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Nonequilibrium Steady States of Boundary Driven Open Spin Chains

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Nonequilibrium steady states of boundary driven open spin chains:
Some exact solutions in the quantum transport problem

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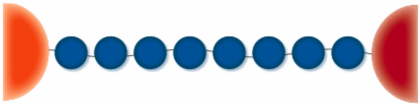
ICTP, 16 October, 2012

Outline

One dimensional open (quantum many body) systems far from equilibrium:



- Quantum: Quasi-free (linear) systems:
  - XY spin 1/2 chain: transition to *long range order* due to local *boundary* opening (TP NJP 2008, TP and I. Pižorn PRL 2008, TP JSTAT 2010)
- Strongly interacting (non-linear) systems
  - NESS via tDMRG: *spin diffusion and quantum Fourier law*, (TP and M. Žnidarič JSTAT 2009), and *long range order far from equilibrium* (numerical examples, TP and M. Žnidarič, PRL 2010)
  - XXZ spin 1/2 chain: exact matrix product NESS and *strict lower bound on spin Drude weight* (TP PRL 2011a, TP PRL 2011b)
  - Exact ansatz for *diffusive* NESS in XX chain /w dephasing noise and boundary driving (M. Žnidarič, JSTAT 2010)
  - Normal spin and charge diffusion in the Hubbard chain at infinite temperature and half-filling (TP and M. Žnidarič, PRB 2012)

The central equation we address is the Lindblad equation for the many-body density operator $\rho(t)$:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \hat{\mathcal{L}}\rho := -\mathrm{i}[H,\rho] + \sum_{\mu}\left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\}\right)$$

where H is a many-body (*Hamiltonian*) with **local couplings**,

$$H = \sum_{j=1}^{n-k+1} h_j$$

and L_μ are *Lindblad operators* which act **locally**, near the **ends** of the chain, say, only on degrees of freedom of sites 1 and n, (e.g. representing the baths).

In the context of 1D quantum transport, the Lindblad model has been carefully derived and discussed in: Wichterich, Herich, Breuer and Gemmer, PRE 2007

Analytical solution for quasi-free fermionic systems

TP, New J. Phys. **10**, 043026 (2008), JSTAT P07020 (2010)

Consider a general solution of the Lindblad equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \hat{\mathcal{L}}\rho := -\mathrm{i}[H,\rho] + \sum_{\mu}\left(2L_\mu \rho L_\mu^\dagger - \{L_\mu^\dagger L_\mu, \rho\}\right)$$

for a general *quadratic system* of n fermions, or n qubits (spins 1/2)

$$H = \sum_{j,k=1}^{2n} w_j H_{jk} w_k = \underline{w} \cdot \mathbf{H}\, \underline{w} \qquad L_\mu = \sum_{j=1}^{2n} l_{\mu,j} w_j = \underline{l}_\mu \cdot \underline{w}$$

where w_j, $j = 1, 2, \ldots, 2n$, are abstract *Hermitian* Majorana operators

$$\{w_j, w_k\} = 2\delta_{j,k} \qquad j,k = 1,2,\ldots,2n$$

Two physical realizations:

- canonical fermions c_m, $w_{2m-1} = c_m + c_m^\dagger$, $w_{2m} = \mathrm{i}(c_m - c_m^\dagger)$, $m = 1, \ldots, n$.
- spins 1/2 with canonical Pauli operators $\vec{\sigma}_m$, $m = 1, \ldots, n$,

$$w_{2m-1} = \sigma_j^{\mathrm{x}} \prod_{m'<m} \sigma_{m'}^{\mathrm{z}} \qquad w_{2m} = \sigma_m^{\mathrm{y}} \prod_{m'<m} \sigma_{m'}^{\mathrm{z}}$$

NESS expectation values of physical observables

The expectation value of *any quadratic observable* $w_j w_k$ in a (unique) NESS can be explicitly computed as

$$\langle w_j w_k \rangle_{\rm NESS} = \delta_{j,k} + \langle 1 | \hat{c}_j \hat{c}_k | {\rm NESS} \rangle = \delta_{j,k} + 4{\rm i} Z_{j,k}$$

where $\mathbf{Z}$ is the solution of the Lyapunov equation

$$\mathbf{X}^T \mathbf{Z} + \mathbf{Z}\mathbf{X} = {\rm Im}\, \mathbf{M}$$

with $\mathbf{X} := -2{\rm i}\mathbf{H} + {\rm Re}\, \mathbf{M}$ where $\mathbf{M} := \sum_\mu \underline{l}_\mu \otimes \bar{\underline{l}}_\mu$.

uniqueness

The NESS is unique iff all eigenvalues of X lie strictly away from the real line.

