



**The Abdus Salam  
International Centre for Theoretical Physics**



2397-3

**Preparatory School to the Winter College on Optics:  
Advances in Nano-Optics and Plasmonics**

*30 January - 3 February, 2012*

**Introduction to Quantum Optics**

I. Ashraf Zahid  
*Quaid-I-Azam University  
Pakistan*

# 1 REVIEW OF QUANTUM MECHANICS

## 1.1 CLASSICAL MECHANICS

Classical mechanics is based on the assumption that any physically interesting variables connected with a system/particle, such as its position, velocity or its energy can be measured with arbitrary precision and without mutual interference from any other such measurement. Laws of classical mechanics can be expressed in various mathematical forms,

1. Newtonian mechanics
2. Hamiltonian mechanics

Hamiltonian of a physical system gives its total energy

$$H = T + V$$

Quantum mechanics is based on the realization that the measuring process may affect the physical system. It is therefore impossible to measure simultaneously certain pairs of variables with precision. In quantum mechanics physical system is described by a state vector or wave function, and variables are represented by operators.

Quantum mechanics can be expressed by,

1. Wave mechanics
2. Dirac's notation

## 1.2 WAVE MECHANICS:

A quantum mechanical system (such as atoms, molecules, ions etc.) are given by its wave function  $\psi(r, t)$ . Itself  $\psi(r, t)$  has no physical meaning but it allows to calculate the expectation values of all observables of interest.

Measurable quantities are called observables and are represented by hermitian operators  $\hat{O}$ . Expectation value is given by

$$\langle \hat{O} \rangle = \int d^3r \psi^*(r, t) \hat{O} \psi(r, t).$$

## 1.3 Probability:

The probability of finding the system in the volume element  $d^3r$  is

$$\psi^*(r, t) \hat{O} \psi(r, t).$$

As the system exist, its probability of being somewhere has to equal 1.

$$\begin{aligned} \int \psi^*(r, t) \psi(r, t) d^3r &= 1 \\ \int \psi_n^*(r, t) \psi_m(r, t) d^3r &= \delta_{nm}. \end{aligned}$$

The time development of wave function is determined by schrodinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi(r, t) = H\psi(r, t),$$

where  $H$  is the Hamiltonian of the system. The energy of the unperturbed system for instance an atom not interacting with light is the sum of its potential and kinetic energies

$$H = \frac{p^2}{2m} + V(r).$$

#### 1.4 Stationary state:

Stationary states of schrodinger equation are those for which space and time dependence are separated,

$$\psi_n(r, t) = U_n(\vec{r})e^{-i\omega t}.$$

Time independent equation,

$$HU_n(\vec{r}) = E_n U_n(\vec{r}) = \hbar\omega_n U_n(\vec{r}),$$

where  $U_n(\vec{r})$  is an eigen function of  $H$  with eigen values  $E_n = \hbar\omega_n$ . The eigen values of hermitian operators are real numbers. The eigen functions of hermitian operators belonging to different eigen values are orthogonal, and eigen functions having same eigen values are normal.

$$\int U_n^*(r)U_m(r)d\vec{r} = \delta_{nm}$$

and complete

$$\sum_n U_n^*(r)U_n(r) = 1.$$

The completeness relation means that any function can be written as a linear combination of the  $U_n(r, t)$ . The wave function

$$\psi(r, t) = \sum_n \psi_n(r, t) = \sum_n C_n(t)U_n(\vec{r})e^{-i\omega_n t},$$

here  $C_n(t)$  are the expansion coefficients.

$C_n(t)$  - constant for problems related to free part of Hamiltonian

$C_n(t)$  - change in time if we include the interaction part of Hamiltonian.

Putting the values of  $\psi(r, t)$  in the normalization condition we get

$$\sum_n |C_n|^2 = 1,$$

gives the probability of finding the system in state  $n$ . The expectation value in terms of  $C_n$

$$\langle \hat{O} \rangle = \sum_{n,m} C_n C_m^* \hat{O}_{nm} e^{-i\omega_{nm} t},$$

Where

$$\hat{O}_{nm} = \int d^3r U_m^*(r) \hat{O} U_n(r),$$

and

$$\omega_{nm} = \omega_n - \omega_m.$$

## 1.5 DIRAC NOTATION:

The wave function of wave mechanics corresponds to the state vector in Dirac's formulation of quantum mechanics. The relation between state vector and wave function is analogous to using vectors instead of coordinates. A vector  $\vec{V}$  can be expanded as,

$$\vec{V} = V_x \hat{x} + V_y \hat{y}.$$

In Dirac's notation

$$|V\rangle = V_x |x\rangle + V_y |y\rangle.$$

x-component of a vector is obtained by

$$\vec{V} \cdot \hat{x} = V_x,$$

in Dirac's notation

$$\langle x|V\rangle = V_x \quad \text{and} \quad \langle y|V\rangle = V_y.$$

Using these Eqn's. we can write

$$\begin{aligned} |V\rangle &= |x\rangle \langle x|V\rangle + |y\rangle \langle y|V\rangle \\ &= (|x\rangle \langle x| + |y\rangle \langle y|) |V\rangle \end{aligned}$$

The identity diadic (outer product of two vectors)

$$|x\rangle \langle x| + |y\rangle \langle y| = 1$$

for  $n$  dimensions

$$\begin{aligned} |V\rangle &= \sum_n |n\rangle \langle n|V\rangle \\ \sum_n |n\rangle \langle n| &= I \end{aligned}$$

where  $\{ |n\rangle \}$  are complete set of vectors, i.e. a basis. The inner products  $\langle n|V\rangle$  are the expansion coefficients of the vector  $|V\rangle$  in this basis. Expansion coefficients are in general complex.

$$\langle k|V\rangle = \langle V|k\rangle^*$$

For continuous basis  $\{ |r\rangle \}$

$$I = \int d^3r |r\rangle \langle r|$$

The wave vector

$$|\psi(t)\rangle = \int d^3r |r\rangle \langle r|\psi\rangle$$

Where the wave function

$$\begin{aligned}\psi(r) &= \langle r|\psi\rangle \\ \psi(x) &= \langle x|\psi\rangle\end{aligned}$$

The expectation value of the operator  $\hat{O}$  is given by,

$$\langle \hat{O} \rangle = \langle \psi(t)|\hat{O}|\psi(t)\rangle$$

Hermitian

$$\begin{aligned}\langle \psi(t)|\hat{O}|\psi(t)\rangle &= \left[ \langle \psi(t)|\hat{O}^\dagger|\psi(t)\rangle \right]^* = \langle \psi(t)|\hat{O}|\psi(t)\rangle^* \\ \hat{O}^\dagger &= \hat{O}\end{aligned}$$

The set of eigen vectors of a hermitian operator is complete. This means that any arbitrary vector  $|\psi(t)\rangle$  can be expressed as a sum of orthogonal eigen vectors.

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n |x_n\rangle$$

Eigen vectors are orthonormal

$$\begin{aligned}\langle X_n|X_m\rangle &= \delta_{nm} \\ \delta_{nm} &= 1 \text{ for } n = m \\ &= 0 \text{ for } n \neq m\end{aligned}$$

Completeness relation for discrete case is

$$\sum_n |X_n\rangle \langle X_n| = I$$

The state vector  $|\psi\rangle$  in terms of position eigen states (which are continuous).

$$\begin{aligned}|\psi(t)\rangle &= \int d\vec{x} |x\rangle \langle x|\psi\rangle \\ \int d\vec{x} |x\rangle \langle x| &= I\end{aligned}$$

The normalization of eigen vectors with a continuous set of eigen values must be normalized with the help of dirac delta function having properties,

$$\begin{aligned}\delta(x - x') &= 0 \quad \text{if } x \neq x' \\ \delta(x - x') &= \infty \quad x = x' \\ \langle x|x'\rangle &= \delta(x - x')\end{aligned}$$

State vectors obey the Schrodinger's equation

$$\begin{aligned} i\hbar \left| \dot{\psi} \right\rangle &= H |\psi\rangle, \\ |\psi\rangle &= \sum_n C_n e^{-i\omega_n t} |n\rangle \end{aligned}$$

Expectation value can be written as

$$|\psi\rangle = \sum_{n,m} C_n^* C_m e^{-i(\omega_n - \omega_m)t} |n\rangle \hat{O}_{nm}$$

Where

$$\hat{O}_{nm} = \langle m | \hat{O} | n \rangle$$

## 1.6 Two level system:

Wave function for two level system is

$$\psi(r, t) = C_a U_a(\vec{r}) e^{-i\omega_a t} + C_b U_b(\vec{r}) e^{-i\omega_b t}$$

State vector

$$|\psi\rangle = C_a e^{-i\omega_a t} |a\rangle + C_b e^{-i\omega_b t} |b\rangle$$

## 1.7 SCHRODINGER, INTERACTION AND HEISENBERG PICTURES:

### 1.8 SCHRODINGER PICTURE:

The interaction of radiation with matter involves a hamiltonian.

$$H = H_o + V$$

$H_o$  - unperturbed energy

$V$  - Interaction energy

The corresponding Schrodinger equation

$$\begin{aligned} \left| \dot{\psi}(t) \right\rangle &= \frac{-i}{\hbar} H |\psi(t)\rangle \\ \left| \dot{\psi}(t) \right\rangle &= \frac{-i}{\hbar} (H_o + V) |\psi(t)\rangle, \\ |\psi(t)\rangle &= e^{\frac{-iHt}{\hbar}} |\psi(0)\rangle \end{aligned}$$

Expectation value of an operator  $\hat{O}$  which represents the observables.

$$\langle \hat{O} \rangle = \langle \psi(t) | \hat{O}(0) | \psi(t) \rangle$$

Operator  $\hat{O}$  is independent of time, but  $|\psi(t)\rangle$  is a function of time. This is the schrodinger picture way of writing the expectation value of an operator.

## 1.9 HEISENBERG PICTURE:

In Heisenberg picture total time dependence goes into operator, so state vector is independent of time, the expectation value of an operator in Schrodinger picture is,

$$\langle \hat{O} \rangle = \langle \psi(t) | \hat{O}(0) | \psi(t) \rangle$$

It can also be written as,

$$\langle \hat{O}(t) \rangle = \langle \psi(t) | e^{-\frac{iHt}{\hbar}} e^{\frac{iHt}{\hbar}} \hat{O}(0) e^{-\frac{iHt}{\hbar}} e^{\frac{iHt}{\hbar}} | \psi(t) \rangle$$

Where  $H$  is the total Hamiltonian. According to Schrodinger equation,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

Integrating

$$\begin{aligned} |\psi(t)\rangle &= e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle \\ e^{\frac{iHt}{\hbar}} |\psi(t)\rangle &= |\psi(0)\rangle \end{aligned}$$

Using this we can write

$$\langle \hat{O}(t) \rangle = \langle \psi(0) | e^{\frac{iHt}{\hbar}} \hat{O}(0) e^{-\frac{iHt}{\hbar}} | \psi(0) \rangle$$

Define

$$\hat{O}(t) = e^{\frac{iHt}{\hbar}} \hat{O}(0) e^{-\frac{iHt}{\hbar}}$$

Then

$$\langle \hat{O}(t) \rangle = \langle \psi(0) | \hat{O}(t) | \psi(0) \rangle$$

Which is called Heisenberg picture. In this all time dependence lies in operator. while wave function is independent of time.

