2384-21

ICTP Latin-American Advanced Course on FPGADesign for Scientific Instrumentation

19 November - 7 December, 2012

Transformada Z

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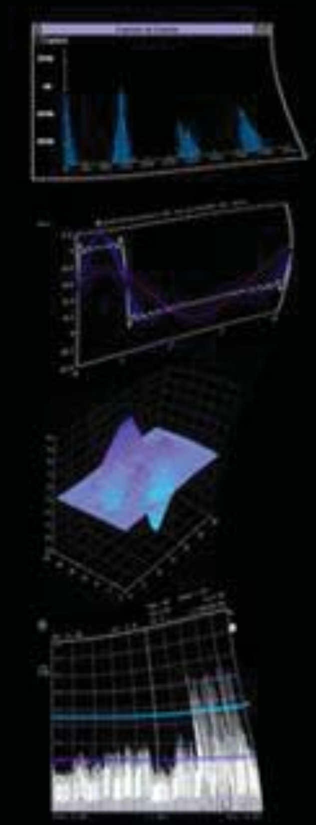
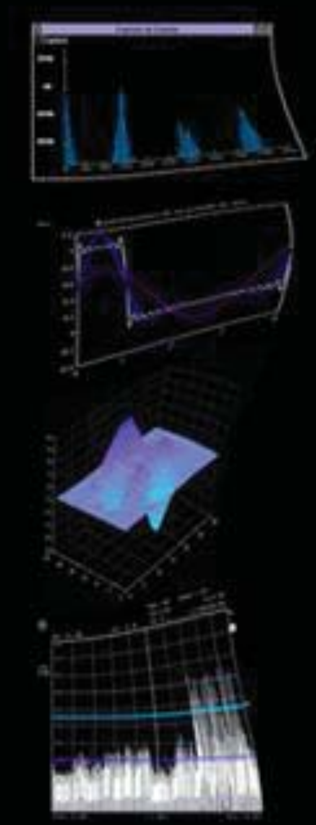
ARGENTINA



Transformada Z

Transformada Z

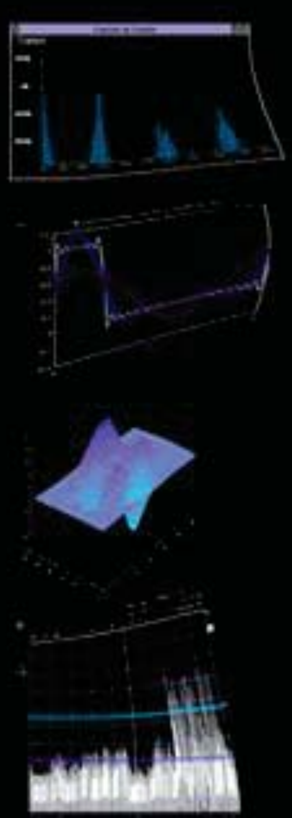
Transformada de Laplace



Transformada	Antitransformada
$TL\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$	$TL^{-1}\{X(s)\} = x(t) = \int_{-\infty}^{\infty} X(s)e^{ts}ds$

Transformada Z

Transformada Z



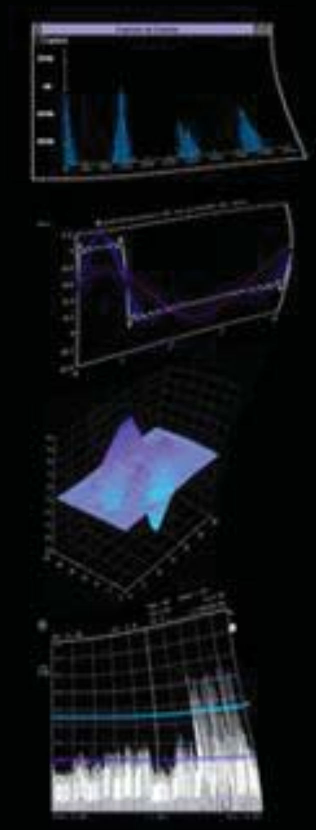
Transformada	Antitransformada
$TZ\left\{x[n]\right\} = X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$	$TZ^{-1}\left\{X(z)\right\} = x[n] = \frac{1}{j2\pi}\oint_C X(z) z^{n-1} dz$

Región de convergencia

$$z \in ROC \Leftrightarrow X(z) = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty$$

Transformada Z

Polos y ceros



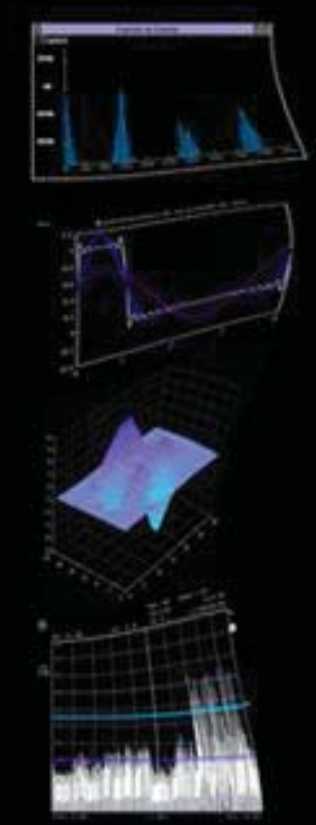
Función polinómica

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = b_0 + b_1 z^{-1} + ... + b_M z^{-M} = \frac{b_0 z^M + ... + b_{M-1} z^1 + b_M}{z^M}$$

$$H(z) = z^{-M}\left(z - c_1\right)...\left(z - c_M\right) = z^{-M} \prod_{k=1}^{M} \left(z - c_k\right)$$

Transformada Z

Polos y ceros



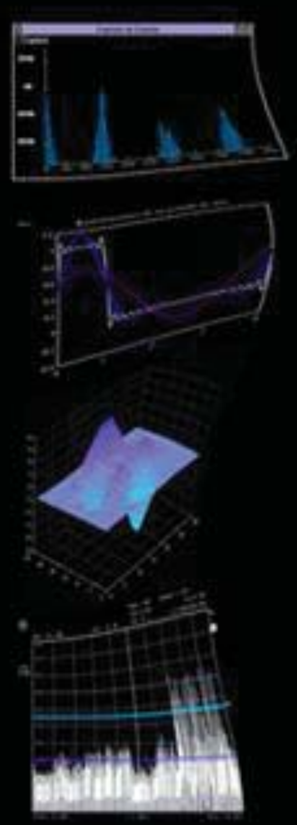
Función polinómica racional

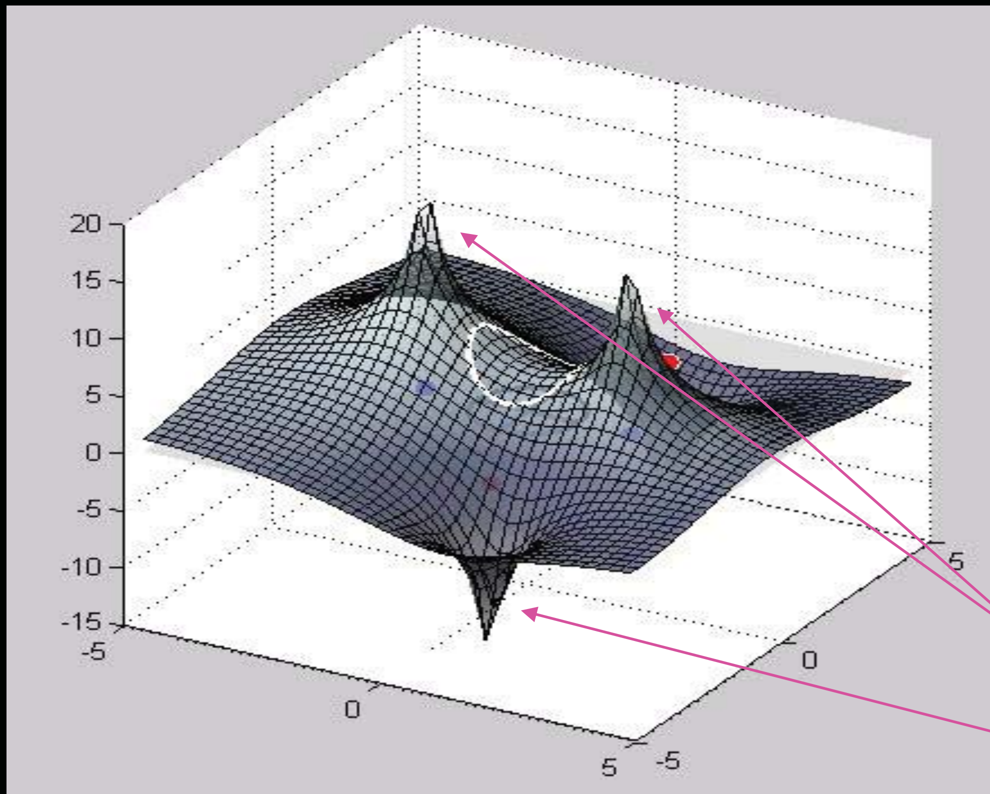
$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{b_0+b_1 z^{-1}+\ldots+b_M z^{-M}}{a_0+a_1 z^{-1}+\ldots+a_N z^{-N}}=\frac{z^N}{z^M}\,\frac{b_0 z^M+\ldots+b_{M-1} z^1+b_M}{a_0 z^N+\ldots+a_{N-1} z^1+a_N}$$

$$H(z)=z^{N-M}\,\frac{(z-c_1)\ldots(z-c_M)}{(z-p_1)\ldots(z-p_N)}=z^{N-M}\,\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-p_k)}$$