Solvable example: open XY quantum spin chains

Consider magnetic and heat transport of a Heisenberg XY spin 1/2 chain, with arbitrary – either homogeneous or positionally dependent (e.g. disordered) – nearest neighbour interaction

$$H = \sum_{m=1}^{n-1} \left(J^{\mathrm{x}}_m \sigma^{\mathrm{x}}_m \sigma^{\mathrm{x}}_{m+1} + J^{\mathrm{y}}_m \sigma^{\mathrm{y}}_m \sigma^{\mathrm{y}}_{m+1} \right) + \sum_{m=1}^{n} h_m \sigma^{\mathrm{z}}_m \qquad (1)$$

which is coupled to *two* thermal/magnetic baths *at the ends* of the chain, generated by two pairs of canonical Lindblad operators

$$\begin{aligned} L_1 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{L}}_1 \sigma^{-}_1} & L_3 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{R}}_1 \sigma^{-}_n} \\ L_2 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{L}}_2 \sigma^{+}_1} & L_4 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{R}}_2 \sigma^{+}_n} \end{aligned} \qquad (2)$$

where $\sigma^{\pm}_m = \sigma^{\mathrm{x}}_m \pm \mathrm{i}\sigma^{\mathrm{y}}_m$ and $\Gamma^{\mathrm{L,R}}_{1,2}$ are positive coupling constants related to bath temperatures/magnetizations. e.g. if spins were non-interacting the bath temperatures $T_{\mathrm{L,R}}$ would be given with $\Gamma^{\mathrm{L,R}}_2/\Gamma^{\mathrm{L,R}}_1 = \exp(-2h_{1,\mathrm{n}}/T_{\mathrm{L,R}})$.

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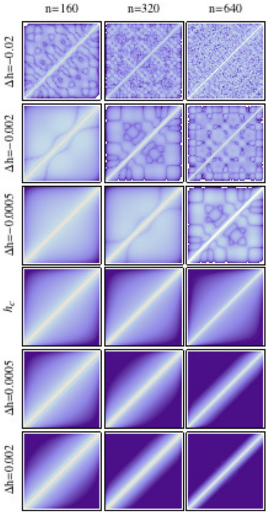
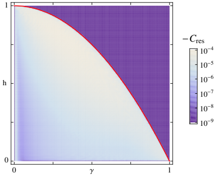
$$\begin{aligned} L_1 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{L}}_1}\sigma^-_1 \qquad & L_3 = \frac{1}{2}\sqrt{\Gamma^{\mathrm{R}}_1}\sigma^-_n \\ L_2 &= \frac{1}{2}\sqrt{\Gamma^{\mathrm{L}}_2}\sigma^+_1 \qquad & L_4 = \frac{1}{2}\sqrt{\Gamma^{\mathrm{R}}_2}\sigma^+_n \end{aligned} \tag{2}$$

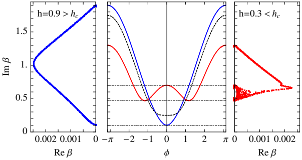
where $\sigma^{\pm}_m = \sigma^{\mathrm{x}}_m \pm \mathrm{i}\sigma^{\mathrm{y}}_m$ and $\Gamma^{\mathrm{L,R}}_{1,2}$ are positive coupling constants related to bath temperatures/magnetizations. e.g. if spins were non-interacting the bath temperatures $T_{\mathrm{L,R}}$ would be given with $\Gamma^{\mathrm{L,R}}_2/\Gamma^{\mathrm{L,R}}_1 = \exp(-2h_{1,\mathrm{n}}/T_{\mathrm{L,R}})$.
Similar models were recently considered e.g. in Karevski and Platini PRL 2009, and Clark, Prior, Hartmann, Jaksch and Plenio PRL2009 & `arXiv:0907.5582`

TP & I. Pižorn, PRL **101**, 105701 (2008)

$$
\begin{aligned}
J^x_m &= (1+\gamma)/2 \\
J^y_m &= (1-\gamma)/2, \\
h_m &= h \\
C(j,k) &= \langle \sigma^z_j \sigma^z_k \rangle - \langle \sigma^z_j \rangle \langle \sigma^z_k \rangle
\end{aligned}
$$





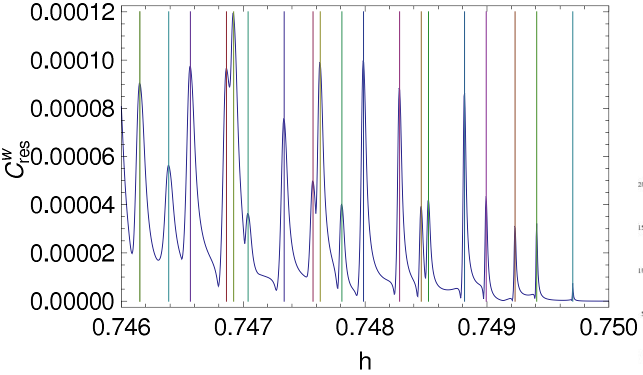


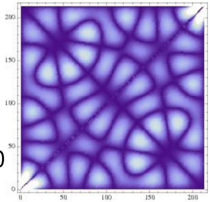
$$h_c = 1 - \gamma^2$$

Near neQPT: Scaling variable $z = (h_{\rm c} - h)n^2$
Scaling ansatz: $C_{2j+\alpha,2k+\beta} = \Psi^{\alpha,\beta}(x = j/n, y = k/n, z)$

Near neQPT: Scaling variable $z = (h_c - h)n^2$
Scaling ansatz: $C_{2j+\alpha,2k+\beta} = \Psi^{\alpha,\beta}(x = j/n, y = k/n, z)$
Certain combination $\Psi(x,y) = (\partial/\partial_x + \partial/\partial_y)(\Psi^{0,0}(x,y) + \Psi^{1,1}(x,y))$ obeys the Helmholtz equation!!!

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4z\right)\Psi = \text{"octopole antenna sources"}$$





Interacting many-body semigroups: quantum diffusion and long range order in NESS

tDMRG simulations of NESS for locally interacting boundary driven spin chains (method as described in TP & M. Žnidarič, JSTAT P02035, 2009).