### 1.9.1 Why called Heisenberg picture?

$$\begin{aligned} \hat{O}(t) &= e^{\frac{iHt}{\hbar}} \hat{O}(0) e^{-\frac{iHt}{\hbar}} \\ \dot{\hat{O}}(t) &= \frac{i}{\hbar} H \hat{O} + \frac{-i}{\hbar} \hat{O} H \\ \dot{\hat{O}}(t) &= \frac{i}{\hbar} [H, \hat{O}] \end{aligned}$$

Which is Heisenberg equation of motion. That is why we call it Heisenberg picture. In between two extremes of Schrodinger and Heisenberg picture. There is an intermediate picture called interaction picture.

## 1.10 INTERACTION PICTURE:

Consider again the equation,

$$\langle \hat{O}(t) \rangle = \langle \psi(0) | e^{\frac{+iHt}{\hbar}} \hat{O}(0) e^{-\frac{iHt}{\hbar}} | \psi(0) \rangle$$

As

$$H = H_o + V$$

$H_o$ - free Hamiltonian,

$V$ - interaction part of Hamiltonian,

If the time dependence created by the interaction energy is only assigned to the state vector and rest of time dependence goes to the operator, then expectation value is written as,

$$\begin{aligned} \langle \hat{O}(t) \rangle &= \langle \psi(0) e^{\frac{+iVt}{\hbar}} | e^{\frac{+iH_o t}{\hbar}} \hat{O}(0) e^{-\frac{iH_o t}{\hbar}} | e^{-\frac{iVt}{\hbar}} \psi(0) \rangle \\ \langle \hat{O}(t) \rangle &= \langle \psi^I(t) | \hat{O}(t) | \psi^I(t) \rangle \end{aligned}$$

Interaction picture state vector

$$|\psi^I(t)\rangle = e^{-\frac{iVt}{\hbar}} |\psi(0)\rangle$$

Equation of motion

$$\left| \dot{\psi}^I(t) \right\rangle = \frac{-iV}{\hbar} |\psi^I(t)\rangle$$

This equation is simpler than ordinary schrodinger equation, but requires the calculation of  $\hat{O}^I(t)$ , where

$$\hat{O}^I(t) = e^{\frac{+iH_o t}{\hbar}} \hat{O}(0) e^{-\frac{iH_o t}{\hbar}}$$

In interaction picture both state vector and operator are time dependent. The interaction picture state vector,

$$|\psi^I(t)\rangle = \sum_n C_n(t) |n\rangle$$

The schrodinger picture state vector is

$$|\psi(t)\rangle = \sum_n c_n(t) |n\rangle = \sum_n C_n(t) e^{-\frac{i\omega_n t}{\hbar}} |n\rangle$$

Where

$$c_n(t) = C_n(t) e^{-\frac{i\omega_n t}{\hbar}}$$

The complete time dependence is given by  $c_n(t)$ , but due to interaction energy is given by  $C_n(t)$ .



## 1.11 PAULI SPIN MATRIX:

Another method to describe two- level atom is a use of  $2 \times 2$  matrix notation. The eigen function  $U_a$  and  $U_b$  or eigen vectors  $|a\rangle$  and  $|b\rangle$  can be represented by the column vectors

$$\begin{aligned}|a\rangle &= U_a \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |b\rangle &= U_b \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}\end{aligned}$$

And the wave function and wave vector by the column vectors

$$\psi(r, t) = \begin{bmatrix} C_a \\ C_b \end{bmatrix}$$

And

$$|\psi(r, t)\rangle = \begin{pmatrix} C_a \\ C_b \end{pmatrix}$$

The energy and electric- dipole operators are written in terms of the Pauli spin matrices as

$$\begin{aligned}\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

These matrices are hermitian, but the spin- flip operators

$$\begin{aligned}\sigma_+ &= \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \text{and } \sigma_- &= \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

are not hermitian,  $\sigma_-$  flips the system from upper- level to a lower- level

$$\sigma_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

While,  $\sigma_+$  flips the system from lower- level to the upper- level

$$\sigma_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

## 1.12 TWO LEVEL ATOMIC SYSTEM AND HAMILTONIAN IN TERMS OF MATRICES:

State vector of a system can be written as,

$$|\psi(r, t)\rangle = C_a e^{-i\omega_a t} |a\rangle + C_b e^{-i\omega_b t} |b\rangle,$$

Which corresponds to the wave function

$$\psi(r, t) = C_a U_a(\vec{r}) e^{-i\omega_a t} + C_b U_b(\vec{r}) e^{-i\omega_b t}$$

The matrix form for the unit vectors (eigen vectors)  $|a\rangle$  and  $|b\rangle$  are

$$\psi(r, t) \leftrightarrow C_a e^{-i\omega_a t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_b e^{-i\omega_b t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_a e^{-i\omega_a t} \\ C_b e^{-i\omega_b t} \end{pmatrix}$$

The two-level Hamiltonian of the semi-classical treatment is given as,

$$H = \hbar\omega_a |a\rangle\langle a| + V_{ab} |a\rangle\langle b| + V_{ba} |b\rangle\langle a| + \hbar\omega_b |b\rangle\langle b|$$

In matrix notation

$$H = \begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix} = \begin{pmatrix} \hbar\omega_a & V_{ab} \\ V_{ba} & \hbar\omega_b \end{pmatrix}$$

Thus the matrix form of the schrodinger equation is

$$i\hbar \frac{d}{dt} \begin{pmatrix} C_a e^{-i\omega_a t} \\ C_b e^{-i\omega_b t} \end{pmatrix} = \begin{pmatrix} \hbar\omega_a & V_{ab} \\ V_{ba} & \hbar\omega_b \end{pmatrix} \begin{pmatrix} C_a e^{-i\omega_a t} \\ C_b e^{-i\omega_b t} \end{pmatrix}$$

### 1.13 EXPECTATION VALUE OF DIPOLE MOMENT OPERATOR FOR A TWO-LEVEL ATOM:

The expectation value of any operator is given by

$$\langle \psi | \hat{O} | \psi \rangle = C_a C_a^* \hat{O}_{aa} + C_b C_b^* \hat{O}_{bb} + \left\{ C_a C_b^* \hat{O}_{ab} e^{-i\omega_{ab}t} + c.c \right\}$$

The expectation value of “ $er$ ” is

$$\langle \psi | er | \psi \rangle = e C_a C_a^* \langle a | r | a \rangle + e C_b C_b^* \langle b | r | b \rangle + \left\{ e C_a C_b^* e^{-i(\omega_a - \omega_b)t} \langle b | r | a \rangle + c.c \right\}$$

As the diagonal matrix element of “ $er$ ” between eigen states of the Hamiltonian generally vanishes.

$$\begin{aligned} er_{aa} &= \langle a | er | a \rangle = e \int d^3r U_a^*(r) r U_a(r) = 0 \\ er_{bb} &= \langle b | er | b \rangle = e \int d^3r U_b^*(r) r U_b(r) = 0 \\ er_{ab} &= \langle a | er | b \rangle = e \int d^3r U_a^*(r) r U_b(r) \end{aligned}$$

In matrix form

$$\begin{aligned} \langle er \rangle &= e \begin{pmatrix} 0 & r_{ab} \\ r_{ba} & 0 \end{pmatrix} \\ \langle er \rangle &= e C_a C_b^* e^{-i(\omega_a - \omega_b)t} r_{ba} + c.c \end{aligned}$$

## 1.14 DENSITY OPERATOR

For a given physical system there exist a state vector  $|\psi\rangle$  which contain all possible information about the system. If we want to extract a piece of information about the system we must calculate the expectation value of the corresponding operator  $\hat{O}$

$$\langle \hat{O} \rangle_{Q M} = \langle \psi | \hat{O} | \psi \rangle$$

In many situations we do not know  $|\psi\rangle$  but we know  $P_\psi$ , probability of finding the system in  $|\psi\rangle$ . For such a situation we not only need to take quantum mechanical average but also the ensemble average over many identical systems that have been similarly prepared

$$\left\langle \langle \hat{O} \rangle_{Q M} \right\rangle_{ensemble} = \sum_{\psi} P_{\psi} \langle \psi | \hat{O} | \psi \rangle$$

It is called a quantum statistical system.

$$\begin{aligned} \sum_n |n\rangle \langle n| &= 1 \\ \left\langle \langle \hat{O} \rangle_{Q M} \right\rangle_{ensemble} &= \sum_n \sum_{\psi} P_{\psi} \langle \psi | n \rangle \langle n | \hat{O} | \psi \rangle \\ &= \sum_n \sum_{\psi} P_{\psi} \langle n | \hat{O} | \psi \rangle \langle \psi | n \rangle \\ &= \sum_n \langle n | \hat{O} \left[ \sum_{\psi} P_{\psi} | \psi \rangle \langle \psi | \right] | n \rangle \\ &= \sum_n \langle n | \hat{O} \rho | n \rangle = \sum_n (\hat{O} \rho)_{nn} \end{aligned}$$

where

$$\rho = \sum_{\psi} P_{\psi} | \psi \rangle \langle \psi |$$

The sum of diagonal elements gives

$$\left\langle \langle \hat{O} \rangle_{Q M} \right\rangle_{ensemble} = Tr(\hat{O} \rho) = Tr(\rho \hat{O})$$

where

$$\rho = \sum_{\psi} P_{\psi} | \psi \rangle \langle \psi |$$

is called density operator. In a particular case where all  $P_{\psi}$  are zero except the one for a state  $|\psi_0\rangle$  then

$$\rho = |\psi_0\rangle \langle \psi_0|$$

and state is called a Pure state. It follows from the conservation of probability that  $Tr(\rho) = 1$ . For a pure state

$$Tr(\rho^2) = 1$$

For mixed case

$$Tr(\rho^2) < 1$$

### 1.15 EQUATION OF MOTION FOR DENSITY OPERATOR

$$\begin{aligned}\rho &= \sum_{\psi} P_{\psi} |\psi\rangle \langle \psi| \\ \dot{\rho} &= \sum_{\psi} P_{\psi} \left[ \left| \dot{\psi} \right\rangle \langle \psi| + |\psi\rangle \left\langle \dot{\psi} \right| \right]\end{aligned}$$

From Schrodinger equation, we know that

$$\left| \dot{\psi} \right\rangle = \frac{1}{i\hbar} H |\psi\rangle$$

Also

$$\left\langle \dot{\psi} \right| = -\frac{1}{i\hbar} \langle \psi| H$$

Using these values, we get

$$\begin{aligned}\dot{\rho} &= \frac{1}{i\hbar} \sum_{\psi} P_{\psi} [H |\psi\rangle \langle \psi| - |\psi\rangle \langle \psi| H] \\ &= \frac{1}{i\hbar} [H\rho - \rho H]\end{aligned}$$

where  $\rho$  and  $H$  are operators.

$$\dot{\rho} = \frac{1}{i\hbar} [H, \rho]$$

It is called “Liouville equation” or “von Neumann equation” and it is equivalent to Schrodinger , but more general (because it has quantum mechanical as well as statistical aspect). In the above equation we have not included the decay of atomic levels due spontaneous emission. The excited atomic levels can also decay because of collision and other phenomena. The finite life time of the atomic level can be dicribed by adding phenominalogical decay terms to the density operator.