Transformada Z

Polos y ceros



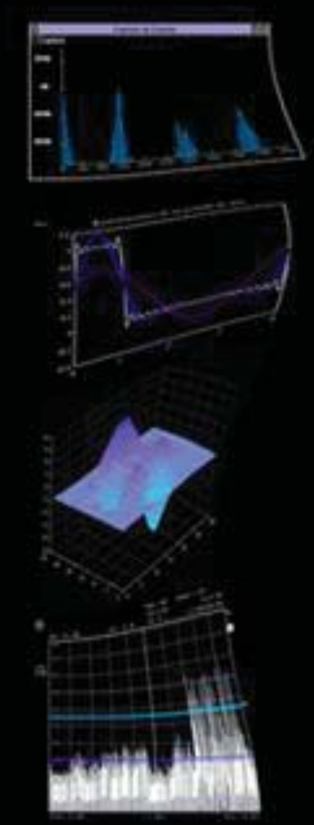


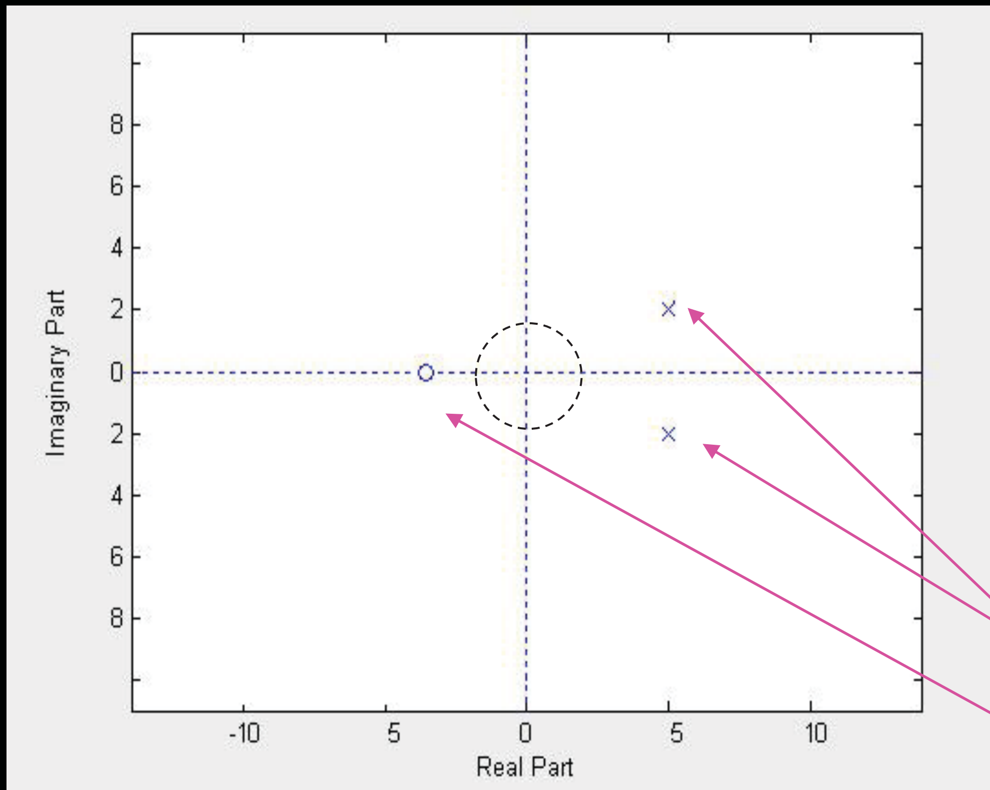
$$\lim_{z \to p_k} H(z) = \infty$$

$$\lim_{z \to z_k} H(z) = 0$$

Transformada Z

Polos y ceros





$$\lim_{z \to p_k} H(z) = \infty$$

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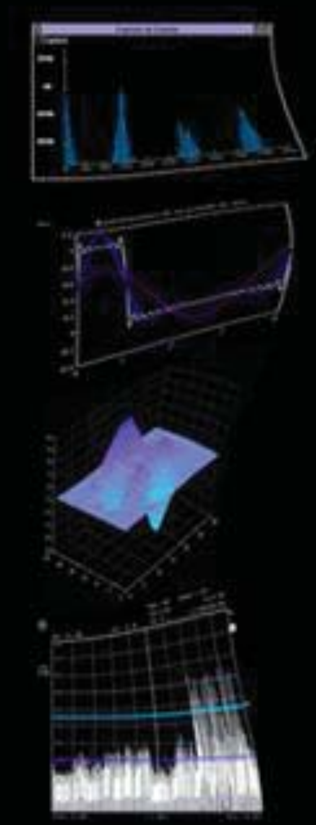
Transformada Z

Polos y ceros



Función racional

Número de polos = número de ceros

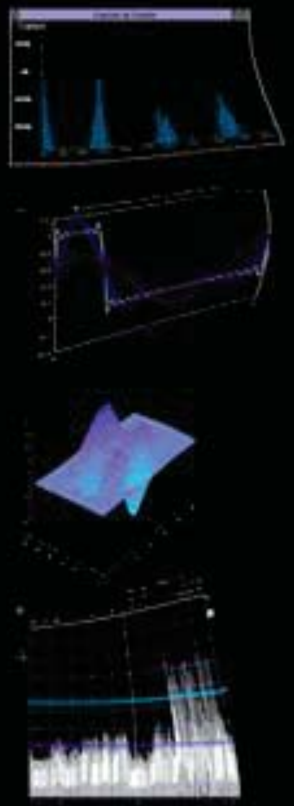
Polos y ceros **triviales**: Ubicados en z=0 ó z=∞

Número de ceros **no triviales**: M

Número de polos **no triviales**: N

Transformada Z

Serie Geométrica



$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} = 1 + a + a^2 + \ldots \quad si \quad |a| < 1$$

$$\sum_{n=1}^{\infty} a^n = a + a^2 + \ldots = \frac{1}{1-a} - 1 = \frac{a}{1-a} \quad si \quad |a| < 1$$

$$\sum_{n=2}^{\infty} a^n = \frac{a^2}{1-a} \quad si \quad |a| < 1$$

$$\vdots$$

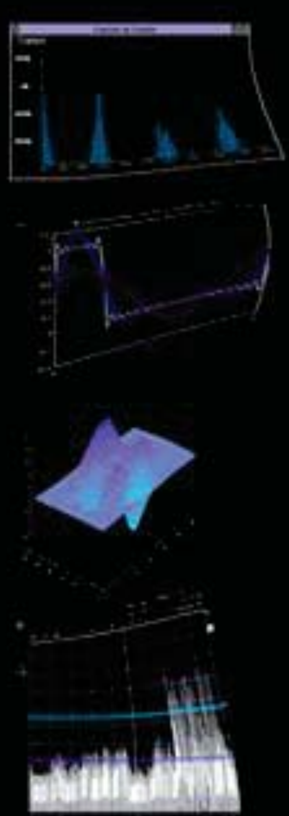
$$\sum_{n=N}^{\infty} a^n = \frac{a^N}{1-a} \quad si \quad |a| < 1$$

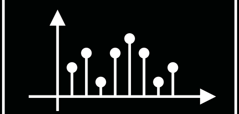
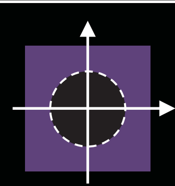
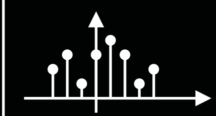
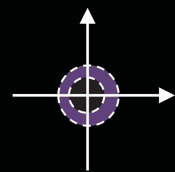
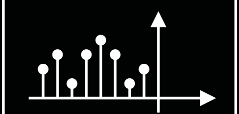
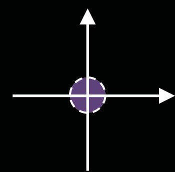
$$\vdots$$

$$\sum_{n=0}^{N-1} a^n = \sum_{n=0}^{\infty} a^n - \sum_{n=N}^{\infty} a^n = \frac{1-a^N}{1-a} \quad si \quad |a| < 1$$