Example, toy model: Locally boundary driven XXZ spin 1/2 chain:

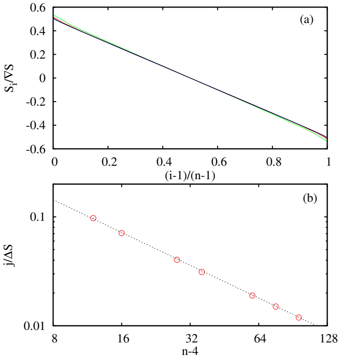
$$H = \sum_{j=1}^{n-1} (\sigma_j^{\mathrm{x}} \sigma_{j+1}^{\mathrm{x}} + \sigma_j^{\mathrm{y}} \sigma_{j+1}^{\mathrm{y}} + \Delta \sigma_j^{\mathrm{z}} \sigma_{j+1}^{\mathrm{z}})$$

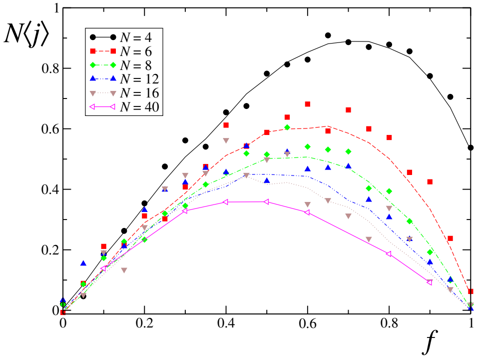
and symmetric magnetic-Lindblad boundary driving:

$$L_1^{\mathrm{L}} = \sqrt{\frac{1}{2}(1-\mu)\varepsilon}\sigma_1^+, \quad L_1^{\mathrm{R}} = \sqrt{\frac{1}{2}(1+\mu)\varepsilon}\sigma_n^+,$$

$$L_2^{\mathrm{L}} = \sqrt{\frac{1}{2}(1+\mu)\varepsilon}\sigma_1^-, \quad L_2^{\mathrm{R}} = \sqrt{\frac{1}{2}(1-\mu)\varepsilon}\sigma_n^-.$$

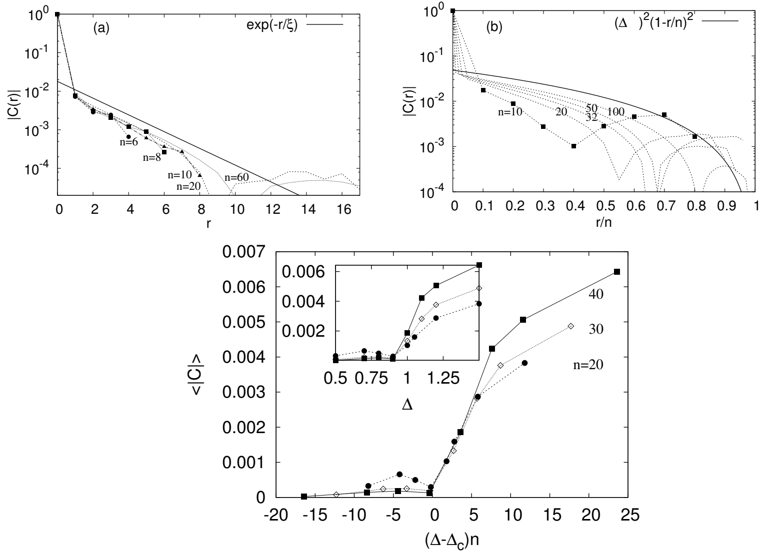
If $\Delta > 1$ the model exhibits diffusive transport for **small driving**, and negative differential conductance for **large driving** $\mu \equiv f$.





Transition to long-range order in NESS (PRL **105**, 060603 (2010))

$C(r) = \langle\sigma^z_{(n+r)/2}\sigma^z_{(n-r)/2}\rangle - \langle\sigma^z_{(n+r)/2}\rangle\langle\sigma^z_{(n-r)/2}\rangle$



Critical anisotropy appears to be $\Delta_c \approx 0.91$ (!?)

Take *boundary driven* XX spin chain ($\Delta = 0$) and in addition put local bulk dephasing with Lindblads $L_j = \gamma \sigma_j^{\mathrm{z}}$. [M. Žnidarič, JSTAT, L05002 (2010)]

$$\rho_{\mathrm{NESS}} = \mathbb{1} + \sum_{j=1}^{n} a_j \sigma^{\mathrm{z}} + b \sum_{j=1}^{n-1} J_j + \mathcal{O}(\mu^2)$$

where $J_j = \sigma_j^{\mathrm{x}} \sigma_{j+1}^{\mathrm{y}} - \sigma_j^{\mathrm{y}} \sigma_{j+1}^{\mathrm{x}}$ is the spin current and

$$a_1 = -b/\varepsilon - \mu, \; a_j = -b(1/\varepsilon + \varepsilon + 2\gamma(j-1)) - \mu, \; a_n = -b(1/\varepsilon + 2\varepsilon + 2(n-1)\gamma) - \mu,$$

$$b = -\frac{\mu}{\varepsilon + 1/\varepsilon + (n-1)\gamma}.$$

The solution yields the spin Fick's law (spin diffusion), $\langle(\sigma_j^{\mathrm{z}} - \sigma_k^{\mathrm{z}})\rangle \propto \frac{\mu(j-k)}{n}$, $\langle J_j \rangle \propto \frac{\mu}{n}$.