The decay rates can be incorporated in equation by a relaxation matrix  $\Gamma$ , which is defined by the equation

$$\Gamma_{nm} = \langle n | \Gamma | m \rangle = \gamma_n \delta_{nm}$$

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] - \frac{1}{2} \{\Gamma, \rho\}; \quad \{\Gamma \rho + \rho \Gamma\} = \{\Gamma, \rho\}$$

Equation of motion for the density matrix elements is

$$\dot{\rho}_{ij} = \frac{1}{i\hbar} \sum_k [H_{ik} \rho_{kj} - \rho_{ik} H_{kj}] - \frac{1}{2} \sum_k [\Gamma_{ik} \rho_{kj} + \rho_{ik} \Gamma_{kj}]$$

## 1.16 TWO LEVEL ATOM

$$\begin{aligned} |\psi\rangle &= c_a(t) |a\rangle + c_b(t) |b\rangle \\ \rho &= |\psi\rangle \langle\psi| \\ \langle\psi| &= c_a^*(t) \langle a| + c_b^*(t) \langle b| \\ \rho &= |\psi\rangle \langle\psi| = |c_a(t)|^2 |a\rangle \langle a| + c_a(t) c_b^*(t) |a\rangle \langle b| + c_a^*(t) c_b(t) |b\rangle \langle a| + |c_b(t)|^2 |b\rangle \langle b| \\ \rho_{aa} &= \langle a | \rho | a \rangle = |c_a|^2 \quad \text{probability of upper state} \\ \rho_{bb} &= \langle b | \rho | b \rangle = |c_b|^2 \quad \text{probability of lower state} \\ \rho_{ab} &= c_a c_b^* \quad \text{is propotional to dipole moment} \\ \rho_{ba} &= c_b c_a^* = \rho_{ab}^* \end{aligned}$$

In matrix form

$$\rho = \begin{pmatrix} |c_a|^2 & c_a c_b^* \\ c_a^* c_b & |c_b|^2 \end{pmatrix}$$

$$P(z, t) = c_a c_b^* p_{ba} + c.c = \rho_{ab}(z, t) p_{ba} + c.c$$

In spiner notation,

$$\begin{aligned} |\psi\rangle &= \begin{pmatrix} c_a \\ c_b \end{pmatrix} & \langle\psi| &= ( c_a^* \quad c_b^* ) \\ \rho &= \begin{pmatrix} c_a \\ c_b \end{pmatrix} ( c_a^* \quad c_b^* ) = \begin{pmatrix} |c_a|^2 & c_a c_b^* \\ c_a^* c_b & |c_b|^2 \end{pmatrix} \end{aligned}$$

## 1.17 EQUATION OF MOTION FOR DENSITY MATIRX ELEMENTS

Consider  $i = a, j = a$

$$\dot{\rho}_{aa} = \frac{1}{i\hbar} \sum_{k=a,b} [H_{ak} \rho_{ka} - \rho_{ak} H_{ka}] - \frac{1}{2} \sum_{k=a,b} [\Gamma_{ak} \rho_{ka} + \rho_{ak} \Gamma_{ka}]$$

using this

$$\begin{aligned}
\Gamma_{nm} &= \langle n|\Gamma|m\rangle = \gamma_n \delta_{nm} \quad \text{and} \\
H &= H_o + \gamma \\
\dot{\rho}_{aa} &= \frac{1}{i\hbar} \sum_{k=a,b} [(H_o)_{ak} \rho_{ka} - \rho_{ak} (H_o)_{ka}] - \frac{1}{2} [\gamma_a \rho_{aa} + \rho_{aa} \gamma_a] \\
&\quad + \frac{1}{i\hbar} \sum_{k=a,b} [V_{ak} \rho_{ka} - \rho_{ak} V_{ka}] \\
(H_o)_{ab} &= \langle a|H_o|b\rangle = E_b \langle a|b\rangle = 0 \\
(H_o)_{aa} &= \langle a|H_o|a\rangle = E_a \quad \text{and} \quad V_{aa}=V_{bb} = 0 \\
\dot{\rho}_{aa} &= -\gamma_a \rho_{aa} + \frac{1}{i\hbar} \sum_{k=a,b} [V_{ab} \rho_{ba} - \rho_{ab} V_{ba}] \\
\dot{\rho}_{bb} &= -\gamma_b \rho_{bb} + \frac{1}{i\hbar} [V_{ba} \rho_{ab} - \rho_{ba} V_{ab}] \\
\dot{\rho}_{ab} &= -(i\omega_o + \gamma_{ab}) \rho_{ab} + \frac{1}{i\hbar} [V_{ab} \rho_{bb} - \rho_{aa} V_{ab}] \\
&\quad \text{where} \\
\gamma_{ab} &= \frac{1}{2} (\gamma_a + \gamma_b) \quad \text{and} \quad \omega_o = \frac{1}{\hbar} (E_a - E_b)
\end{aligned}$$

The population of excited level decays in time because of spontaneous emission. In some cases the upper level decays to ground state lower level then

$$\dot{\rho}_{aa} = -\Gamma \rho_{aa} + \frac{1}{i\hbar} [V_{ab} \rho_{ba} - \rho_{ab} V_{ba}]$$

and

$$\dot{\rho}_{bb} = \Gamma \rho_{bb} + \frac{1}{i\hbar} [V_{ba} \rho_{ab} - \rho_{ba} V_{ab}]$$

Where  $\Gamma$  is the upper-to-lower level decay constant.

# 1 Quantization of the Free Electromagnetic Field

Dirac combined the wave and particle like aspects of light. Wave nature shows all the interference phenomena. Particle nature shows the excitation of a specific atom absorbing one photon of energy.

## Classical field fails to explain

1. Spontaneous emission
2. Atomic decay
3. Lamb shift
4. Photon statistics

An interesting consequences of the quantization of the radiation is the fluctuations associated with the zero-point energy or so called vacuum fluctuations. These fluctuations have no classical analog and are responsible for many interacting phenomena in quantum optics.

## 1.1 Spontaneous Emission and Atomic Decay

A phenomena which we described phenomenologically in our treatment of semi-classical theory requires a quantum field. Spontaneous emission is often said to be the result of stimulating the atom by vacuum fluctuations.

## 1.2 Lamb Shift

According to the classical description of the field (*Dirac theory*) the  $2S_{\frac{1}{2}}$  and  $2P_{\frac{1}{2}}$  states in the hydrogen atom should have equal energies. Experimentally the two levels differ by approximately 1057 MHz. a fully quantized treatment of the field and atomic systems gives impressive agreement with the experimentally observed shift, because of the radiative correction due to the interaction between the atomic electron and the vacuum shift the  $2S_{\frac{1}{2}}$  level higher in energy by around 1057 MHz relative to the  $2P_{\frac{1}{2}}$  level.

## 1.3 Photon Statistics

In order to explain the photon statistics the concept of a particle, the photon is either necessary or convenient. For the quantization of the electromagnetic field in free space, it is convenient to begin with the classical description of the field based on Maxwell's equations. In MKS system

$$\begin{aligned}\nabla \cdot D &= 0 \\ \nabla \cdot B &= 0 \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \times H &= \frac{\partial D}{\partial t}\end{aligned}$$



where in free space

$$\begin{aligned} D &= \epsilon_0 E \\ B &= \mu_0 H \end{aligned}$$

here  $\epsilon_0$  and  $\mu_0$  are the free space permittivity and permeability respectively and

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

where  $c$  is the speed of light in vacuum. Using these Maxwell's equations we know that  $E(r, t)$  and also  $B(r, t)$  satisfies the wave equation

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

Following the Dirac approach we associate each mode of the radiation field with a quantized simple Harmonic oscillator. Energy of the Harmonic oscillator (classically) is given by the hamiltonian

$$H = \frac{1}{2} m \omega^2 x^2 + \frac{p^2}{2m}$$

and quantum mechanically it is written as

$$H = \frac{1}{2} m \omega^2 \hat{x}^2 + \frac{\hat{p}^2}{2m}$$

## 2 Mode Expansion of The Field

### 2.1 Quantization of Field Inside the Cavity of Length L

Electric field is linearly polarized in the x direction. Expanding the field in the normal modes of the cavity

$$E_x(z, t) = \sum_j A_j q_j(t) \sin(k_j z)$$

$j$  corresponds to different modes such that

$$L = j \frac{\lambda}{2}; \quad \lambda = \frac{2\pi}{k} \quad \text{and} \quad L = \frac{j\pi}{k_j}, \quad \text{where } j = 1, 2, 3, \dots$$

Where  $q_j$  is the normal mode amplitude with the dimensions of length (position) and

$$\begin{aligned} A_j &= \left( \frac{2\omega_j^2 m_j}{V \epsilon_0} \right)^{1/2} \\ \text{where } \omega_j &= ck_j = \frac{j\pi c}{L} \end{aligned}$$

is the cavity eigen frequency.  $V = LA$  is the volume ( $A$  is the transverse area of the optical resonator)  $m_j$  is a constant with the dimensions of mass, included to make an analogy with SHO nothing to do with mass of photon. The E.M.F is assumed to be transverse with electric field polarized in the x-direction. Such field satisfies

$$\nabla \cdot E = 0$$

The nonvanishing component of the magnetic field in the cavity is obtained by using Maxwell's 4th equation i.e.,

$$\nabla \times H = \frac{\partial D}{\partial t} = \epsilon_0 \frac{\partial E}{\partial t}$$

$$\begin{aligned} \nabla \times H &= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{pmatrix} \\ &= \hat{i} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + \hat{j} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + \hat{k} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \end{aligned}$$

x-component of  $(\nabla \times H)_x$  is written as

$$\left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right)_x = \epsilon_0 \frac{\partial E_x}{\partial t}$$

$H_z = 0$ ,  $z$  is the direction of propagation

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= \epsilon_0 \frac{\partial}{\partial t} \left( \sum_j A_j q_j(t) \sin(k_j z) \right) \\ &= \epsilon_0 \sum_j A_j \dot{q}_j(t) \sin(k_j z). \end{aligned}$$

As

$$\sin(k_j z) = -\frac{1}{k_j} \frac{\partial}{\partial z} \cos(k_j z),$$

putting the value of  $\sin(k_j z)$  in the above equation we can write

$$\begin{aligned} -\frac{\partial H_y}{\partial z} &= -\frac{\partial}{\partial z} \left( \sum_j A_j \left( \frac{\dot{q}_j(t) \epsilon_0}{k_j} \right) \cos(k_j z) \right), \\ H_y &= \sum_j A_j \left( \frac{\dot{q}_j(t) \epsilon_0}{k_j} \right) \cos(k_j z). \end{aligned}$$

The classical Hamiltonian of the field i.e, the (total energy of the field ) is

$$H = \frac{1}{2} \int_v d\tau (\epsilon_0 E_x^2 + \mu_0 H_y^2)$$

where the integration is over the volume of the cavity. Substituting the values of  $E_x$  and  $H_y$  in the above equation and performing the integration we get,

$$\begin{aligned} H &= \frac{1}{2} \sum_j \left( m_j \omega_j^2 q_j^2 + m_j \dot{q}_j^2 \right) \\ &= \frac{1}{2} \sum_j \left( m_j \omega_j^2 q_j^2 + \frac{p_j^2}{m_j} \right) \end{aligned}$$

where  $p_j = m_j \dot{q}_j$  is the canonical momentum of the  $j$ th mode. The above equation expresses the hamiltonian of the radiation field as a sum of independent oscillator energies. Each mode of the field is dynamically equivalent to a mechanical harmonic oscillator.