Transformada Z

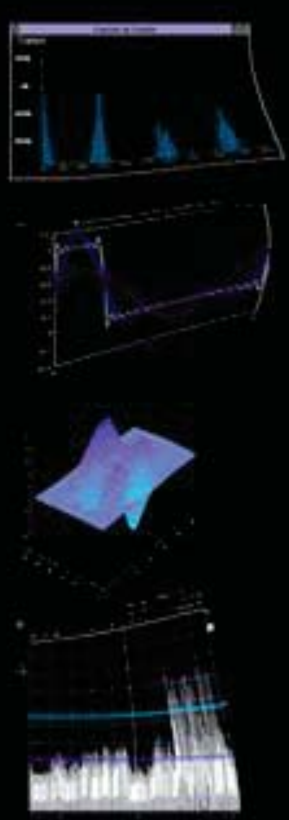
ROC

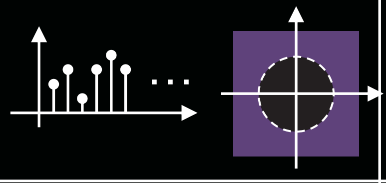
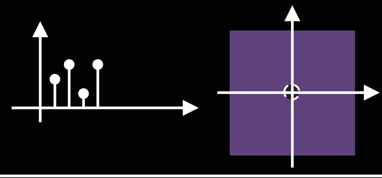
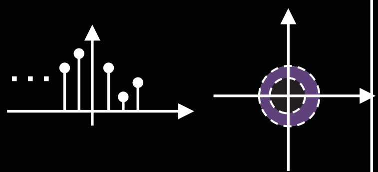
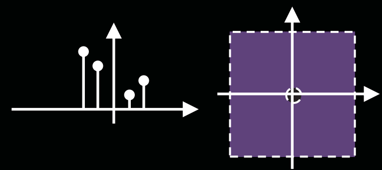
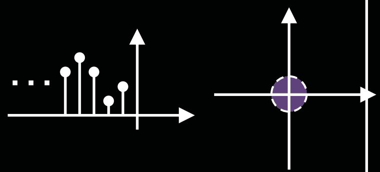
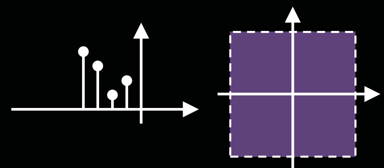


 unilateral derecha	$X(z)=\sum_{n=n_0}^{\infty} x[n]z^{-n} < \infty \quad \forall \quad \lvert a\rvert < \lvert z\rvert \quad ; \quad n_0 \ge 0$	
 bilateral	$X(z)=\sum_{n=n_1}^{n_2} x[n]z^{-n} < \infty \quad \forall \quad \lvert a_1\rvert < \lvert z\rvert < \lvert a_2\rvert \quad ; \quad n_1 < 0 < n_2$	
 unilateral izquierda	$X(z)=\sum_{n=-\infty}^{n_0} x[n]z^{-n} \quad \forall \quad \lvert z\rvert < \lvert a\rvert \quad ; \quad n_0 < 0$	

Transformada Z

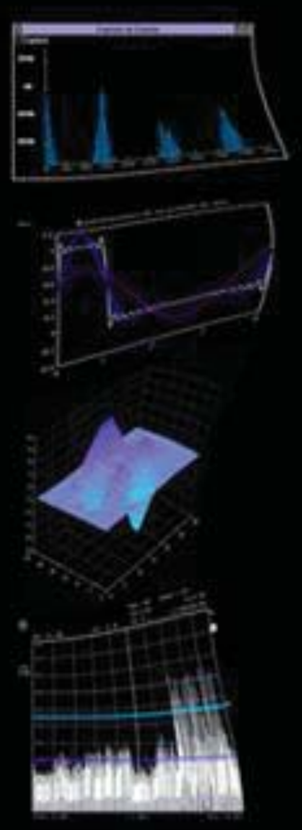
ROC



Lateralidad	Longitud infinita	Longitud finita
Derecha		
Bilateral		
Izquierda		

Transformada Z

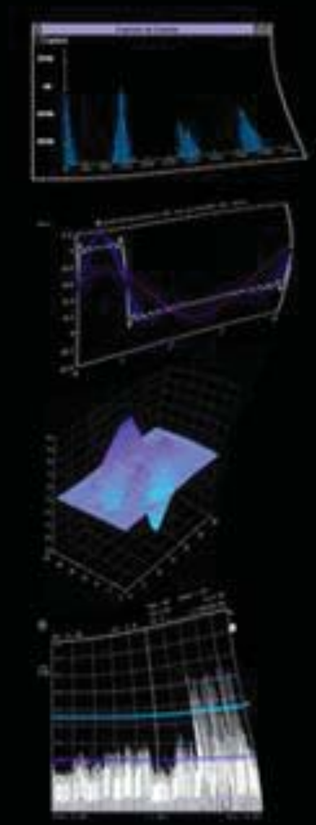
Tabla de pares de funciones y transformadas



Antitransformada	Transformada	ROC
$\delta[n]$	1	Z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$\cos[\omega n]u[n]$	$\frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z\rvert > 1$
$sen[\omega n]u[n]$	$\frac{z^{-1}\,\mathrm{sen}\,\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos[\omega n]u[n]$	$\frac{az^{-1}\,\mathrm{sen}\,\omega}{1-2az^{-1}\cos\omega+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n sen[\omega n]u[n]$	$\frac{1-az^{-1}\cos\omega}{1-2az^{-1}\cos\omega+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Transformada Z

Propiedades



Linealidad

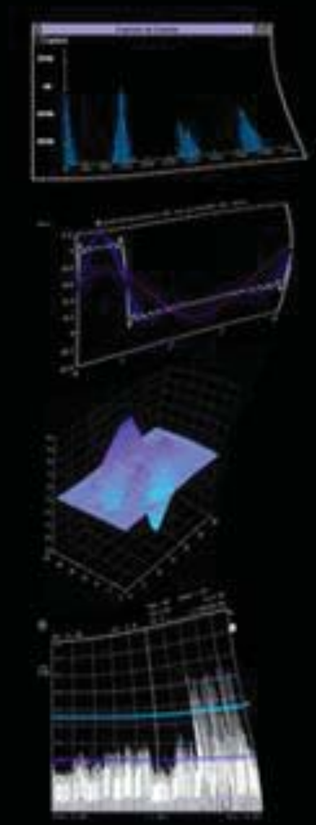
$$c_1 x_1[n] + c_2 x_2[n] \overset{TZ}{\leftrightarrow} c_1 X_1(z) + c_2 X_2(z) \quad ; \quad ROC = ROC_1 \cap ROC_2$$

Convolución

$$x_1[n] * x_2[n] \overset{TZ}{\leftrightarrow} X_1(z) X_2(z) \quad ; \quad ROC = ROC_1 \cap ROC_2$$

Transformada Z

Propiedades



Desplazamiento Temporal

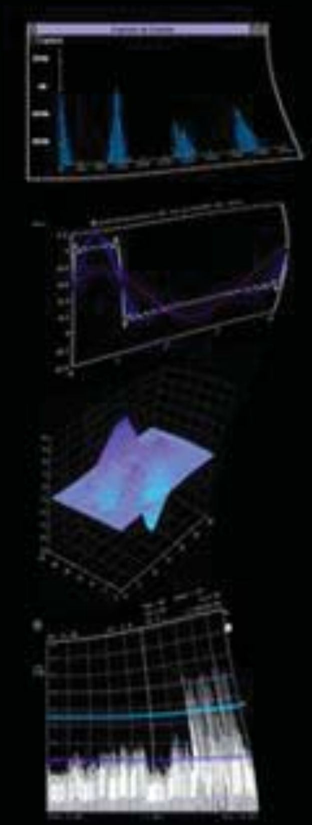
$$x[n-k] \overset{TZ}{\leftrightarrow} z^{-k}X(z) \quad ; \quad = ROC \vee ROC \not\subset 0 \vee ROC \not\subset \infty$$

Reflexión Temporal

$$x[-n] \overset{TZ}{\leftrightarrow} X\left(z^{-1}\right) \quad ; \quad ROC : \frac{1}{r_{\text{sup}}} < |z| < \frac{1}{r_{\text{inf}}};$$

Transformada Z

Propiedades



Escalado en el Dominio Z

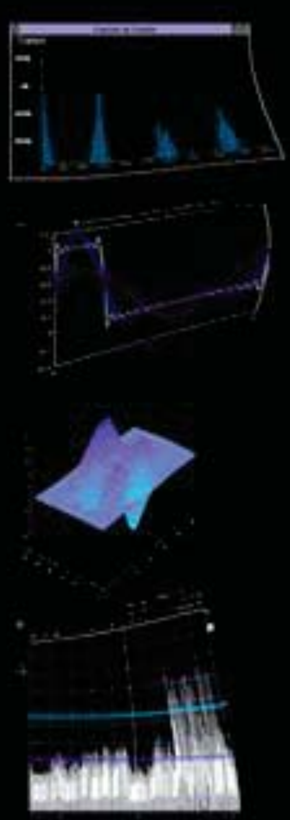
$$a^n x[n] \overset{TZ}{\leftrightarrow} X\left(\frac{z}{a}\right) \quad ; \quad ROC: |a| r_{\text{inf}} < |z| < |a| r_{\text{sup}}$$

Diferenciación en el Dominio Z

$$nx[n] \overset{TZ}{\leftrightarrow} -z\frac{\partial X(z)}{\partial z} \quad ; \quad = ROC \vee ROC \not\subset 0 \vee ROC \not\subset \infty$$