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$$b = -\frac{\mu}{\varepsilon + 1/\varepsilon + (n-1)\gamma}.$$

The solution yields the spin Fick's law (spin diffusion), $\langle(\sigma^{\rm z}_j - \sigma^{\rm z}_k)\rangle \propto \frac{\mu(j-k)}{n}$, $\langle J_j \rangle \propto \frac{\mu}{n}$.
The higher orders, say $\mathcal{O}(\mu^2)$ have also been calculated analytically and predict *'hydrodynamic long range order'* [observed in nonequilibrium classical exclusion processes (see e.g. Derrida JSTAT 2007)]

$$C_{j=xn,k=yn} = \frac{(2\mu)^2}{n} x(1-y)$$

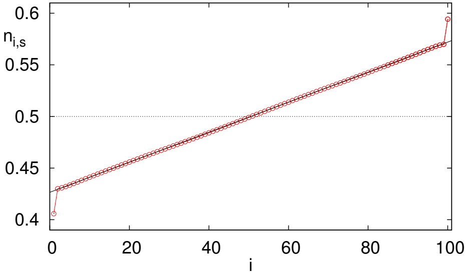
Hamiltonian is rewritten from fermionic to spin-ladder formulation:

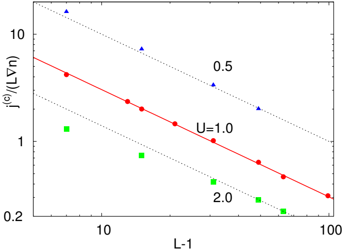
$$\begin{aligned} H &= -t\sum_{i=1}^{L-1}\sum_{s\in\{\uparrow,\downarrow\}}(c^\dagger_{i,s}c_{i+1,s}+\mathrm{h.c.})+U\sum_{i=1}^{L}n_{i\uparrow}n_{i\downarrow}, \\ &= -\frac{t}{2}\sum_{i=1}^{L-1}(\sigma^{\mathrm{x}}_i\sigma^{\mathrm{x}}_{i+1}+\sigma^{\mathrm{y}}_i\sigma^{\mathrm{y}}_{i+1}+\tau^{\mathrm{x}}_i\tau^{\mathrm{x}}_{i+1}+\tau^{\mathrm{y}}_i\tau^{\mathrm{y}}_{i+1})+\frac{U}{4}\sum_{i=1}^{L}(\sigma^{\mathrm{z}}_i+1)(\tau^{\mathrm{z}}_i+1). \end{aligned}$$

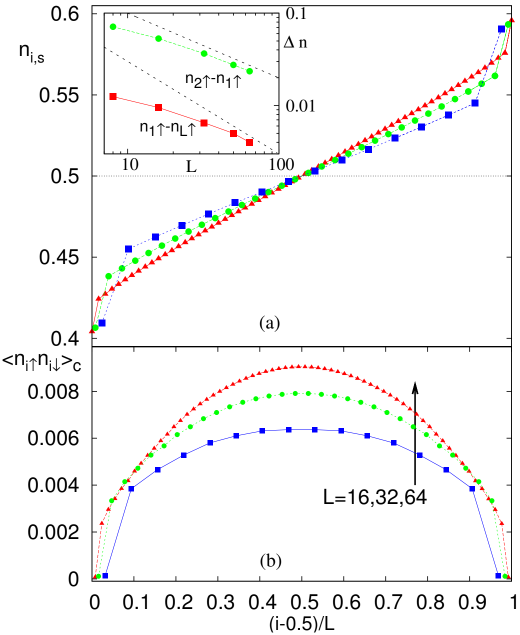
And the following simplest boundary driving channels are considered:

$$\begin{aligned} L_{1,2} &= \sqrt{\varepsilon(1\mp\mu)}\,\sigma^{\pm}_1, \quad L_{3,4}=\sqrt{\varepsilon(1\pm\mu)}\,\sigma^{\pm}_L \\ L_{5,6} &= \sqrt{\varepsilon(1\mp\mu)}\,\tau^{\pm}_1, \quad L_{7,8}=\sqrt{\varepsilon(1\pm\mu)}\,\tau^{\pm}_L \end{aligned}$$

[TP and M. Žnidarič, PRB 86, 125118 (2012)]


0.6
$n_{i,s}$
0.55
0.5
0.45
0.4
0
20
40
60
80
100
i


10
$j^{(c)}/(L\nabla n)$
1
0.2
0.5
U=1.0
2.0
10
100
L-1



0.6
0.55
0.5
0.45
0.4
n_{i,s}
0.1
Δ n
0.01
n2↑-n1↑
n1↑-nL↑
10
L
100
(a)
<n_{i↑}n_{i↓}>_c
0.008
0.006
0.004
0.002
0
L=16,32,64
(b)
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
(i-0.5)/L

Explicit solutions possible in two regimes!

PRL **107,** 137201 (2011) PHYSICAL REVIEW LETTERS week ending 23 SEPTEMBER 2011

Exact Nonequilibrium Steady State of a Strongly Driven Open *XXZ* Chain

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(Received 15 June 2011; published 19 September 2011)

An exact and explicit ladder-tensor-network ansatz is presented for the nonequilibrium steady state of an anisotropic Heisenberg *XXZ* spin-$1/2$ chain which is driven far from equilibrium with a pair of Lindblad operators acting on the edges of the chain only. We show that the steady-state density operator of a finite system of size n is—apart from a normalization constant—a polynomial of degree $2n-2$ in the coupling constant. Efficient computation of physical observables is facilitated in terms of a transfer operator reminiscent of a classical Markov process. In the isotropic case we find cosine spin profiles, $1/n^2$ scaling of the spin current, and long-range correlations in the steady state. This is a fully nonperturbative extension of a recent result [Phys. Rev. Lett. **106**, 217206 (2011)].