### 3 Quantization

The present dynamical problem can be quantized by identifying  $q_j$  and  $p_j$  as operators, which obey the commutation relations

$$\begin{aligned} [\hat{q}_j, \hat{p}_{j'}] &= i\hbar \delta_{jj'}. \\ [\hat{q}_j, \hat{q}_{j'}] &= [\hat{p}_j, \hat{p}_{j'}] = 0. \end{aligned}$$

It can be transformed as

$$\hat{a}_j = \frac{1}{\sqrt{2m_j\hbar\omega_j}} (m_j\omega_j\hat{q}_j + i\hat{p}_j) \exp(i\omega_j t)$$

and

$$\hat{a}_j^\dagger = \frac{1}{\sqrt{2m_j\hbar\omega_j}} (m_j\omega_j\hat{q}_j - i\hat{p}_j) \exp(-i\omega_j t)$$

$$\begin{aligned} \hat{q}_j &= \left( \hat{a}_j \exp(-i\omega_j t) + \hat{a}_j^\dagger \exp(i\omega_j t) \right) \sqrt{\frac{\hbar}{2m_j\omega_j}} \\ \hat{p}_j &= -i\sqrt{\frac{m_j\omega_j\hbar}{2}} \left( \hat{a}_j \exp(-i\omega_j t) - \hat{a}_j^\dagger \exp(i\omega_j t) \right) \end{aligned}$$

The commutation relations between  $\hat{a}_j$  and  $\hat{a}_j^\dagger$  follow from those between  $\hat{q}_j$  and  $\hat{p}_j$ ,

$$\begin{aligned} [\hat{a}_j, \hat{a}_j^\dagger] &= \frac{1}{2m_j\hbar\omega_j} \left[ -im_j\omega_j [\hat{q}_j, \hat{p}_j] + im_j\omega_j [\hat{p}_j, \hat{q}_j] \right] \\ &= \frac{1}{2m_j\hbar\omega_j} \left[ -im_j\omega_j (i\hbar) + im_j\omega_j (-i\hbar) \right] \\ &= 1. \end{aligned}$$

Similarly

$$\begin{aligned} [\hat{a}_j, \hat{a}_{j'}] &= [\hat{a}_j^\dagger, \hat{a}_{j'}^\dagger] = 0 \\ [\hat{a}_j, \hat{a}_{j'}^\dagger] &= \delta_{jj'} \end{aligned}$$

The operators  $\hat{a}$  and  $\hat{a}^\dagger$  are referred to as the destruction and creation operators, they are not hermitian. Substituting the value of  $\hat{q}_j$  and  $\hat{p}_j$  in the equation for Hamiltonian we get

$$\begin{aligned} H &= \sum_j \frac{1}{2} m_j \omega_j^2 \left( \frac{\hbar}{2m_j \omega_j} \right) \left( a_j^2 \exp(-2i\omega_j t) + a_j^{\dagger 2} \exp(2i\omega_j t) + \hat{a}_j \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_j \right) \\ &\quad + \frac{1}{2m_j} \left( -\frac{m_j \hbar \omega_j}{2} \right) \left( a_j^2 \exp(-2i\omega_j t) + a_j^{\dagger 2} \exp(2i\omega_j t) - \hat{a}_j \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j \right) \\ &= \sum_j \left( \frac{\hbar \omega_j}{2} \right) \left( \hat{a}_j \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_j \right) \end{aligned}$$

As

$$\begin{aligned} [\hat{a}_j, \hat{a}_j^\dagger] &= 1 \\ \left( \hat{a}_j \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j \right) &= 1 \\ \hat{a}_j \hat{a}_j^\dagger &= \hat{a}_j^\dagger \hat{a}_j + 1 \end{aligned}$$

thus we can write

$$H = \sum_j \hbar \omega_j \left( \hat{a}_j^\dagger \hat{a}_j + \frac{1}{2} \right)$$

In terms of  $\hat{a}_j$  and  $\hat{a}_j^\dagger$ , the electric and magnetic fields can be written as

$$\begin{aligned} E_x(z, t) &= \sum_j \varepsilon_j \left( \hat{a}_j \exp(-i\omega_j t) + \hat{a}_j^\dagger \exp(i\omega_j t) \right) \sin(k_j z) \\ H_y(z, t) &= -i\epsilon_0 c \sum_j \varepsilon_j \left( \hat{a}_j \exp(-i\omega_j t) - \hat{a}_j^\dagger \exp(i\omega_j t) \right) \cos(k_j z) \end{aligned}$$

where the quantity

$$\varepsilon_j = \left( \frac{\hbar \omega_j}{\epsilon_0 V} \right)^{1/2}$$

has the dimensions of an electric field.

### 3.1 Quantization of Field Inside a Large Cavity of Finite Length L

Consider the field in a large but finite cubic cavity of side  $L$ . We consider the running wave solutions instead of the standing wave solutions. The classical electric and magnetic field can be expanded in terms of plane waves.

$$E(r, t) = \sum_k \hat{\epsilon}_k \epsilon_k \alpha_k \exp(-i\omega_k t + ik \cdot r) + c.c$$

using Maxwell's equation i.e,

$$\nabla \times H = \frac{\partial D}{\partial t}$$

we get

$$H(r, t) = \frac{1}{\mu_0} \sum_k \frac{k \times \hat{\epsilon}_k}{\omega_k} \epsilon_k \alpha_k \exp(-i\omega_k t + ik \cdot r) + c.c$$

where the summation is taken over an infinite discrete set of values of wave vector  $k = (k_x, k_y, k_z)$ ,  $\epsilon_k$  is a unit polarization vector,  $\alpha_k$  is a dimensionless amplitude and

$$\epsilon_k = \left( \frac{\hbar \omega_k}{2\epsilon_0 V} \right)^{1/2}$$

Periodic boundary conditions require that

$$k_x = \frac{2\pi n_x}{L}, k_y = \frac{2\pi n_y}{L}, k_z = \frac{2\pi n_z}{L}$$

where  $n_x, n_y, n_z$  are integers  $0, \pm 1, \pm 2, \dots$ . A set of numbers  $(n_x, n_y, n_z)$  defines a mode of electromagnetic field. For transverse field

$$\nabla \cdot D = 0$$

which requires

$$\vec{k} \cdot \hat{\epsilon}_k = 0$$

There are two independent polarization directions of  $\hat{\epsilon}_k$  for each  $k$ . Changing from a discrete distribution of modes to a continuous distribution i.e,

$$\sum_k \implies 2 \left( \frac{L}{2\pi} \right)^3 \int d^3k$$

where factor of 2 accounts for two possible states of polarization. The number of modes available in a cavity is infinite, however the number of modes whose frequency lies between  $\omega$  and  $\omega + d\omega$  is finite. This is the same number of field modes having the magnitude of  $k$ , between  $k$  and  $k + dk$ . Making

transformation from rectangular coordinates  $(k_x, k_y, k_z)$  to the polar coordinates  $(k \sin \theta \cos \phi, k \sin \theta \sin \phi, k \cos \theta)$ , the volume element in  $k$ -space is written as

$$\begin{aligned} d^3k &= k^2 dk \sin \theta d\theta d\phi \\ &= \frac{\omega^2}{c^3} d\omega \sin \theta d\theta d\phi. \end{aligned}$$

The total number of modes in the volume  $L^3$  in the range between  $\omega$  and  $\omega + d\omega$  is given by

$$\begin{aligned} dN &= 2 \left( \frac{L}{2\pi} \right)^3 \frac{\omega^2 d\omega}{c^3} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \\ &= \frac{L^3 \omega^2}{\pi^2 c^3} d\omega \end{aligned}$$

Radiation field is quantized by identifying  $\alpha_k$  and  $\alpha_k^*$  by the harmonic oscillator operators  $\hat{a}_k$  and  $\hat{a}_k^\dagger$  respectively, which satisfy the commutation relation

$$\left[ \hat{a}_k, \hat{a}_k^\dagger \right] = 1$$

The quantized electric and magnetic fields takes the form

$$\begin{aligned} E(r, t) &= \sum_k \hat{\epsilon}_k \epsilon_k \hat{a}_k \exp(-i\omega_k t + ik \cdot r) + H.C \\ H(r, t) &= \frac{1}{\mu_0} \sum_k \frac{k \times \hat{\epsilon}_k}{\omega_k} \epsilon_k \hat{a}_k \exp(-i\omega_k t + ik \cdot r) + H.C \end{aligned}$$

where H.C is Hermetian conjugate. Separating positive and negative frequency parts of these field operators

$$\begin{aligned} E^+(r, t) &= \sum_k \hat{\epsilon}_k \epsilon_k \hat{a}_k \exp(-i\omega_k t + ik \cdot r) \\ E^-(r, t) &= \sum_k \hat{\epsilon}_k \epsilon_k \hat{a}_k^\dagger \exp(i\omega_k t - ik \cdot r) \end{aligned}$$

where  $E^+(r, t)$  is the annihilation operator and  $E^-(r, t)$  is the creation operator.

## 4 Fock or Number States of Radiation Field

Consider a single mode of the field of frequency  $\omega$  having creation and annihilation operators  $\hat{a}^\dagger$  and  $\hat{a}$  respectively. Let  $|n\rangle$  be the energy eigen state corresponding to the energy eigen value  $E_n$ , i.e.

$$\begin{aligned} H|n\rangle &= \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) |n\rangle \\ &= E_n |n\rangle \end{aligned} \quad (1)$$

applying operator  $\hat{a}$  from the left of the eigenstates we have

$$\begin{aligned} Ha|n\rangle &= \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) a|n\rangle \\ [a, a^\dagger] &= aa^\dagger - a^\dagger a = 1 \\ \implies aa^\dagger - 1 &= a^\dagger a \end{aligned} \quad (2)$$

Putting in Eq. (2) we get

$$\begin{aligned} Ha|n\rangle &= \hbar\omega \left( aa^\dagger - 1 + \frac{1}{2} \right) a|n\rangle \\ &= \hbar\omega \left( aa^\dagger a - a + \frac{a}{2} \right) |n\rangle \\ &= a\hbar\omega \left( a^\dagger a + \frac{1}{2} - 1 \right) |n\rangle \\ &= a(\hbar\omega(a^\dagger a + \frac{1}{2}) - \hbar\omega) |n\rangle \\ &= a(E_n - \hbar\omega) |n\rangle \\ &= (E_n - \hbar\omega)a|n\rangle \end{aligned}$$

where  $a|n\rangle$  is an energy eigen state with eigen value  $(E_n - \hbar\omega)$ . Operator  $a$  lowers the energy and therefore it is called annihilation, destruction or absorption operator.

$$\implies |n-1\rangle = \frac{a}{\alpha_n} |n\rangle,$$

is an energy eigen state but with the reduced eigen value i.e.

$$\begin{aligned} E_{n-1} &= (E_n - \hbar\omega) \\ H|n-1\rangle &= E_{n-1}|n-1\rangle, \end{aligned}$$

and  $\alpha_n$  is a constant which will be determined from the normalization condition,

$$\langle n-1 | n-1 \rangle = 1.$$

If we repeat this procedure  $n$  times we move down the energy ladder in steps of  $\hbar\omega$  until we obtain

$$Ha|0\rangle = (E_0 - \hbar\omega)a|0\rangle$$

$E_0$  is the ground state energy .  $E_n - \hbar\omega$  is smaller than  $E_0$  i.e,  $E_n - \hbar\omega$  is negative. Since energy eigen value cannot be negative

$$a |0\rangle = 0$$

The state  $|0\rangle$  is called the vacuum state. (in which no photon is excited).

$$\begin{aligned} H |0\rangle &= \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) |0\rangle \\ &= \frac{1}{2} \hbar\omega |0\rangle \\ \implies E_0 &= \frac{1}{2} \hbar\omega \end{aligned}$$

is the energy of the ground state. Now we go step by step up as

$$\begin{aligned} E_{n-1} &= E_n - \hbar\omega \\ E_n &= E_{n-1} + \hbar\omega \end{aligned}$$