Transformada Z

Antitransformada Z

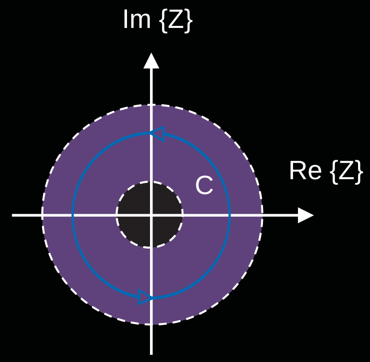


Cálculo directo

Integración en contorno C en sentido antihorario

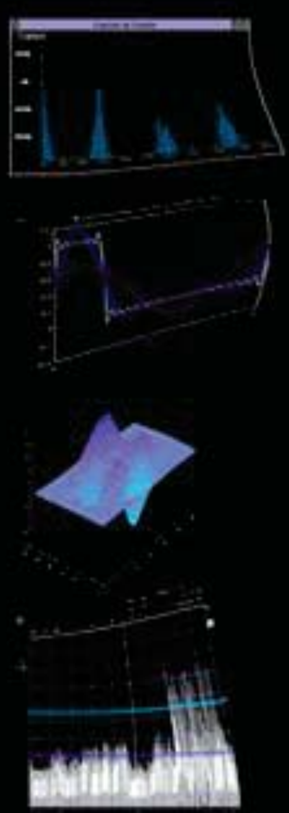
ROC $\subset$ C y z=0

ROC $\subset$ z_i simples



Transformada Z

Antitransformada Z



Cálculo directo

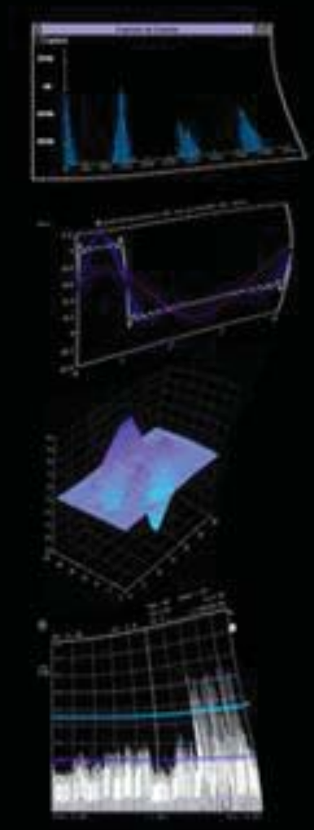
$$h[n] = \frac{1}{j2\pi} \oint_C H(z) z^{n-1} dz =$$

$$= \sum_{\text{polos simples } \{z_i\} \text{ en } C} \text{residuos}\left\{H(z) z^{n-1}\right\}\Big|_{z=z_i} =$$

$$= \sum_i (z - z_i) H(z) z^{n-1}\Big|_{z=z_i}$$

Transformada Z

Antitransformada Z



Expansión en Serie de Potencias

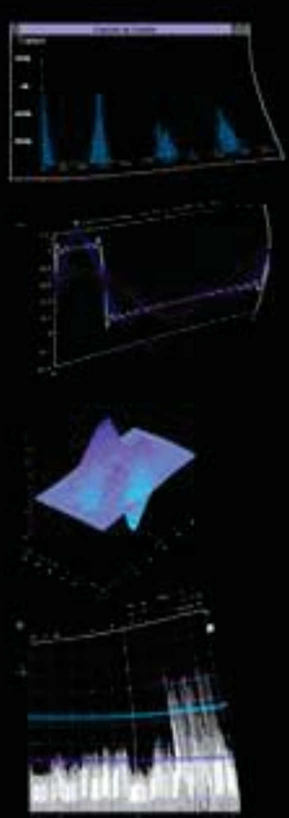
$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{b_0+b_1 z^{-1}+...+b_M z^{-M}}{a_0+a_1 z^{-1}+...+a_N z^{-N}}=\frac{b_M z^{-M}+...+b_1 z^{-1}+b_0}{a_N z^{-N}+...+a_1 z^{-1}+a_0}$$

División de polinomios en orden creciente o decreciente

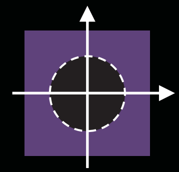
Por inspección se determina la antitransformada

Transformada Z

Antitransformada Z



Expansión en Series de z^{-k}



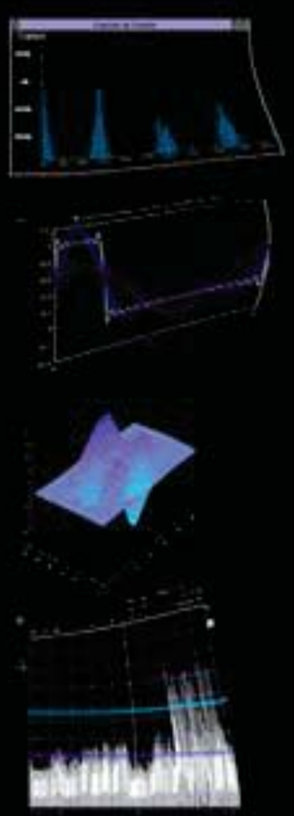
$$si \quad ROC: \quad |p_k| < |z|$$

División de polinomios en orden decreciente

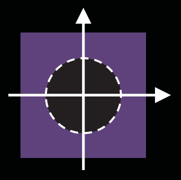
$$
\begin{array}{rl|l}
 & b_0 + b_1 z^{-1} + \ldots + b_M z^{-M} & a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N} \\
- & b_0 + d_1 z^{-1} + \ldots + d_M z^{-M} + \ldots + d_N z^{-N} & c_0 + c_1 z^{-1} + \ldots \\
\hline
 & e_1 z^{-1} + \ldots + e_M z^{-M} + \ldots + e_N z^{-N} & \\
- & e_1 z^{-1} + \ldots + f_M z^{-M} + \ldots + f_N z^{-N} + \ldots\ldots & \\
\hline
 & \vdots &
\end{array}
$$

Transformada Z

Antitransformada Z



Expansión en Series de z^{-k}



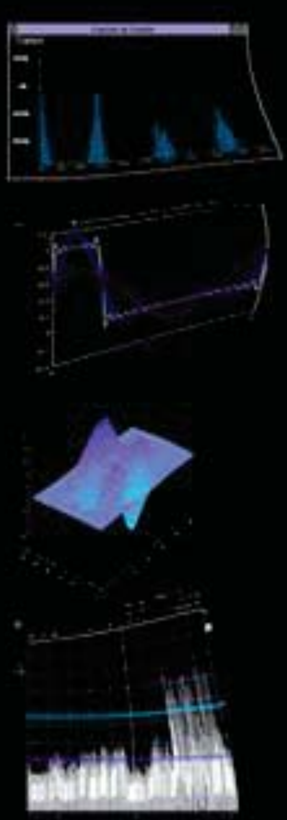
Por inspección se determina la antitransformada

$$H(z)=\frac{b_0+b_1z^{-1}+\ldots+b_Mz^{-M}}{a_0+a_1z^{-1}+\ldots+a_Nz^{-N}}=c_0+c_1z^{-1}+c_2z^{-2}+\ldots$$

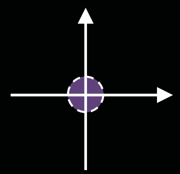
$$H(z)=c_0+c_1z^{-1}+\ldots\overset{Z}{\leftrightarrow}\left\{\underset{\uparrow}{c_0};c_1;c_2;\ldots;c_k;\ldots\right\}=h[n]$$

Transformada Z

Antitransformada Z



Expansión en Series de z^{+k}



$$si \quad ROC: \quad |z| < |p_k|$$

División de polinomios en orden creciente

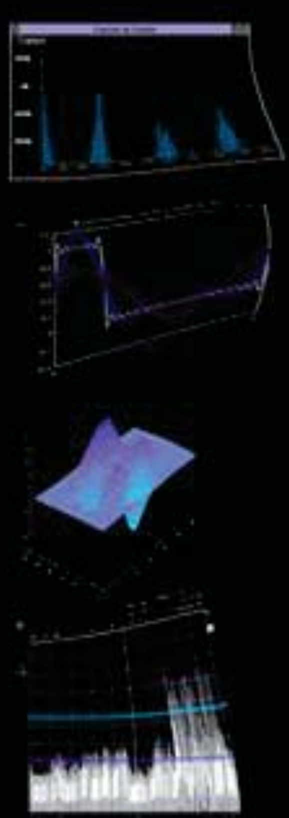
$$\begin{array}{rl} & b_M z^{-M} + \ldots + b_0 \\ - & b_M z^{-M} + \ldots + d_0 + \ldots + d_{-M+N} z^{-M+N} \\ \hline & +\ldots + e_0 + \ldots + e_{-M+N} z^{-M+N} \\ - & +\ldots + f_0 + \ldots + f_{-M+N} z^{-M+N} + \ldots.. \\ \hline & \vdots \end{array}$$

$$\left| \, a_N z^{-N} + \ldots + a_2 z^{-2} + a_1 z^{-1} + a_0 \right.$$

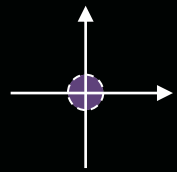
$$c_{M-N} z^{N-M} + c_{M+1-N} z^{N-M+1} + \ldots$$