DOI: 10.1103/PhysRevLett.107.137201 PACS numbers: 75.10.Pq, 02.30.Ik, 03.65.Yz, 05.60.Gg

PRL **106,** 217206 (2011) PHYSICAL REVIEW LETTERS week ending 27 MAY 2011

Open *XXZ* Spin Chain: Nonequilibrium Steady State and a Strict Bound on Ballistic Transport

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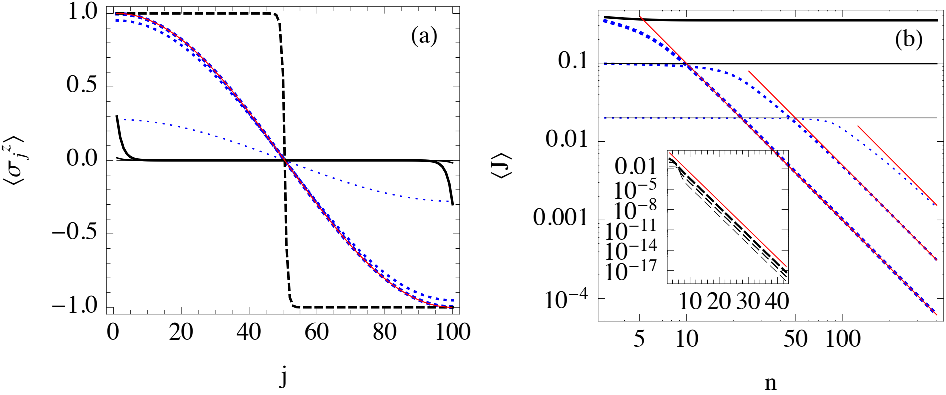
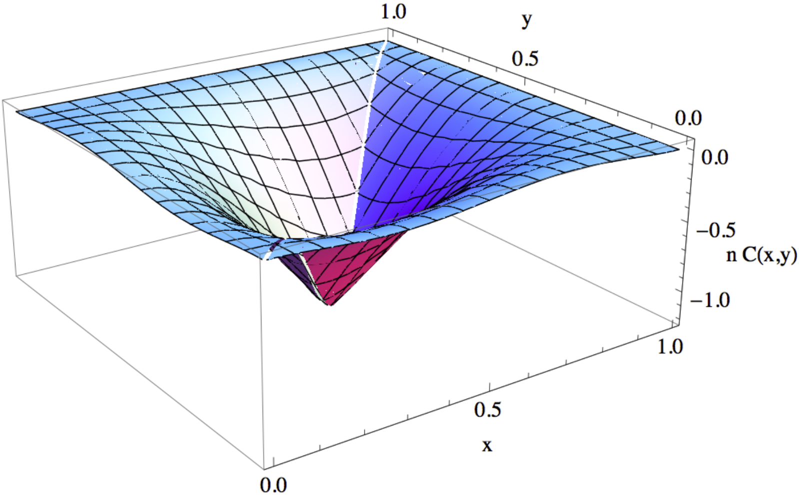
(Received 7 March 2011; revised manuscript received 11 April 2011; published 27 May 2011)

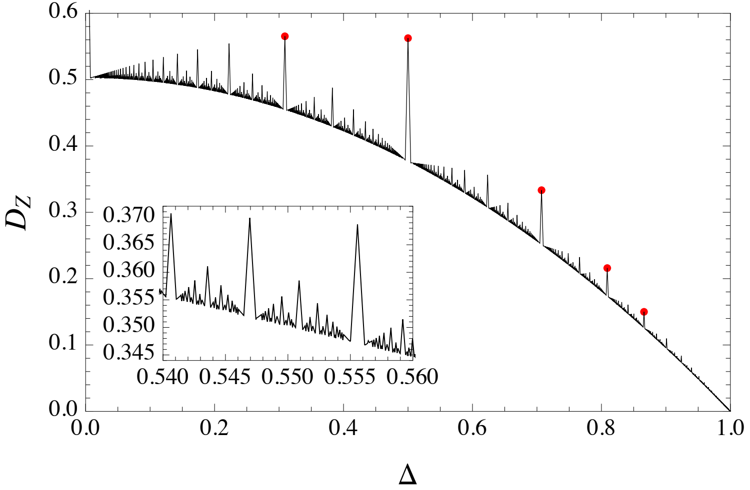
An explicit matrix product ansatz is presented, in the first two orders in the (weak) coupling parameter, for the nonequilibrium steady state of the homogeneous, nearest neighbor Heisenberg *XXZ* spin $1/2$ chain driven by Lindblad operators which act only at the edges of the chain. The first order of the density operator becomes, in the thermodynamic limit, an exact pseudolocal conservation law and yields—via the Mazur inequality—a rigorous lower bound on the high-temperature spin Drude weight. Such a Mazur bound is a nonvanishing fractal function of the anisotropy parameter Δ for $|\Delta| < 1$.