For  $n = 1$  we can write

$$\begin{aligned} E_1 &= E_0 + \hbar\omega \\ &= \frac{1}{2} \hbar\omega + \hbar\omega = \frac{3}{2} \hbar\omega \end{aligned}$$

Similarly

$$\begin{aligned} E_2 &= E_1 + \hbar\omega \\ &= \frac{3}{2} \hbar\omega + \hbar\omega \\ &= \frac{5}{2} \hbar\omega \end{aligned}$$

It can also be written as

$$\begin{aligned} E_2 &= \left( 2 + \frac{1}{2} \right) \hbar\omega, \\ &\vdots \\ E_n &= \left( n + \frac{1}{2} \right) \hbar\omega \\ \implies H |n\rangle &= E_n |n\rangle \end{aligned}$$

$$\begin{aligned} \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) |n\rangle &= \left( n + \frac{1}{2} \right) \hbar\omega |n\rangle \\ \implies a^\dagger a |n\rangle &= n |n\rangle \end{aligned}$$



$|n\rangle$  is also an energy eigen state of the number operator

$$n = a^\dagger a$$

The normalization constant can be now calculated

$$\langle n-1 | n-1 \rangle = 1$$

as

$$\begin{aligned} |n-1\rangle &= \frac{\hat{a}}{\alpha_n} |n\rangle \\ \langle n-1| &= \frac{\langle n| a^\dagger}{\alpha_n^*} \\ \langle n-1 | n-1 \rangle &= \frac{\langle n| a^\dagger a |n\rangle}{\alpha_n^* \alpha_n} \\ &= \frac{1}{|\alpha_n|^2} \langle n| a^\dagger a |n\rangle \\ 1 &= \frac{n}{|\alpha_n|^2} \langle n | n \rangle = |\alpha_n|^2 = n \\ \alpha &= \sqrt{n} e^{i\phi} \end{aligned}$$

If we take the phase of the normalization constant  $\alpha_n$  to be zero then  $\alpha_n = \sqrt{n}$

$$\begin{aligned} a |n\rangle &= \alpha_n |n-1\rangle \\ &= \sqrt{n} |n-1\rangle \end{aligned}$$

now for operator  $a^\dagger$

$$\begin{aligned} H a^\dagger |n\rangle &= \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) a^\dagger |n\rangle \\ &= \hbar\omega \left( a^\dagger a a^\dagger + \frac{a^\dagger}{2} \right) |n\rangle \end{aligned}$$

using  $aa^\dagger = a^\dagger a + 1$ ,

$$\begin{aligned} H a^\dagger |n\rangle &= \hbar\omega \left( a^\dagger a^\dagger a + a^\dagger + \frac{a^\dagger}{2} \right) |n\rangle \\ &= a^\dagger (E_n + \hbar\omega) |n\rangle \\ \left( \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) a^\dagger |n\rangle \right) &= (E_n + \hbar\omega) a^\dagger |n\rangle \end{aligned}$$

Thus  $a^\dagger |n\rangle$  is also an energy eigen state of the field with eigen value  $E_n + \hbar\omega$ . We define

$$\begin{aligned} |n+1\rangle &= \frac{\hat{a}^\dagger}{\beta_n} |n\rangle \\ E_{n+1} &= E_n + \hbar\omega \\ \implies H |n+1\rangle &= E_{n+1} |n+1\rangle \end{aligned}$$

using the same procedure we get

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

A repeated use of the above equation gives,

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

The energy eigen states  $|n\rangle$  are called fock states or photon number states. They form a complete set of state i.e.

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = 1$$

The energy eigen value are discrete in contrast to classical electromagnetic theory where energy can have any value. State vector is written as the superposition of energy eigen states. i.e

$$|\Psi\rangle = \sum_n c_n |n\rangle$$

where  $c_n$  are complex coefficients. The energy  $E_0 = \frac{1}{2}\hbar\omega$  is called zero-point energy. The energy levels for Q.M oscillations associated with the electromagnetic field are given as

$$\begin{array}{l} \text{-----} \xrightarrow{a^\dagger \uparrow} \text{-----} E_{n+1} = \left(n + \frac{3}{2}\right) \hbar\omega \\ \text{-----} \xrightarrow{a \downarrow} \text{-----} E_n = \left(n + \frac{1}{2}\right) \hbar\omega \\ \text{-----} \text{-----} E_{n-1} = \left(n - \frac{1}{2}\right) \hbar\omega \\ \vdots \\ \text{-----} \text{-----} E_2 = \frac{5}{2} \hbar\omega \\ \text{-----} \text{-----} E_1 = \frac{3}{2} \hbar\omega \\ \text{-----} \text{-----} E_0 = \frac{1}{2} \hbar\omega \end{array}$$

The operators  $a$  and  $a^\dagger$  are not hermitian but some of the combinations are Hermitian such as,

$$\begin{aligned} a_1 &= (a + a^\dagger) / 2, \\ a_2 &= (a - a^\dagger) / 2i. \end{aligned}$$

Different energy eigen states of the field are orthogonal. The only non-vanishing matrix elements of  $a$  and  $a^\dagger$  are of the types;

$$\begin{aligned}\langle n-1 \mid a \mid n \rangle &= \sqrt{n} \\ \langle n+1 \mid a^\dagger \mid n \rangle &= \sqrt{n+1}\end{aligned}$$

An important property of  $\mid n \rangle$  is that the expectation value of the single mode linearly polarized field operator vanishes. Using

$$E_x(z, t) = \varepsilon(ae^{-i\omega t} + a^\dagger e^{i\omega t}) \sin kz,$$

or

$$E(r, t) = \varepsilon a e^{-i\omega t + ik \cdot r} + \varepsilon^* a^\dagger e^{i\omega t - ik \cdot r}$$

$$\langle n \mid E(r, t) \mid n \rangle = \varepsilon \langle n \mid a \mid n \rangle e^{-i\omega t + ik \cdot r} + \varepsilon^* \langle n \mid a^\dagger \mid n \rangle e^{i\omega t - ik \cdot r} = 0$$

Now in order to find the average value of  $\langle E^2 \rangle$ , we write

$$\begin{aligned}\langle n \mid E^2 \mid n \rangle &= |\varepsilon|^2 \langle n \mid aa^\dagger + a^\dagger a \mid n \rangle + \varepsilon^2 e^{-2i\omega t + 2ik \cdot r} \langle n \mid a^2 \mid n \rangle \\ &+ \varepsilon^2 e^{2i\omega t - 2ik \cdot r} \langle n \mid a^{\dagger 2} \mid n \rangle.\end{aligned}$$

$$\begin{aligned}\langle E^2 \rangle &= (2n+1)|\varepsilon|^2 = 2\left(n + \frac{1}{2}\right)|\varepsilon|^2 \\ \Delta E^2 &= \langle E^2 \rangle - \langle E \rangle^2 \\ &= 2\left(n + \frac{1}{2}\right)|\varepsilon|^2,\end{aligned}$$

as  $\langle E \rangle^2 = 0$ . For  $n = 0$  i.e. in vacuum

$$\Delta E^2 \neq 0,$$

but is equal to

$$\Delta E^2 = |\varepsilon|^2$$

From these equations we conclude that the mean value is zero but fluctuations are present. These fluctuations are considered to be responsible for spontaneous emission, Lamb shift etc.

## 5 Multimode Field

The wave function of a single mode field can be written as a linear superposition of photon number states  $|n\rangle$ .

$$|\Psi\rangle = \sum_n c_n |n\rangle$$

For multimode field

$$H = \sum_k H_k$$

where

$$H_k = \hbar\omega_k \left( a_k^\dagger a_k + \frac{1}{2} \right)$$

The energy eigen state of  $H_k$  is

$$H_k |n_k\rangle = \hbar\omega_k \left( n_k + \frac{1}{2} \right) |n_k\rangle$$

The general eigen state of  $H = \sum_k H_k$  can therefore have  $n_{k1}$  photon in the 1st mode,  $n_{k2}$  in the 2nd mode,  $n_{kl}$  in the  $l$ th mode and so on and can be written as

$$|n_{k1}, n_{k2}, \dots, n_{kl}, \dots\rangle$$

Different cavity modes are independent, the state of the total field can be written as the product of the individual modes i.e.

$$|n_{k1}, n_{k2}, \dots, n_{kl}, \dots\rangle = |n_{k1}\rangle |n_{k2}\rangle |n_{k3}\rangle \dots$$

As the states of the individual cavity modes are normalized, this implies that the total state of the field is also normalized. This state can be written in short as

$$|n_{k1}\rangle |n_{k2}\rangle |n_{k3}\rangle \dots = \{|n_k\rangle\}$$

The symbols  $\{n_k\}$  denote the complete set of numbers that specify the excitation levels of the harmonic oscillators with the cavity modes. The annihilation and creation operators  $a_{kl}$  and  $a_{kl}^\dagger$  lower and raise the  $n_l$ th entry alone.

$$\begin{aligned} a_{kl} |n_{k1}, n_{k2}, \dots, n_{kl}, \dots\rangle &= \sqrt{n_{kl}} |n_{k1}, n_{k2}, \dots, n_{kl}-1, \dots\rangle \\ a_{kl}^\dagger |n_{k1}, n_{k2}, \dots, n_{kl}, \dots\rangle &= \sqrt{n_{kl}+1} |n_{k1}, n_{k2}, \dots, n_{kl}+1, \dots\rangle \end{aligned}$$

The general state vector of the field is a linear superposition of these states.

$$\begin{aligned} |\Psi\rangle &= \sum_{n_{k1}} \sum_{n_{k2}} \dots \sum_{n_{kl}} \dots C_{n_{k1}, n_{k2}, \dots, n_{kl}} |n_{k1}, n_{k2}, \dots, n_{kl}\rangle \\ &= \sum_{\{n_k\}} C_{\{n_k\}} |\{n_k\}\rangle \end{aligned}$$

It can also be written as

$$|\Psi\rangle = |\Psi_{k1}\rangle |\Psi_{k2}\rangle \dots \dots \dots |\Psi_{kl}\rangle$$

## 6 State of the Field

### 6.1 The Mode Phase Operators

Classically an electromagnetic field consists of waves with well defined amplitude and phase

$$E(r, t) = \varepsilon_0 \exp i(k.r - \omega t + \phi) + c.c$$

Quantized fields has fluctuations associated with both amplitude and phase of the field. An electromagnetic field in a number state  $|n\rangle$  has a well defined amplitude but completely uncertain phase

$$\hat{n} |n\rangle = n |n\rangle$$

where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle$$

The single mode quantum mechanical electric field operator is

$$\hat{E} = \hat{\epsilon} \left( \frac{\hbar\omega}{2\epsilon_0 V} \right)^{1/2} \left( a \exp(-i\omega t + ik.r) + a^\dagger \exp(i\omega t - ik.r) \right)$$

where  $\epsilon$  is a unit polarization vector. In an analogy to classical field separate  $a$  and  $a^\dagger$  into a product of amplitude and phase operators. In quantum mechanics there is no unique way or definition of the quantum mechanical phase operators, But there are some conditions on the phase operator.

- 1- It should have same significance as classical phase in the appropriate limit.
- 2- It should be associated with hermitian operators (observable quantity).