Transformada Z

Antitransformada Z



Expansión en Series de z^{+k}



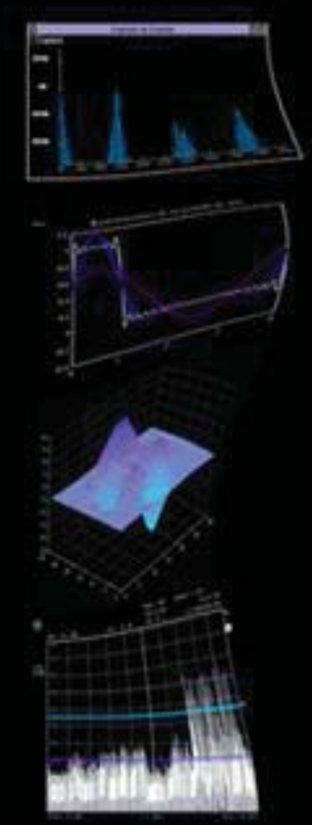
Por inspección se determina la antitransformada

$$H(z) = \frac{b_M z^{-M} + \ldots + b_1 z^{-1} + b_0}{a_N z^{-N} + \ldots + a_1 z^{-1} + a_0} = c_{M-N} z^{N-M} + c_{M+1-N} z^{N-M+1} + \ldots + c_{M+1-N} + \ldots$$

$$H(z) = c_{M-N} z^{N-M} + c_{M+1-N} z^{N-M+1} + \ldots \overset{Z}{\leftrightarrow} \left\{ \ldots; c_{M+1-N}; c_{M-N}; \ldots; \underset{\uparrow}{0} \right\} = h[n]$$

Transformada Z

Antitransformada Z



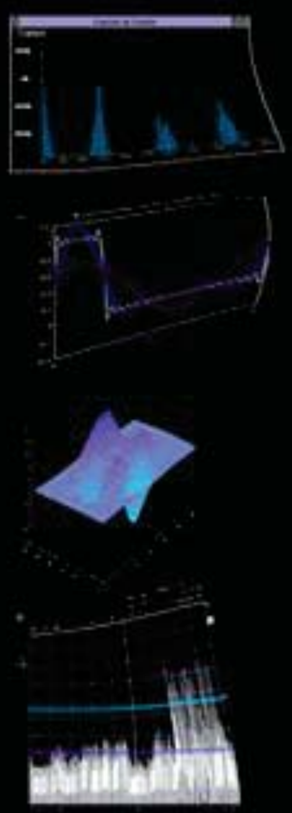
Expansión en Serie de Potencias

Una función H(v) puede expresarse como serie de potencias

$$H(v) = \sum_{n=0}^{\infty} \frac{H^{(n)}(v_0)(v - v_0)^n}{n!}$$

Transformada Z

Antitransformada Z



Expansión en Serie de Potencias

Una función H(v) puede expresarse como serie de potencias

$$H(v) = \sum_{n=0}^{\infty} \frac{H^{(n)}(v_0)(v - v_0)^n}{n!}$$

$$= H(v_0) + H^{'}(v_0)(v - v_0) + \frac{H^{''}(v_0)(v - v_0)^2}{2} + \ldots + \frac{H^{(k)}(v_0)(v - v_0)^k}{k!} + \ldots$$

Para eso se calculan las derivadas y se evaluan en v=v_0

$$H(v) \Rightarrow H(v_0)$$

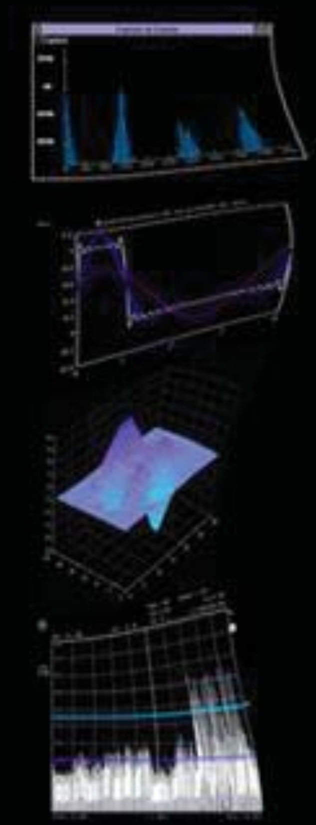
$$H^{'}(v) \Rightarrow H^{'}(v_0)$$

$$H^{''}(v) \Rightarrow H^{''}(v_0)$$

$$\vdots$$

Transformada Z

Antitransformada Z



Serie de Potencias

Si $v_0=0$

$$H(v)=\sum_{n=0}^{\infty}\frac{H^{(n)}(0)(v)^n}{n!}=H(0)+H^{'}(0)v+\frac{H^{''}(0)v^2}{2}+...+\frac{H^{(k)}(0)v^k}{k!}+...$$

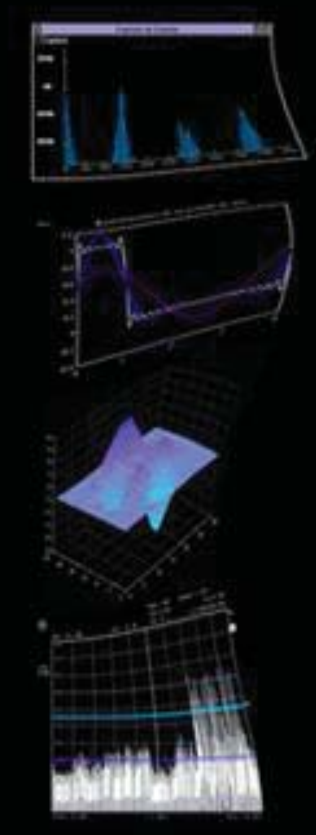
Si $v=z^{-1}$

$$H(z)=\sum_{n=0}^{\infty}\frac{H^{(n)}(0)z^{-n}}{n!}=H(0)+H^{'}(0)z^{-1}+\frac{H^{''}(0)z^{-2}}{2}+...+\frac{H^{(k)}(0)z^{-k}}{n!}+...$$

$$H(z)=\sum_{n=0}^{\infty}\frac{H^{(n)}(0)z^{-n}}{n!}\overset{z}{\leftrightarrow}\left\{\underset{\uparrow}{H(0)};H^{'}(0);\frac{H^{''}(0)}{2};...;\frac{H^{(k)}(0)}{k!};...\right\}=h\left[n\right]$$

Transformada Z

Antitransformada Z



Expansión en fracciones simples

Grados de los polinomios: °N(z)=M y °D(z)=N

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

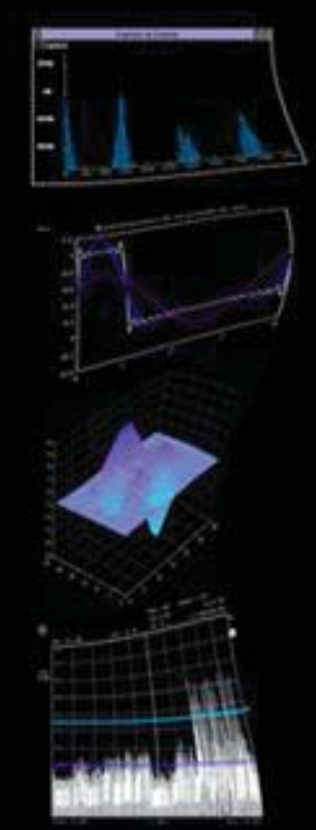
Si M≥N $\longrightarrow$ división previa para reducir el grado

Si M<N y no ∃ z_i de orden s≠1 $\longrightarrow$ cálculo directo

Si M<N y ∃ z_i de orden s≠1 $\longrightarrow$ cálculo separado

Transformada Z

Antitransformada Z



Expansión en fracciones simples

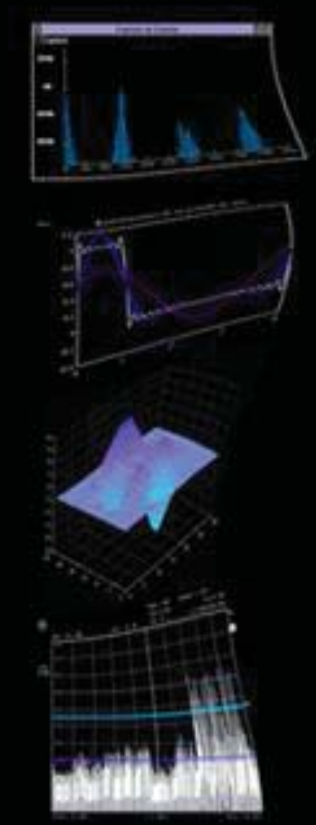
Si M≥N $\rightarrow$ división previa para reducir el grado

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + \ldots + c_{M-N} z^{-M-N} + \frac{R(z)}{D(z)}$$

donde °R(z)<°D(z)