DOI: 10.1103/PhysRevLett.106.217206 PACS numbers: 75.10.Pq, 02.30.Ik, 03.65.Yz, 05.60.Gg



1.0
0.5
0.0
−0.5
−1.0
(a)
$\langle\sigma_j^z\rangle$
0
20
40
60
80
100
j
(b)
0.1
0.01
0.001
10^{-4}
⟨J⟩
0.01
10^{-5}
10^{-8}
10^{-11}
10^{-14}
10^{-17}
10 20 30 40
5
10
50
100
n


1.0
y
0.5
0.0
0.0
−0.5
n C(x,y)
−1.0
1.0
0.5
x
0.0



0.6
0.5
0.4
0.3
0.2
0.1
0.0
D_Z
0.0
0.2
0.4
0.6
0.8
1.0
Δ
0.370
0.365
0.360
0.355
0.350
0.345
0.540
0.545
0.550
0.555
0.560

The formal expansion

$$\begin{aligned}
\hat{\mathcal{L}}\rho_\infty &= 0, \\
\hat{\mathcal{L}} &= -\mathrm{i}\,\mathrm{ad}\,H + \varepsilon\hat{\mathcal{D}}, \\
\rho_\infty &= \sum_{p=0}^{\infty}(\mathrm{i}\varepsilon)^p \rho^{(p)}
\end{aligned}$$

implies an operator-valued recurrence:

The formal expansion

$$\begin{aligned}\hat{\mathcal{L}}\rho_\infty &= 0,\\ \hat{\mathcal{L}} &= -\mathrm{i}\,\mathrm{ad}\,H + \varepsilon\hat{\mathcal{D}},\\ \rho_\infty &= \sum_{p=0}^{\infty}(\mathrm{i}\varepsilon)^p\rho^{(p)}\end{aligned}$$

implies an operator-valued recurrence:

$$\begin{aligned}[H,\rho^{(0)}] &= 0,\\ (\,\mathrm{ad}\,H)\rho^{(p+1)} &= -\hat{\mathcal{D}}(\rho^{(p)}), \quad p = 0,1,2,\ldots\end{aligned}$$

NESS: The weak coupling perturbation expansion: explicit solution

$$
\begin{aligned}
2^0 \rho^{(0)} &= \mathbb{1}, \\
2^n \rho^{(1)} &= \mu (Z - Z^\dagger), \\
2^n \rho^{(2)} &= \frac{\mu^2}{2} (Z - Z^\dagger)^2 - \frac{\mu}{2} [Z, Z^\dagger].
\end{aligned}
$$

$$
\begin{aligned}
2^0 \rho^{(0)} &= \mathbb{1}, \\
2^n \rho^{(1)} &= \mu (Z - Z^\dagger), \\
2^n \rho^{(2)} &= \frac{\mu^2}{2}(Z - Z^\dagger)^2 - \frac{\mu}{2}[Z, Z^\dagger].
\end{aligned}
$$

$$
Z = \sum_{(s_1,\ldots,s_n)\in\{+,-,0\}^n} \langle \mathrm{L} | \mathbf{A}_{s_1} \mathbf{A}_{s_2} \cdots \mathbf{A}_{s_n} | \mathrm{R} \rangle \sigma^{s_2} \otimes \sigma^{s_2} \cdot \otimes \sigma^{s_n}
$$

$$
\begin{aligned}
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2^n \rho^{(1)} &= \mu (Z - Z^\dagger), \\
2^n \rho^{(2)} &= \frac{\mu^2}{2} (Z - Z^\dagger)^2 - \frac{\mu}{2} [Z, Z^\dagger].
\end{aligned}
$$

$$
Z = \sum_{(s_1,\ldots,s_n)\in\{+,-,0\}^n} \langle \mathrm{L} | \mathbf{A}_{s_1} \mathbf{A}_{s_2} \cdots \mathbf{A}_{s_n} | \mathrm{R} \rangle \sigma^{s_2} \otimes \sigma^{s_2} \cdot \otimes \sigma^{s_n}
$$

$$
\begin{aligned}
\mathbf{A}_0 &= |\mathrm{L}\rangle\langle\mathrm{L}| + |\mathrm{R}\rangle\langle\mathrm{R}| + \sum_{r=1}^{\infty} \cos(r\lambda) \, |r\rangle\langle r|, \qquad \cos\lambda \equiv \Delta \\
\mathbf{A}_+ &= |\mathrm{L}\rangle\langle 1| + c \sum_{r=1}^{\infty} \sin\left(2 \left\lfloor \frac{r+1}{2} \right\rfloor \lambda\right) |r\rangle\langle r+1|, \\
\mathbf{A}_- &= |1\rangle\langle\mathrm{R}| - c^{-1} \sum_{r=1}^{\infty} \sin\left(\left(2\left\lfloor \frac{r}{2} \right\rfloor + 1\right)\lambda\right) |r+1\rangle\langle r|,
\end{aligned}
$$

The cubic algebra

$$
\begin{aligned}
[\mathbf{A}_0, \mathbf{A}_\pm \mathbf{A}_\mp] &= 0, \\
\{\mathbf{A}_0, \mathbf{A}_\pm^2\} &= 2\Delta \mathbf{A}_\pm \mathbf{A}_0 \mathbf{A}_\pm, \\
2\Delta\{\mathbf{A}_0^2, \mathbf{A}_\pm\} - 4\mathbf{A}_0 \mathbf{A}_\pm \mathbf{A}_0 &= \{\mathbf{A}_\mp, \mathbf{A}_\pm^2\} - 2\mathbf{A}_\pm \mathbf{A}_\mp \mathbf{A}_\pm, \\
2\Delta[\mathbf{A}_0^2, \mathbf{A}_\pm] &= [\mathbf{A}_\mp, \mathbf{A}_\pm^2].
\end{aligned}
$$