Consider the phase operator  $e^{i\phi}$  defined by the relation (Ref: Susskind and Glogower (1964))

$$a = \sqrt{\hat{n} + 1} e^{i\phi}$$

The hermitian conjugate relation is

$$a^\dagger = e^{-i\phi} \sqrt{\hat{n} + 1}$$

The exponential operator defined, describe the quantum mechanical phase of electromagnetic field,

$$\begin{aligned} e^{i\phi} &= (\hat{n} + 1)^{-1/2} a \\ \text{and, } e^{-i\phi} &= a^\dagger (\hat{n} + 1)^{-1/2} \end{aligned}$$

Now from the relation

$$\begin{aligned} [a, a^\dagger] &= aa^\dagger - a^\dagger a = 1 \\ \implies e^{i\phi} e^{-i\phi} &= 1 \end{aligned}$$

The reverse order product of exponential operator is not equal to unity. Now we have to check whether these phase operators are hermitian or not. Find  $e^{i\phi} | n \rangle$  and  $e^{-i\phi} | n \rangle$

$$\begin{aligned} e^{i\phi} | n \rangle &= (\hat{n} + 1)^{-1/2} a | n \rangle \\ &= (\hat{n} + 1)^{-1/2} \sqrt{n} | n - 1 \rangle \end{aligned}$$

As we know that  $f(\hat{n}) | n \rangle = f(n) | n \rangle$ , we can write

$$\begin{aligned} e^{i\phi} | n \rangle &= (n - 1 + 1)^{-1/2} \sqrt{n} | n - 1 \rangle = | n - 1 \rangle \\ e^{-i\phi} | n \rangle &= a^\dagger (\hat{n} + 1)^{-1/2} | n \rangle = | n + 1 \rangle \end{aligned}$$

The non-vanishing matrix elements of phase operators are

$$\begin{aligned} \langle n - 1 | e^{i\phi} | n \rangle &= 1 \\ \langle n + 1 | e^{-i\phi} | n \rangle &= 1 \end{aligned}$$

and all other types of matrix elements are zero. It implies that  $e^{i\phi}$  and  $e^{-i\phi}$  do not satisfy the relation.

$$\langle i | \hat{O} | j \rangle = \langle j | \hat{O} | i \rangle^*$$

where  $O = O^\dagger$ . They are not hermitian operators but can be combined to produce another pair of operators.

$$\begin{aligned} \hat{\cos}\phi &= \frac{1}{2} \{ e^{i\phi} + e^{-i\phi} \} \\ \hat{\sin}\phi &= \frac{1}{2} \{ e^{i\phi} - e^{-i\phi} \} \end{aligned}$$

whose non-vanishing matrix elements are

$$\langle n - 1 | \hat{\cos}\phi | n \rangle = \langle n | \hat{\cos}\phi | n - 1 \rangle = \frac{1}{2}$$

and

$$\langle n - 1 | \hat{\sin}\phi | n \rangle = -\langle n | \hat{\sin}\phi | n - 1 \rangle = \frac{1}{2i}$$

These matrix elements do satisfy equation

$$\langle i | \hat{O} | j \rangle = \langle j | \hat{O} | i \rangle^*$$

and operators  $\hat{\cos}\phi$  and  $\hat{\sin}\phi$  are hermitian. They represent the observable phase properties of the electromagnetic field.

$$\begin{aligned} [n, \hat{\cos}\phi] &= -i\hat{\sin}\phi \\ [n, \hat{\sin}\phi] &= i\hat{\cos}\phi \end{aligned}$$

Number and phase operators do not commute therefore it is not possible to setup states of the radiation field that are simultaneous eigen states of the operator. The amplitude of an electromagnetic wave associated with  $\hat{n}$  and phase associated with  $\hat{\cos}\phi$  or  $\hat{\sin}\phi$  cannot both be precisely specified. The uncertainty relation

$$\begin{aligned} [q, p] &= i\hbar \\ \Delta q \Delta p &\geq \frac{1}{2} |\langle \hbar \rangle| \\ &\geq \frac{\hbar}{2} \end{aligned}$$

The uncertainty relation between  $\hat{n}$  and phase operators are

$$\begin{aligned} \Delta n \Delta \cos \phi &\geq \frac{1}{2} \left| \langle \hat{\sin}\phi \rangle \right| \\ \Delta n \Delta \sin \phi &\geq \frac{1}{2} \left| \langle \hat{\cos}\phi \rangle \right| \end{aligned}$$

where  $\Delta$  indicates the root mean square deviation. The uncertainty in the photon number is zero for the state  $|n\rangle$  i.e.

$$\begin{aligned} \Delta n &= \sqrt{\langle n^2 \rangle - \langle n \rangle^2} \\ \langle n \mid \hat{n} \mid n \rangle &= \langle n \mid a^\dagger a \mid n \rangle = n \langle n \mid n \rangle = n \\ \Delta n &= \sqrt{n^2 - n^2} = 0 \end{aligned}$$

For the phase operators,

$$\langle n \mid \hat{\cos}\phi \mid n \rangle = \langle n \mid \hat{\sin}\phi \mid n \rangle = 0$$

Diagonal elements of the phase operators are zero. Now the expectation values of phase operators are

$$\begin{aligned} \langle n \mid \hat{\cos}\phi \mid n \rangle &= \langle \hat{\cos}\phi \rangle = 0 \\ \langle n \mid \hat{\sin}\phi \mid n \rangle &= \langle \hat{\sin}\phi \rangle = 0 \end{aligned}$$

and

$$\begin{aligned} \langle n \mid \hat{\cos}^2\phi \mid n \rangle &= \langle n \mid \hat{\sin}^2\phi \mid n \rangle = \frac{1}{2}, & \text{for } n \neq 0 \\ &= \frac{1}{4} & \text{for } n = 0 \end{aligned}$$

The root mean square deviation or uncertainties in phase are

$$\Delta \cos \phi = \Delta \sin \phi = \frac{1}{\sqrt{2}} \quad \text{for } n \neq 0$$

This value shows that phase angle can have any value between 0 to  $2\pi$ . The expectation value of field operator is zero

$$\langle E \rangle = \langle n | E | n \rangle = 0$$

and the expectation value of  $E^2$  is given as

$$\begin{aligned} \langle n | E^2 | n \rangle &= 2|\varepsilon|^2 (n + 1/2) \\ \varepsilon &= \left( \frac{\hbar\omega}{2\varepsilon_0 V} \right)^{1/2} \end{aligned}$$

$$\begin{aligned} \Delta E &= \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \\ \Delta E &= \left( \frac{\hbar\omega}{\varepsilon_0 V} \right)^{1/2} \left( n + \frac{1}{2} \right)^{\frac{1}{2}} \end{aligned}$$

Field amplitude of electromagnetic wave is given by

$$E_0 = \left( \frac{\hbar\omega}{\varepsilon_0 V} \right)^{1/2} \left( n + \frac{1}{2} \right)^{\frac{1}{2}}$$

The actual position of the wave along the horizontal axis is completely undetermined owing to the complete uncertainty in the phase angle. Field oscillates like a sine wave of known frequency  $\omega$ .

Number states form a useful representation for high energy photons, e.g,  $\gamma$ -rays where the number of photons is very small. They are not the most suitable representation for optical fields where the total number of photons is large. Experimental difficulties have prevented the generation of photon number states with more than a small number of photons.

Most optical fields are either a superposition of number states (pure state) or a mixture of number states (mixed state). Despite this the number states of the electromagnetic field have been used as a basis for several problems in quantum optics including some laser theories. A more appropriate basis for many optical fields are the coherent states. The coherent states have indefinite number of photons which allow them to have a more precisely defined phase than a number state where the phase is completely random.



# 1 The Coherent Photon States

The single-mode states of physical importance are not the individual number states  $|n\rangle$  (because the electromagnetic wave generated by practical light source do not have definite numbers of photons), but the linear superposition of states  $|n\rangle$ . There is a wide variety of possible superposition states.

A superposition state can be constructed for which uncertainties in the expectation values of the phase operators  $\hat{\cos}\phi$  and  $\hat{\sin}\phi$  are both equal to zero. Such states have  $\Delta n = \infty$ . They cannot be excited in any real experiment. Another kind is the coherent state. A coherent state has equal amount of uncertainties in amplitude and phase. A field in coherent state is in a minimum uncertainty state. For coherent state an electric field variation approaches that of classical wave of stable amplitude and fixed phase, in the limit of high excitation. They are important because, they are the closest quantum mechanical approach to a classical electromagnetic wave. A single mode laser operated well above threshold generates a coherent state excitation.

The coherent state  $|\alpha\rangle$  is the eigen state of the positive frequency part of the electric field operator or the eigen state of the destruction operator of the field.

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle,$$

where  $\alpha$  is complex,  $|\alpha\rangle$  in terms of linear superposition of number state  $|n\rangle$  is given by

$$\begin{aligned} |\alpha\rangle &= \sum_{n=0}^{\infty} C_n |n\rangle & (1) \\ \hat{a} |\alpha\rangle &= \sum_{n=0}^{\infty} C_n \hat{a} |n\rangle \\ &= \sum_{n=0}^{\infty} C_n \sqrt{n} |n-1\rangle \\ &= 0 + C_1 \sqrt{1} |0\rangle + C_2 \sqrt{2} |1\rangle + \dots \\ &= \sum_{n=0}^{\infty} C_{n+1} \sqrt{n+1} |n\rangle & (a) \end{aligned}$$

From eqn(1) multiplying with  $\alpha$  we can write

$$\begin{aligned} \alpha |\alpha\rangle &= \sum_{n=0}^{\infty} C_n \alpha |n\rangle \\ \hat{a} |\alpha\rangle &= \sum_{n=0}^{\infty} C_n \alpha |n\rangle & (b) \end{aligned}$$

comparing eqn(a) and (b)

$$\begin{aligned}
C_{n+1}\sqrt{n+1} &= C_n\alpha \\
C_n\sqrt{n} &= C_{n-1}\alpha \\
C_n &= \frac{\alpha}{\sqrt{n}}C_{n-1} \\
C_n &= \frac{\alpha}{\sqrt{n}}\frac{\alpha}{\sqrt{n-1}}\frac{\alpha}{\sqrt{n-2}}\cdots\frac{\alpha}{\sqrt{1}}C_0 \\
&= \frac{\alpha^n}{\sqrt{n!}}C_0
\end{aligned}$$

putting in Eqn(1),

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

the constant  $C_0$  can be found by normalization

$$\begin{aligned}
\langle\alpha | \alpha\rangle &= C_0^*C_0 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} \frac{\alpha^m}{\sqrt{m!}} \langle n | m\rangle \\
\langle\alpha | \alpha\rangle &= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}
\end{aligned}$$

using

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

we can write

$$\begin{aligned}
1 &= |C_0|^2 e^{|\alpha|^2} \\
|C_0|^2 &= e^{-|\alpha|^2} \\
C_0 &= e^{-\frac{|\alpha|^2}{2}}
\end{aligned}$$

Therefore

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Another way of proving the above relation which interpret  $|\alpha\rangle$  as a superposition of number state is,

$$|\alpha\rangle = \sum_n |n\rangle \langle n | \alpha\rangle \quad (1)$$

where  $\sum_n |n\rangle \langle n| = 1$  is the completeness relation for number state. As

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \quad (2)$$

$$\langle n | = \langle 0 | \frac{(a)^n}{\sqrt{n!}} \quad (3)$$

Putting in Eqn(1) we can write as

$$| \alpha \rangle = \sum_n | n \rangle \langle 0 | \frac{(a)^n}{\sqrt{n!}} | \alpha \rangle$$

we know that

$$\begin{aligned} \hat{a} | \alpha \rangle &= \alpha | \alpha \rangle \\ (\hat{a})^n | \alpha \rangle &= \alpha^n | \alpha \rangle \\ \Rightarrow | \alpha \rangle &= \sum_n | n \rangle \frac{(\alpha)^n}{\sqrt{n!}} \langle 0 | \alpha \rangle \end{aligned} \quad (4)$$

The value of  $\langle 0 | \alpha \rangle$  is obtained by normalization i,e

$$\langle \alpha | \alpha \rangle = 1$$

$$\begin{aligned} \langle \alpha | \alpha \rangle &= \sum_n \sum_m \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{(\alpha)^n}{\sqrt{n!}} \langle m | n \rangle |\langle 0 | \alpha \rangle|^2 \\ &= \sum_n \frac{(\alpha^* \alpha)^n}{n!} |\langle 0 | \alpha \rangle|^2 \\ &= \sum_n \frac{(|\alpha|^2)^n}{n!} |\langle 0 | \alpha \rangle|^2 \\ \Rightarrow 1 &= e^{|\alpha|^2} |\langle 0 | \alpha \rangle|^2 \\ |\langle 0 | \alpha \rangle|^2 &= e^{-|\alpha|^2} \\ \langle 0 | \alpha \rangle &= e^{-\frac{|\alpha|^2}{2}} \end{aligned}$$

Putting in Eqn(4) we get

$$| \alpha \rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle$$

hence proved.