Transformada Z

Antitransformada Z



Expansión en fracciones simples

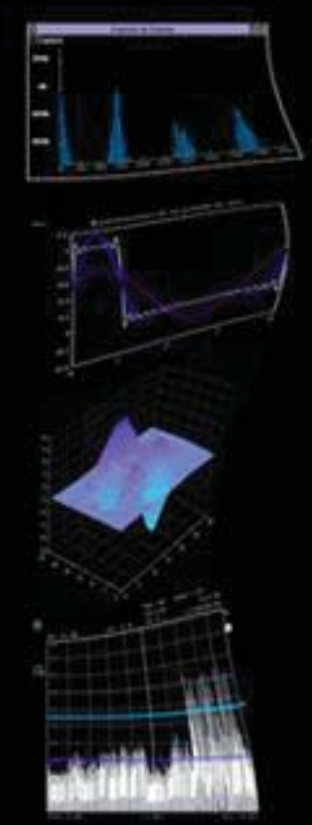
Si M<N y no ∃ z_i de orden s≠1 → cálculo directo

$$H(z) = \sum_{k=1}^{M} \frac{A_k}{1 - p_k z^{-1}}$$

$$\frac{A_k}{1 - p_k z^{-1}} \overset{TZ}{\leftrightarrow} \begin{cases} A_k [p_k]^n u[n] & si \quad ROC: \ |p_k| < |z| \\ -A_k [p_k]^n u[-n-1] & si \quad ROC: \ |z| < |p_k| \end{cases}$$

Transformada Z

Antitransformada Z



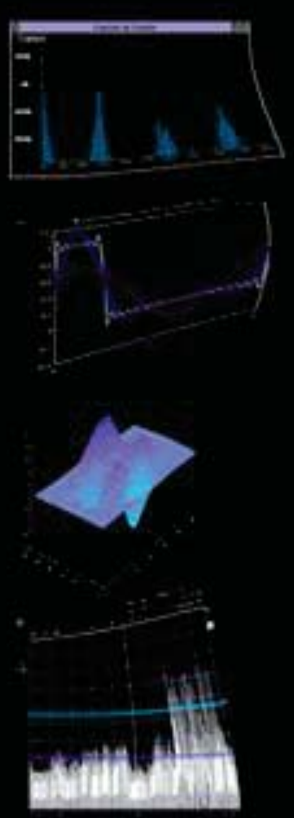
Expansión en fracciones simples

Polos complejos conjugados ↔ residuos complejos conjugados

$$\frac{A_k}{1-p_k z^{-1}}+\frac{A_k^*}{1-p_k^* z^{-1}}=\frac{b_0+b_1 z^{-1}}{a_0+a_1 z^{-1}+a_2 z^{-2}}$$

Transformada Z

Análisis de sistemas LIT



$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Updownarrow TZ \qquad \Updownarrow TZ$$

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

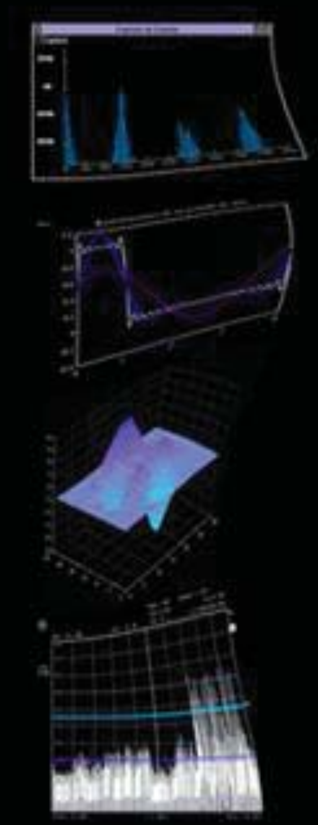
$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

$$\Downarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Transformada Z

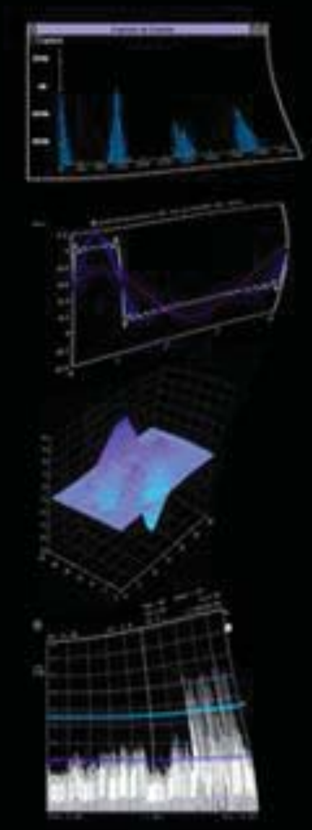
Análisis de sistemas LIT



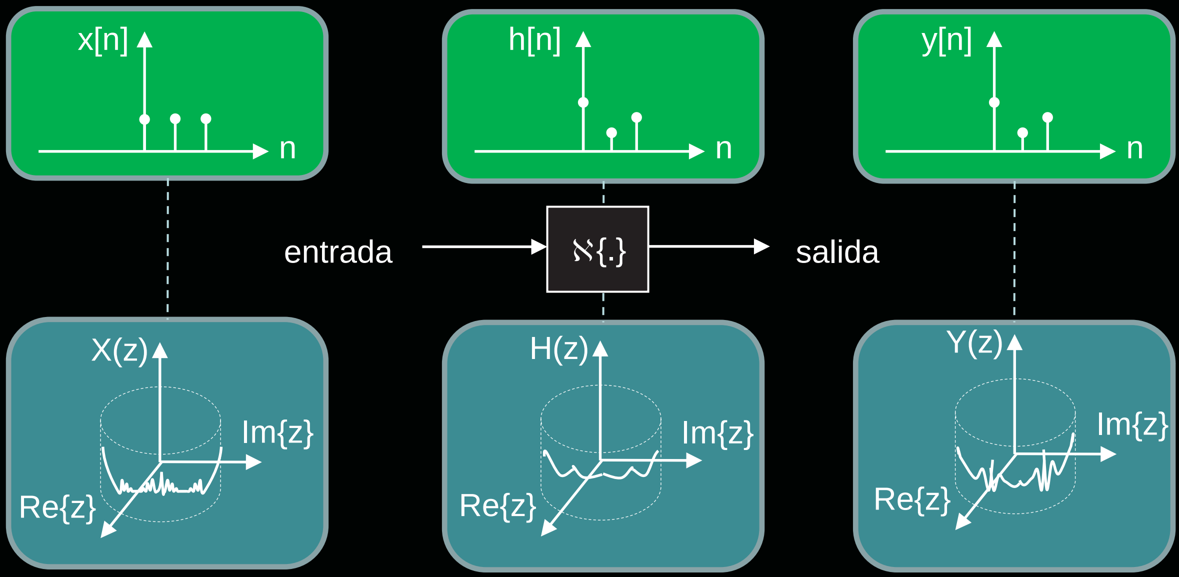
<table>
<tr><th>FIR</th><th>IIR</th></tr>
<tr><td rowspan="2">MA (Promediador Móvil)
$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$
M ceros y 1 polo de orden M</td><td>AR (Autoregresivo)
$H(z) = \frac{b_0}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}}$
N polos y 1 cero de orden N</td></tr>
<tr><td>ARMA (Promediador Móvil Autoregresivo)
$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$
M ceros y N polos no triviales</td></tr>
</table>