The boundary relations:

$$
\begin{aligned}
\langle \mathrm{L}|\mathbf{A}_- &= \langle \mathrm{L}|\mathbf{A}_+ \mathbf{A}_- \mathbf{A}_+ = \langle \mathrm{L}|\mathbf{A}_+ \mathbf{A}_-^2 = 0, \\
\mathbf{A}_+|\mathrm{R}\rangle &= \mathbf{A}_- \mathbf{A}_+ \mathbf{A}_-|\mathrm{R}\rangle = \mathbf{A}_+^2 \mathbf{A}_-|\mathrm{R}\rangle = 0, \\
\langle \mathrm{L}|\mathbf{A}_0 &= \langle \mathrm{L}|, \quad \mathbf{A}_0|\mathrm{R}\rangle = |\mathrm{R}\rangle. \quad \langle \mathrm{L}|\mathbf{A}_+ \mathbf{A}_-|\mathrm{R}\rangle = 1.
\end{aligned}
$$

Transfer matrix calculation of observables

$$
\begin{aligned}
M_j &= \langle \sigma_j^{\rm z} \rangle = \varepsilon^2 \mu \langle {\rm L} | \mathbf{T}^{j-1} \mathbf{V} \mathbf{T}^{n-j} | {\rm R} \rangle, \\
C_{j,k} &= \langle \sigma_j^{\rm z} \sigma_k^{\rm z} \rangle = \varepsilon^2 \mu^2 \langle {\rm L} | \mathbf{T}^{j-1} \mathbf{V} \mathbf{T}^{k-j-1} \mathbf{V} \mathbf{T}^{n-k} | {\rm R} \rangle, \; j < k,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T} &= |{\rm L}\rangle\langle {\rm L}| + |{\rm R}\rangle\langle {\rm R}| + \frac{1}{2}(|{\rm L}\rangle\langle 1| + |1\rangle\langle {\rm R}|) \\
&+ \sum_{r=1}^{\infty} \Bigg\{ \cos^2(r\lambda)\, |r\rangle\langle r| + \frac{c^2}{2} \sin^2\left(2 \left\lfloor \frac{r+1}{2} \right\rfloor \lambda\right) |r\rangle\langle r+1| \\
&\qquad + \frac{c^{-2}}{2} \sin^2\left(\left(2\left\lfloor \frac{r}{2} \right\rfloor + 1\right)\lambda\right) |r+1\rangle\langle r| \Bigg\}, \\
\mathbf{V} &= \frac{|{\rm L}\rangle\langle 1|}{2} - \frac{|1\rangle\langle {\rm R}|}{2} + \sum_{r=1}^{\infty} \Bigg\{ \frac{c^2}{2} \sin^2\left(2\left\lfloor \frac{r+1}{2} \right\rfloor \lambda\right) |r\rangle\langle r+1| \\
&\qquad - \frac{c^{-2}}{2} \sin^2\left(\left(2\left\lfloor \frac{r}{2} \right\rfloor + 1\right)\lambda\right) |r+1\rangle\langle r| \Bigg\},
\end{aligned}
$$

Mazur bound on infinite temperature spin Drude weight

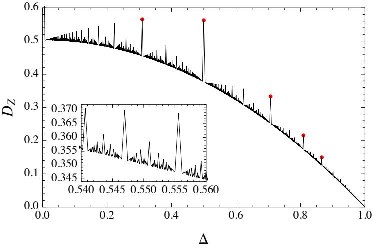
$$D_n = \lim_{t\to\infty} \frac{\beta}{2nt}\int_0^t \mathrm{d}t' \langle J(t')J\rangle \geq \frac{\beta}{4} D_{\mathrm{Mazur}} := \frac{\beta}{2n}\sum_k \frac{(J,Q_k)^2}{(Q_k,Q_k)},$$

We take $Q = \mathrm{i}(Z - Z^\dagger)$ satisfying $[H,Q] = -2\mathrm{i}\sigma_1^{\mathrm{z}} + 2\mathrm{i}\sigma_n^{\mathrm{z}}$

$$D_{\mathrm{Mazur}} = \frac{1}{4}\lim_{n\to\infty}\frac{n}{\langle \mathtt{L}|\mathbf{T}^n|\mathtt{R}\rangle}.$$

Jordan decomposition of the transfer matrix $\mathbf{T}$ yields explicit fractal dependence:

$$D_{\mathrm{Mazur}}\left(\Delta = \cos(\pi l/m)\right) = \frac{1}{2}(1-\Delta^2)\frac{m}{m-1}$$



Non-perturbative NESS (arbitrary coupling)

$$\rho_\infty = (\operatorname{tr} R)^{-1} R, \quad R = SS^\dagger, \quad S = \sum_{(s_1,\ldots,s_n)\in\{+,-,0\}^n} \langle 0|\mathbf{A}_{s_1}\mathbf{A}_{s_2}\cdots\mathbf{A}_{s_n}|0\rangle \sigma^{s_1}\otimes\sigma^{s_2}\cdots\otimes\sigma^{s_n}$$