Some other representations of the coherent state

$$\begin{aligned}
|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
|n\rangle &= \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \\
|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \\
|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0\rangle
\end{aligned}$$

since we know that  $e^{-\alpha^* a} |0\rangle = |0\rangle$

$$\begin{aligned}
&\implies |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a} |0\rangle \\
D(\alpha) &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}
\end{aligned}$$

where  $D(\alpha)$  is called the displacement operator.

$$|\alpha\rangle = D(\alpha) |0\rangle$$

## 1.1 Baker-Hausdorff identity

If

$$[[A, B], A] = [[A, B], B] = 0$$

then

$$e^{A+B} = e^{-[A,B]/2} e^A e^B$$

Let we have  $A = \alpha a^\dagger$  and  $B = -\alpha^* a$

$$\begin{aligned}
e^{\alpha a^\dagger - \alpha^* a} &= e^{-\frac{1}{2}[-\alpha a^\dagger \alpha^* a + \alpha^* a \alpha a^\dagger]} e^{\alpha a^\dagger} e^{-\alpha^* a} \\
&= e^{-\frac{1}{2}|\alpha|^2[-a^\dagger a + a a^\dagger]} e^{\alpha a^\dagger} e^{-\alpha^* a} \\
&= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger - \alpha^* a}
\end{aligned}$$

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

another definition

$$\implies |\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle$$

The other equivalent antinormal form of  $D(\alpha)$  is obtained by using  $A = -\alpha^* a$  and  $B = \alpha a^\dagger$ , then we get

$$D(\alpha) = e^{\frac{|\alpha|^2}{2}} e^{-\alpha^* a} e^{\alpha a^\dagger}$$

or by using

$$\begin{aligned} e^{A+B} &= e^{\frac{1}{2}[A,B]} e^B e^A \\ D(\alpha) &= e^{\frac{|\alpha|^2}{2}} e^{-\alpha^* a} e^{\alpha a^\dagger} \end{aligned}$$

The operator  $D(\alpha)$  is a unitary operator. i.e.

$$D^\dagger(\alpha) = D(-\alpha) = D^{-1}(\alpha)$$

It acts as a displacement operator upon the amplitudes  $a$  and  $a^\dagger$  i.e.

$$\begin{aligned} D^{-1}(\alpha) \hat{a} D(\alpha) &= a + \alpha \\ D^{-1}(\alpha) \hat{a}^\dagger D(\alpha) &= a^\dagger + \alpha^* \end{aligned}$$

This can be proved by writing

$$\begin{aligned} D(\alpha) &= e^{\alpha a^\dagger - \alpha^* a} \\ D^\dagger(\alpha) &= e^{\alpha^* a - \alpha a^\dagger} = D^{-1}(\alpha) \end{aligned}$$

using these equations we get

$$D^{-1}(\alpha) \hat{a} D(\alpha) = e^{\alpha^* a} e^{-\alpha a^\dagger} \hat{a} e^{\alpha a^\dagger} e^{-\alpha^* a}$$

For any operators A and B we have

$$e^{-\alpha A} B e^{\alpha A} = B - \alpha [A, B] + \frac{\alpha^2}{2!} [A, [A, B]] + \dots$$

For  $A = \hat{a}^\dagger$  and  $B = \hat{a}$

$$\begin{aligned} e^{-\alpha \hat{a}^\dagger} \hat{a} e^{\alpha \hat{a}^\dagger} &= a + \alpha \\ D^{-1}(\alpha) \hat{a} D(\alpha) &= a + \alpha \end{aligned}$$

Similarly for

$$D^{-1}(\alpha) \hat{a}^\dagger D(\alpha) = e^{\alpha^* a} \hat{a}^\dagger e^{-\alpha^* a}$$

here  $A = \hat{a}$  and  $B = \hat{a}^\dagger$

$$\implies D^{-1}(\alpha) \hat{a}^\dagger D(\alpha) = a^\dagger + \alpha^*$$

Prove

$$\begin{aligned} e^{-\alpha A} B e^{\alpha A} &= B - \alpha [A, B] + \frac{\alpha^2}{2!} [A, [A, B]] + \dots \\ &\left( 1 - \alpha A + \frac{\alpha^2 A^2}{2!} + \dots \right) B \left( 1 + \alpha A + \frac{\alpha^2 A^2}{2!} + \dots \right) \\ &= B - \alpha (AB - BA) + \frac{\alpha^2}{2!} (\dots) \\ e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} &= a - \alpha [a^\dagger, a] + \frac{\alpha^2}{2!} [a^\dagger, [a^\dagger, a]] + \dots \\ &= a^\dagger + \alpha \end{aligned}$$

## 1.2 Properties of coherent states

Properties of a cavity mode excited to a coherent state  $|\alpha\rangle$  can be determined by the method applied to the number state  $|n\rangle$ .

1- The mean number of photon in the coherent state  $|\alpha\rangle$  is given by

$$\begin{aligned}\langle n \rangle &= \langle \alpha | \hat{n} | \alpha \rangle = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2 \\ | \alpha \rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n \rangle \\ \langle \alpha | &= e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m | \end{aligned}$$

therefore

$$\begin{aligned}\langle n \rangle &= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} \langle m | a^\dagger a | n \rangle \\ &= e^{-|\alpha|^2} \sum_n \frac{(\alpha^* \alpha)^n}{n!} n \\ &= e^{-|\alpha|^2} \sum_n \frac{(|\alpha|^2)^n}{n!} n \end{aligned}$$

Let  $x = |\alpha|^2$ , and also

$$\begin{aligned}x \frac{\partial}{\partial x} \sum_n \frac{x^n}{n!} &= \sum_n \frac{x^n}{n!} n \\ x \frac{\partial}{\partial x} e^x &= \sum_n \frac{x^n}{n!} n \end{aligned}$$

therefore we can write

$$\begin{aligned}\langle n \rangle &= e^{-|\alpha|^2} |\alpha|^2 \frac{\partial}{\partial |\alpha|^2} e^{|\alpha|^2} \\ &= |\alpha|^2 \end{aligned}$$

Find

$$\begin{aligned}\langle \alpha | \hat{n}^2 | \alpha \rangle &= e^{-|\alpha|^2} \sum_n \frac{(\alpha^* \alpha)^n}{n!} n^2 \\ &= e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} \{n(n-1) + n\} \end{aligned}$$

Let again  $x = |\alpha|^2$ , by the definition

$$\begin{aligned}\sum_n \frac{x^n}{n!} n(n-1) &= x^2 \frac{\partial^2}{\partial x^2} \sum_n \frac{x^n}{n!} \\ &= x^2 \frac{\partial^2}{\partial x^2} e^x \end{aligned}$$

so we can write

$$\langle \alpha | \hat{n}^2 | \alpha \rangle = e^{-|\alpha|^2} e^{|\alpha|^2} (|\alpha|^4 + |\alpha|^2)$$

Root-mean-square deviation is

$$\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2} = \sqrt{|\alpha|^2}$$

Where  $|\alpha|^2$  is the mean number of photons in the cavity mode and uncertainty spread about the mean is equal to the square root of the mean number of photons.

ii)- Photon statistics: photon distribution function:  
The probability of finding n-photons in the field  $|\alpha\rangle$  is

$$p(n) = |\langle n | \alpha \rangle|^2$$

where

$$\langle n | \alpha \rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{(\alpha)^m}{\sqrt{m!}} \langle n | m \rangle$$

$$\begin{aligned} p(n) &= \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!} \\ &= \frac{e^{-\langle n \rangle} \langle n \rangle^n}{n!} \end{aligned}$$

is a poisson distribution.

iii)- Coherent state is the minimum energy state: i.e

$$\Delta p \Delta q = \frac{\hbar}{2}$$

$\hat{a}$  and  $\hat{a}^\dagger$  are not hermitian but their combinations are.  
Let  $m_j = 1$  at  $t = 0$   $r = 0$

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q} + i\hat{p}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q} - i\hat{p}) \end{aligned}$$

adding these two we get

$$\begin{aligned} \frac{1}{2} (\hat{a} + \hat{a}^\dagger) &= \sqrt{\frac{\omega}{2\hbar}} \hat{q} \\ \hat{a} + \hat{a}^\dagger &= \sqrt{\frac{2\omega}{\hbar}} \hat{q} \end{aligned}$$

and

$$\begin{aligned}\hat{a} - \hat{a}^\dagger &= i\sqrt{\frac{2}{\hbar\omega}}\hat{p} \\ \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) &= \sqrt{\frac{1}{2\hbar\omega}}\hat{p}\end{aligned}$$

$\hat{p}$  and  $\hat{q}$  are hermitian and represent observable quantities,

$$\begin{aligned}[\hat{q}, \hat{p}] &= i\hbar \\ \Delta\hat{q}\Delta\hat{p} &\geq \frac{\hbar}{2}\end{aligned}$$

for a coherent state we have to prove that

$$\Delta p \Delta q = \frac{\hbar}{2}$$

$$\begin{aligned}\langle \hat{p} \rangle &= \langle \alpha | \hat{p} | \alpha \rangle = \frac{\sqrt{2\hbar\omega}}{2i} (\langle a \rangle - \langle a^\dagger \rangle) \\ \langle a \rangle &= \langle \alpha | a | \alpha \rangle = \alpha \langle \alpha | \alpha \rangle\end{aligned}$$

$$\begin{aligned}\langle p \rangle &= \frac{\sqrt{2\hbar\omega}}{2i} (\alpha - \alpha^*) \\ \langle p^2 \rangle &= \left( \frac{\sqrt{2\hbar\omega}}{2i} \right)^2 \langle (a - a^\dagger)(a - a^\dagger) \rangle \\ &= -\frac{2\hbar\omega}{4} \langle \alpha | a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a | \alpha \rangle \\ &= -\frac{2\hbar\omega}{4} (\alpha^2 + \alpha^{*2} - 2\alpha^* \alpha - 1)\end{aligned}$$

$$\begin{aligned}\Delta p^2 &= \langle p^2 \rangle - \langle p \rangle^2 \\ &= -\frac{\hbar\omega}{2} (\alpha^2 + \alpha^{*2} - 2\alpha^* \alpha - 1) + \frac{\hbar\omega}{2} (\alpha^2 + \alpha^{*2} - 2\alpha^* \alpha) \\ &= \frac{\hbar\omega}{2}\end{aligned}$$

Similarly

$$\begin{aligned}\Delta q^2 &= \frac{\hbar}{2\omega} \\ \Delta q \Delta p &= \frac{\hbar}{2}\end{aligned}$$



iv)- Coherent states are not orthogonal, but are normalized. i.e.

$$\langle \alpha | \alpha \rangle = e^{-|\alpha|^2} \sum_n \frac{(\alpha^* \alpha)^n}{n!} = 1$$

For two different complex numbers  $\alpha$  and  $\beta$ .

$$\begin{aligned} | \alpha \rangle &= e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} | n \rangle \\ | \beta \rangle &= e^{-\frac{|\beta|^2}{2}} \sum_m \frac{\beta^m}{\sqrt{m!}} | m \rangle \end{aligned}$$

$$\begin{aligned} \langle \alpha | \beta \rangle &= e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} e^{\alpha \times \beta} \neq 0 \\ \implies |\langle \alpha | \beta \rangle|^2 &= e^{-|\alpha - \beta|^2}. \end{aligned}$$

The  $| \alpha \rangle$  form an over complete set of states and lack of orthogonality is a consequence of this.  $i, j, k$  are orthogonal and independent of each other. If we divide space in 5 directions they would not be orthogonal and independent of each other. Therefore over complete. i.e. there are many more coherent states  $| \alpha \rangle$  than there are states  $| n \rangle$ .

