}{\sqrt{3}} s^e_{12}} \approx 0.642 \qquad \theta_{12} \approx 32.7^\circ$$

Zero CP phase $\delta \approx 0$

Summary

- Simple patterns BM, TB, GR excluded, TBC mixing OK
- Expand about TB mixing, data prefers: $a<0$, $s<0$, $r=0.22$
- TM1: $s=0$, $a=r.\cos\delta$, TM2: $s=0$, $a=-r/2.\cos\delta$
- Two theory approaches: Symmetry or Anarchy
- Family Symmetry implemented indirectly or directly
- Indirect models: CSD→TB, PCSD→TBR, CSD2→TM1
- Direct models: A4,S4,A5→BM,TB,GR, u breaking→TM2
- Delta(96) → $\theta_{13}\sim 12^{\circ}$, $\theta_{12}=\theta_{23}\sim 36^{\circ}$ excluded
- Delta(96)xSU(5) → $\theta_{13}\sim 9.6^{\circ}$, $\theta_{12}\sim 32.7^{\circ}$, $\theta_{23}\sim 36.9^{\circ}$ OK