Transformada Z

Análisis de Sistemas LIT

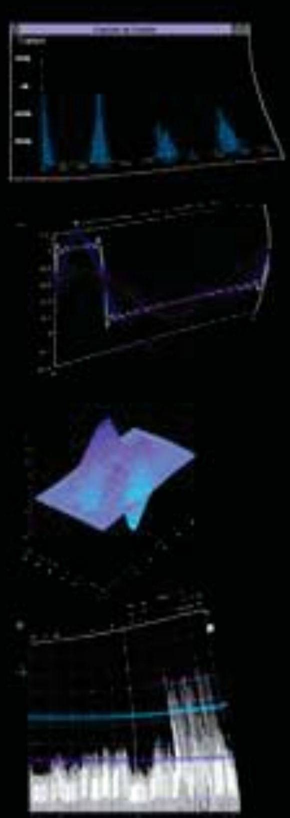


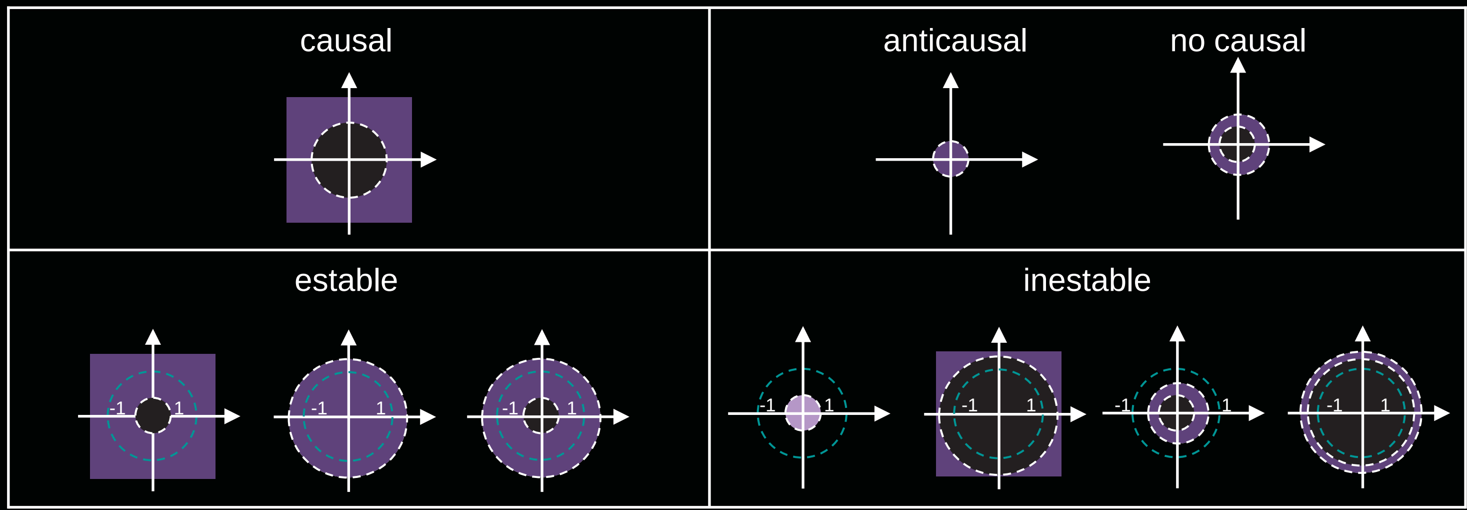
Transferencia H(z)



Transformada Z

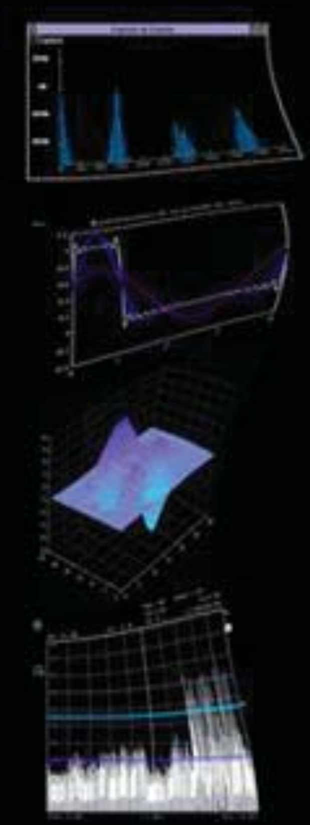
Análisis de sistemas LIT





Transformada Z

Análisis de sistemas LIT



Estabilidad

$$ROC: 1 < |z| \Rightarrow \quad H(z) \quad es \quad SE$$

Causalidad

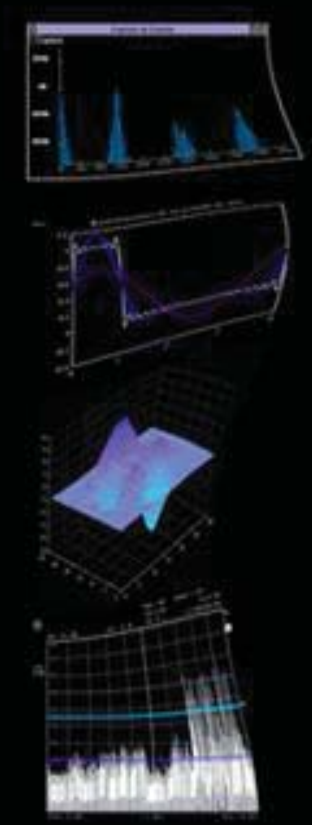
$$ROC: a < |z| \Rightarrow \quad H(z) \quad es \quad SC$$

Respuesta espectral

$$ROC \subset |z| = 1 \Rightarrow H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\prod_{s=1}^{M}(z - c_s)}{\prod_{k=1}^{N}(z - p_k)}$$

Transformada Z

Transformada Z unilateral



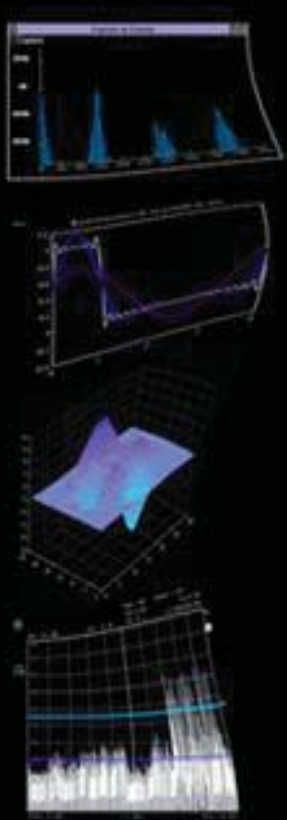
Transformada
$TZ^{+}\{x[n]\} = X^{+}(z) = \sum_{n=0}^{+\infty} x[n] z^{-n}$

Región de convergencia

$$ROC : a < |z|$$

Transformada Z

Propiedades



Retardo temporal

$$x[n-k] \overset{TZ^{+}}{\longleftrightarrow} z^{-k}\left[X^{+}(z)+\sum_{n=1}^{k} x[-n]z^{n}\right] \quad ; \quad k>0$$

Avance Temporal

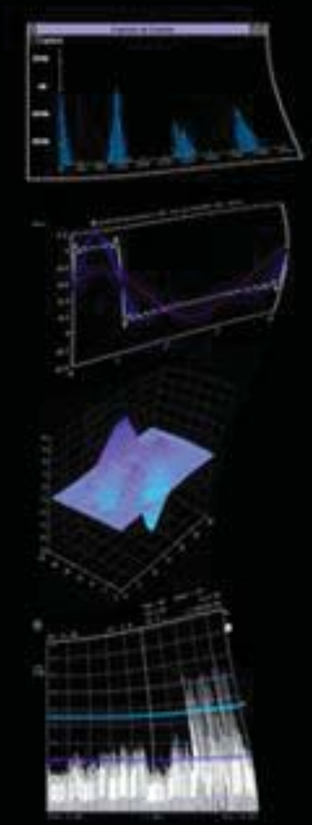
$$x[n+k] \overset{TZ^{+}}{\longleftrightarrow} z^{k}\left[X^{+}(z)-\sum_{n=0}^{k-1} x[n]z^{-n}\right] \quad ; \quad k>0$$

Teorema del Valor Final

$$\underset{n\to\infty}{\text{lím}}\, x[n] = \underset{z\to 1}{\text{lím}}\,(z-1)\,X^{+}(z)$$

Transformada Z

Propiedades



Transferencia

$$H(z) = \frac{B(z)}{A(z)} \quad con \quad polos \quad p_k \quad k = 0;1;\ldots;N$$

Entrada

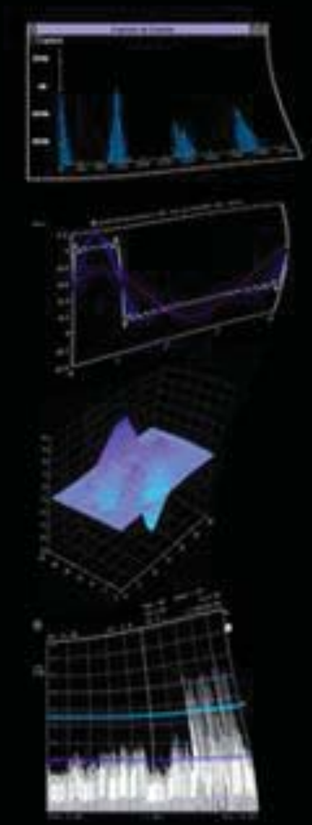
$$X(z) = \frac{Q(z)}{N(z)} \quad con \quad polos \quad q_k \neq p_k \quad k = 0;1;\ldots;L$$

Salida

$$Y(z) = X(z)H(z) = \frac{Q(z)}{N(z)}\frac{B(z)}{A(z)}$$

Transformada Z

Propiedades



Salida

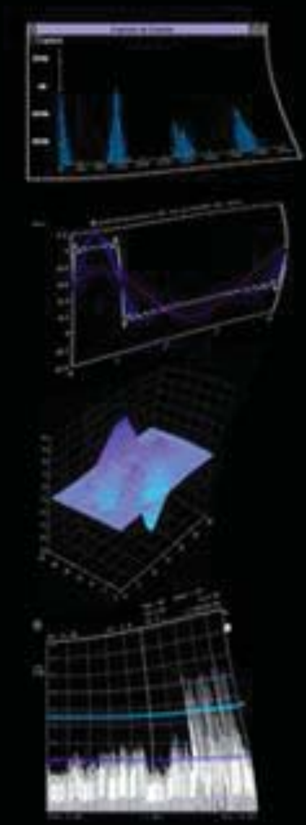
$$Y(z) = X(z)H(z) = \frac{B(z)}{A(z)}\frac{Q(z)}{N(z)} = \sum_{k=1}^{N}\frac{A_k}{1-p_k z^{-1}} + \sum_{k=1}^{L}\frac{Q_k}{1-q_k z^{-1}}$$

$$y[n] = \sum_{k=1}^{N} A_k {p_k}^n u[n] + \sum_{k=1}^{L} Q_k {q_k}^n u[n]$$

Respuesta Natural (first sum) — **Respuesta forzada** (second sum)