$$\begin{aligned}
\mathbf{A}_0 &= |0\rangle\langle 0| + \sum_{r=1}^{\infty} a_r^0 |r\rangle\langle r|, \\
\mathbf{A}_+ &= \mathrm{i}\varepsilon|0\rangle\langle 1| + \sum_{r=1}^{\infty} a_r^+ |r\rangle\langle r+1|, \\
\mathbf{A}_- &= |1\rangle\langle 0| + \sum_{r=1}^{\infty} a_r^- |r+1\rangle\langle r|, \\
a_r^0 &= \cos(r\lambda) + \mathrm{i}\varepsilon\frac{\sin(r\lambda)}{2\sin\lambda}, \\
a_{2k-1}^+ &= c\sin(2k\lambda) + \mathrm{i}\varepsilon\frac{c\sin((2k-1)\lambda)\sin(2k\lambda)}{2(\cos((2k-1)\lambda)+\tau_{2k-1})\sin\lambda}, \\
a_{2k}^+ &= c\sin(2k\lambda) - \mathrm{i}\varepsilon\frac{c(\cos(2k\lambda)+\tau_{2k})}{2\sin\lambda}, \\
a_{2k-1}^- &= -\frac{\sin((2k-1)\lambda)}{c} + \mathrm{i}\varepsilon\frac{\cos((2k-1)\lambda)+\tau_{2k-1}}{2c\sin\lambda}, \\
a_{2k}^- &= -\frac{\sin((2k+1)\lambda)}{c} - \mathrm{i}\varepsilon\frac{\sin(2k\lambda)\sin((2k+1)\lambda)}{2c(\cos(2k\lambda)+\tau_{2k})\sin\lambda}.
\end{aligned}$$

Exactly the same cubic algebra (!)

$$\begin{aligned}
[\mathbf{A}_0, \mathbf{A}_\pm \mathbf{A}_\mp] &= 0, \\
\{\mathbf{A}_0, \mathbf{A}_\pm^2\} &= 2\Delta \mathbf{A}_\pm \mathbf{A}_0 \mathbf{A}_\pm, \\
2\Delta\{\mathbf{A}_0^2, \mathbf{A}_\pm\} - 4\mathbf{A}_0 \mathbf{A}_\pm \mathbf{A}_0 &= \{\mathbf{A}_\mp, \mathbf{A}_\pm^2\} - 2\mathbf{A}_\pm \mathbf{A}_\mp \mathbf{A}_\pm, \\
2\Delta[\mathbf{A}_0^2, \mathbf{A}_\pm] &= [\mathbf{A}_\mp, \mathbf{A}_\pm^2].
\end{aligned}$$

with modified boundary relations:

$$\begin{aligned}
\langle 0|\mathbf{A}_- &= \langle 0|\mathbf{A}_+(\mathbf{A}_-\mathbf{A}_+ - \mathrm{i}\varepsilon\mathbb{1}) = \langle 0|\mathbf{A}_+\mathbf{A}_-^2 = 0, \\
\mathbf{A}_+|0\rangle &= (\mathbf{A}_-\mathbf{A}_+ - \mathrm{i}\varepsilon\mathbb{1})\mathbf{A}_-|0\rangle = \mathbf{A}_+^2\mathbf{A}_-|0\rangle = 0, \\
\langle 0|\mathbf{A}_0 &= \langle 0|, \quad \mathbf{A}_0|0\rangle = |0\rangle, \quad \langle 0|\mathbf{A}_+\mathbf{A}_-|0\rangle = \mathrm{i}\varepsilon.
\end{aligned}$$

$$[\mathbf{T},[\mathbf{T},\mathbf{V}]] = -\frac{\varepsilon^2}{4}(2\mathbf{V} + \{\mathbf{T},\mathbf{V}\}),$$
$$\langle 0|(\mathbf{T}-\mathbf{V}) = \langle 0|, \quad (\mathbf{T}+\mathbf{V})|0\rangle = |0\rangle,$$
$$\frac{\langle 0|\mathbf{T}^n|0\rangle}{\langle 0|\mathbf{T}^{n-1}|0\rangle} \simeq \varepsilon^2\left(\frac{(4n-3)^2}{32\pi^2} - \alpha\right) + 1 + \mathcal{O}(n^{-1}),$$

In the continuum limit $M(x \equiv \frac{j-1}{n-1}) := \langle \sigma^{\mathrm{z}}_j \rangle$ we get ODE $M'' = -\pi^2 M$ /w b.c. $M(0) = -M(1) = 1$

$$M(x) = \cos(\pi x) + \mathcal{O}(\frac{1}{n})$$

Similarly for the 2-point correlator $C(x \equiv \frac{j-1}{n-1}, y \equiv \frac{k-1}{n-1}) := \langle \sigma^{\mathrm{z}}_j \sigma^{\mathrm{z}}_k \rangle - \langle \sigma^{\mathrm{z}}_j \rangle \langle \sigma^{\mathrm{z}}_k \rangle$,

$$\begin{aligned} C(x,y) &= \frac{\pi}{4n} f(\min(x,y),\max(x,y)) + \mathcal{O}(\frac{1}{n^2}) \\ f(x,y) &= 2\pi x(y-1)\sin(\pi x)\sin(\pi y) \\ &+ \cos(\pi x)((1-2y)\sin(\pi y) + \pi(y-1)y\cos(\pi y)). \end{aligned}$$

Conclusions

- t-DMRG in Liouville space is an efficient simulation technique to capture non-equilibrium steady states of 1D quantum chains
- Non-equilibrium boundary driving allows for exact solutions in some cases: emerging "non-equilibrium integrability" (?)
- Spin and charge diffusion observed in certain fully coherent strongly interacting systems, even in some cases where the bulk is Bethe Ansatz integrable (!)