Transformada Z

Propiedades



Salida

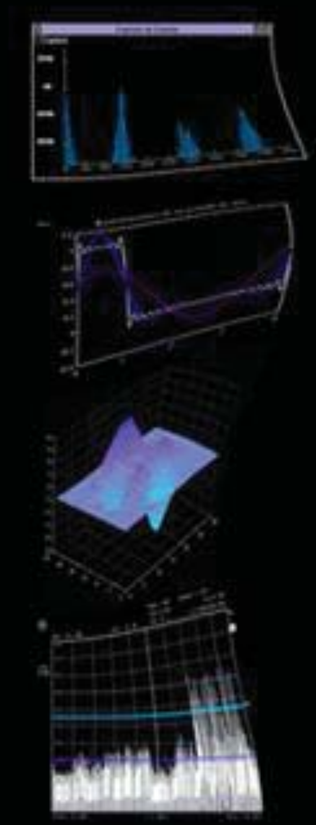
$$Y(z) = \sum_{k=0}^{M} b_k X(z) z^{-k} - \sum_{k=1}^{N} a_k Y(z) z^{-k}$$

$$Y^{+}(z) = \sum_{k=0}^{M} b_k X^{+}(z) z^{-k} - \sum_{k=1}^{N} a_k \left[Y^{+}(z) + \sum_{n=1}^{k} y[-n] z^{n} \right] z^{-k}$$

$$Y^{+}(z) = \frac{\sum_{k=0}^{M} b_k X(z) z^{-k} - \sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y[-n] z^{n}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)} X(z) \frac{N_0(z)}{A(z)}$$

Transformada Z

Propiedades



Salida

$$Y^{+}(z)=\sum_{k=1}^{N}\frac{A_k+D_k}{1-p_k z^{-1}}+\sum_{k=1}^{L}\frac{Q_k}{1-q_k z^{-1}}$$

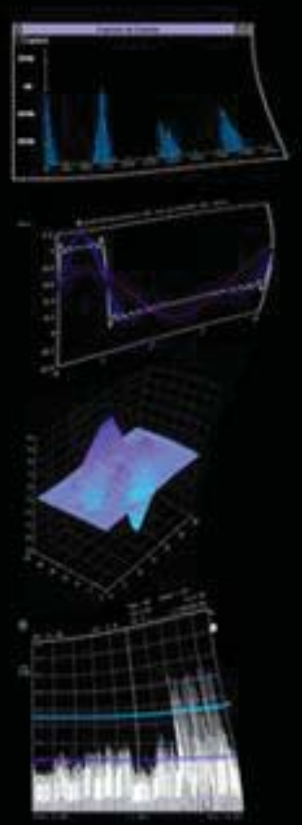
$$y[n]=\sum_{k=1}^{N}\left(A_k+D_k\right)p_k^{\,n}u[n]+\sum_{k=1}^{L}Q_k q_k^{\,n}u[n]$$

Respuesta natural (ZS + ZI)

Respuesta forzada

Transformada Z

Análisis de sistemas LIT



$$H(z)=\frac{\prod_{s=1}^{M}(z-c_s)}{\prod_{k=1}^{N}(z-p_k)}$$

Módulo

Es la relación entre los productos
de las distancias del círculo unitario $e^{j\omega}$
a cada cero y a cada polo

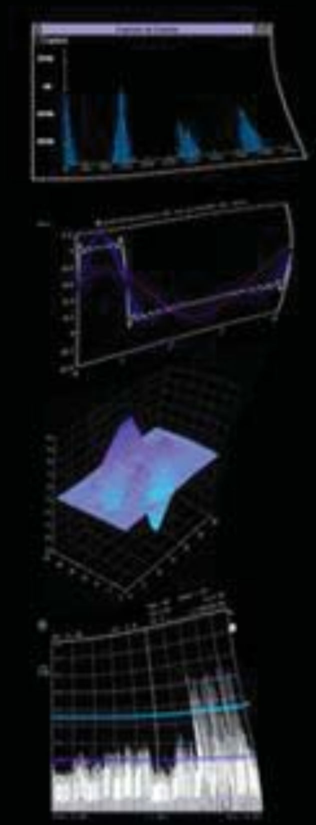
$$\left|H(e^{j\omega})\right|=\frac{\prod_{s=1}^{M}\left|e^{j\omega}-c_s\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-p_k\right|}$$

Fase

Es la resultante de las fases de los
vectores que van de los ceros y polos
al número complejo $e^{j\omega}$

$$\angle H(e^{j\omega})=\sum_{s=1}^{M}\angle\left(e^{j\omega}-c_s\right)-\sum_{k=1}^{N}\angle\left(e^{j\omega}-p_k\right)$$

Transformada Z

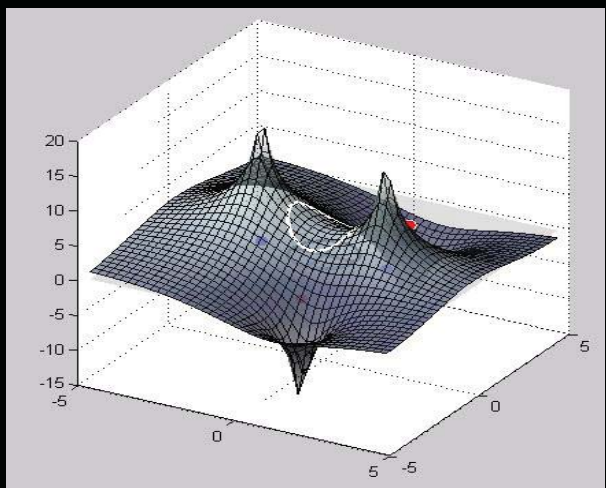


zt

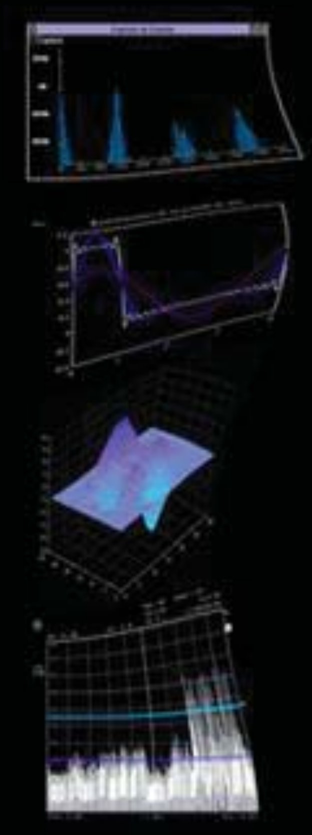
Computa y grafica la Transformada Z en dB de una transferencia

Sintaxis

```
[Hz,Hw,z,w,c,p]=zt(b,a,f,graficar)
```



Transformada Z

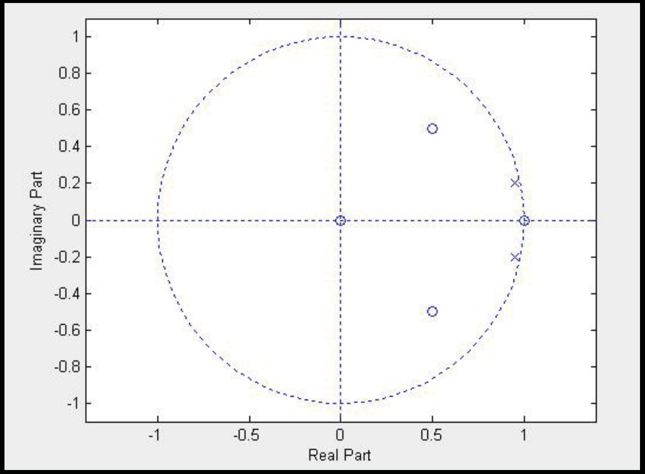


zplane

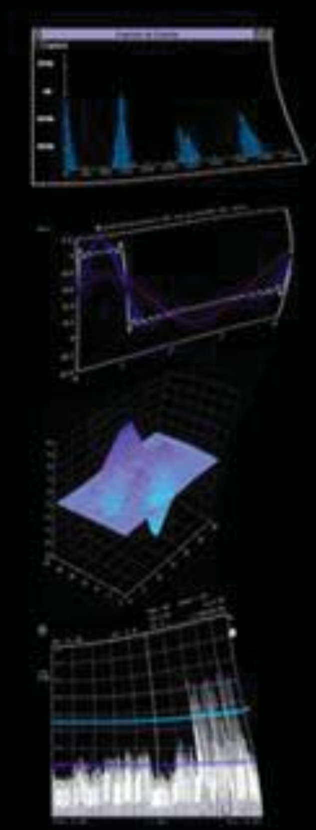
Grafica el mapa de polos y ceros en el plano Z

Sintaxis

```
zplane[z,p]
```



Transformada Z



impz

Computa la respuesta al impulso de un sistema discreto

Sintaxis

```
[h,t] = impz(b,a,n,fs)
impz(b,a,n,fs)
```