Completeness relation:

For number states

$$\sum_n | n \rangle \langle n | = 1$$

Similarly the set of all coherent states  $| \alpha \rangle$  is a complete set and satisfy the completeness relation.

$$\frac{1}{\pi} \int d^2 \alpha | \alpha \rangle \langle \alpha | = 1$$

Let

$$\begin{aligned} \alpha &= r e^{i\theta} \\ d^2 \alpha &= r dr d\theta \end{aligned}$$

$$\begin{aligned} \int d^2 \alpha | \alpha \rangle \langle \alpha | &= \int e^{-\frac{|\alpha|^2}{2}} \sum_m \frac{\alpha^m}{\sqrt{m!}} | m \rangle e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^{*n}}{\sqrt{n!}} \langle n | d^2 \alpha \\ &= \int_0^\infty \int_0^{2\pi} e^{-|\alpha|^2} \sum_{n,m} \frac{|\alpha|^{n+m+1}}{\sqrt{n!m!}} e^{i(m-n)\theta} | m \rangle \langle n | d|\alpha| d\theta \end{aligned}$$

as

$$\int_0^{2\pi} d\theta e^{i(m-n)\theta} = 2\pi \delta_{nm}$$

therefore

$$\begin{aligned} \int d^2\alpha \quad | \quad \alpha \rangle \langle \alpha | &= 2\pi \int_0^\infty \sum_n \frac{1}{n!} |\alpha|^{2n+1} e^{-|\alpha|^2} |n\rangle \langle n| d(|\alpha|) \\ &= \pi \sum_n \frac{1}{n!} \int_0^\infty 2|\alpha| d(|\alpha|) |\alpha|^{2n} e^{-|\alpha|^2} |n\rangle \langle n| \end{aligned}$$

putting

$$\begin{aligned} x &= |\alpha|^2 \\ dx &= 2|\alpha| d|\alpha| \end{aligned}$$

$$\int d^2\alpha \quad | \quad \alpha \rangle \langle \alpha | = \pi \sum_n \frac{1}{n!} \int_0^\infty dx x^n e^{-x} |n\rangle \langle n|$$

$$\int_0^\infty dx x^n e^{-x} = n!$$

$$\int d^2\alpha \quad | \quad \alpha \rangle \langle \alpha | = \pi \sum_n \frac{1}{n!} n! |n\rangle \langle n|$$

$$= \pi \sum_n |n\rangle \langle n|$$

$$\frac{1}{\pi} \int d^2\alpha \quad | \quad \alpha \rangle \langle \alpha | = 1$$

The completeness property is essential for the utility of a set of states.  $| \alpha \rangle$  is a complete but not orthogonal. As a result any coherent state can be expanded in terms of other states.

$$\begin{aligned} | \quad \alpha \rangle &= \frac{1}{\pi} \int d^2\alpha' \quad | \alpha' \rangle \langle \alpha' | \alpha \rangle \\ &= \frac{1}{\pi} \int d^2\alpha \quad | \alpha' \rangle \exp \left[ -\frac{1}{2} |\alpha|^2 + \alpha' \alpha^* - \frac{1}{2} |\alpha'|^2 \right] \end{aligned}$$

This shows that the coherent states are overcomplete.

## 2 Squeezed states of the radiation field

It is possible to generate states in which fluctuations are reduced below the symmetric quantum limit in one quadrature component, at the expense of enhanced fluctuations in the canonically conjugate quadrature such that the Heisenberg uncertainty principle is not violated. Such states of the radiation field are called squeezed states.

Consider two hermitian operators  $A$  and  $B$  which satisfy the commutation relation

$$[A, B] = iC$$

According to the Heisenberg uncertainty principle, the product of the uncertainties in determining the expectation values of two variables  $A$  and  $B$  is given by.

$$\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$$

A state of the system is called a squeezed state if the uncertainty in one of the observables (say  $A$ ) satisfies the relation

$$(\Delta A)^2 < \frac{1}{2} |\langle C \rangle|$$

If in addition to the above condition the variances satisfy the minimum uncertainty relation i.e.

$$\Delta A \Delta B = \frac{1}{2} |\langle C \rangle|$$

then the state is called an ideal squeezed state.

In a squeezed state, therefore the quantum fluctuations in one variable are reduced below their value in a symmetric minimum uncertainty state.

$$(\Delta A)^2 = (\Delta B)^2 = \frac{1}{2} |\langle C \rangle|$$

at the expense of the corresponding increased fluctuations in the conjugate variable such that the uncertainty relation is not violated.

### 2.1 Quadrature amplitude operators

Let us define Hermitian amplitude operators.

$$\begin{aligned} X_1 &= \frac{1}{2} (a + a^\dagger) \\ X_2 &= \frac{1}{2i} (a - a^\dagger) \end{aligned}$$

$X_1$  and  $X_2$  are dimensionless position and momentum operators

$$\begin{aligned} q &= \frac{\sqrt{2\hbar/\omega m}}{2} (a + a^\dagger) \\ p &= \frac{\sqrt{2\hbar\omega m}}{2i} (a - a^\dagger) \end{aligned}$$

$$\begin{aligned}\hat{a} &= X_1 + iX_2 \\ \hat{a}^\dagger &= X_1 - iX_2\end{aligned}$$

The operators  $X_1$  and  $X_2$  are Hermitian and satisfy the commutation relation

$$[X_1, X_2] = \frac{i}{2}$$

These operators are also called quadrature operators. In terms of  $X_1$  and  $X_2$  quantized single mode field can be written as

$$E(t) = 2\varepsilon\hat{\epsilon}(X_1 \cos \omega t + X_2 \sin \omega t)$$

The Hermitian operators  $X_1$  and  $X_2$  are the amplitudes of the two quadratures of the field having a phase difference  $\pi/2$ .

From the commutation relation of  $X_1$  and  $X_2$ , we get the uncertainty relation for the two amplitudes i.e.

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

A squeezed state of the radiation field is obtained if

$$(\Delta X_i)^2 < \frac{1}{4}, \quad \text{for } i = 1 \text{ or } 2$$

An ideal squeezed state is obtained if in addition to the above equation, the relation

$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

also holds. Example of an ideal squeezed state is the two-photon coherent state. The quadrature operators allow us to represent a beam of light graphically. These so called phasor diagrams are very popular in quantum optics. Any state of light can be represented on a phasor diagram of the operators, i.e. a plot of  $X_1$  versus  $X_2$ . Unlike classical vacuum, quantum mechanical vacuum is represented by a circle at the origin. ( $\langle X_1 \rangle, \langle X_2 \rangle$ ) every point in this circle will represent a wave. In quadrature representation each point represents a wave.

Now consider how does different states look like pictorially

i)- Vacuum

$$\Delta X_1 = \sqrt{\langle X_1^2 \rangle - \langle X_1 \rangle^2}$$

$$\begin{aligned}\langle X_1 \rangle &= \langle 0 | \frac{a + a^\dagger}{2} | 0 \rangle = 0 \\ \langle X_1^2 \rangle &= \langle 0 | \frac{(a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a)}{4} | 0 \rangle \\ &= \langle 0 | \frac{(a^2 + a^{\dagger 2} + 2a^\dagger a + 1)}{4} | 0 \rangle \\ &= \frac{1}{4}\end{aligned}$$

$$\implies \Delta X_1^2 = \frac{1}{4}$$

Similarly

$$\Delta X_2^2 = \frac{1}{4}$$

$$\begin{aligned} \Delta X_1 &= \frac{1}{2}, & \Delta X_2 &= \frac{1}{2} \\ \Delta X_1 \Delta X_2 &= \frac{1}{4} \end{aligned}$$

ii)- Coherent States:

$$\begin{aligned} \langle X_1 \rangle &= \frac{1}{2} \langle \alpha | a + a^\dagger | \alpha \rangle \\ &= \frac{1}{2} (\alpha + \alpha^*) \\ \alpha &= |\alpha| e^{i\phi} \end{aligned}$$

Therefore

$$\begin{aligned} \langle X_1 \rangle &= |\alpha| \cos \phi \\ \langle X_2 \rangle &= \frac{1}{2i} \langle \alpha | a - a^\dagger | \alpha \rangle \\ &= \frac{1}{2i} (\alpha - \alpha^*) \\ &= |\alpha| \sin \phi \end{aligned}$$

$$\begin{aligned} \langle X_1^2 \rangle &= \frac{1}{4} \langle \alpha | a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a | \alpha \rangle \\ &= \frac{1}{4} (\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1) \\ \langle X_2^2 \rangle &= -\frac{1}{4} (\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1) \end{aligned}$$

$$\begin{aligned} \implies \Delta X_1^2 &= \langle X_1^2 \rangle - \langle X_1 \rangle^2 \\ &= \frac{1}{4} (\alpha^2 + \alpha^{*2} + 2\alpha\alpha^* + 1) - \frac{1}{4} (\alpha^2 + \alpha^{*2} + 2\alpha\alpha^*) \\ &= \frac{1}{4} \end{aligned}$$

Similarly

$$\Delta X_2^2 = \frac{1}{4}$$

which means that there is equal amount of uncertainty in both the quadratures. It represent a circle with displacement  $|\alpha|$ . Displacement operator displaced the vacuum by an amount  $\alpha$  that creates a coherent state. Therefore coherent state is a displaced vacuum state.

$$|\alpha\rangle = D(\alpha) |0\rangle$$

Coherent state has both amplitude and phase uncertainties.

Is Coherent and Fock state are squeezed states?

First consider the coherent state

$$\begin{aligned} (\Delta X_1)^2 &= \langle \alpha | X_1^2 | \alpha \rangle - (\langle \alpha | X_1 | \alpha \rangle)^2 \\ &= \frac{1}{4} \langle \alpha | a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a | \alpha \rangle - \frac{1}{4} (\langle \alpha | (a + a^\dagger) | \alpha \rangle)^2 \\ &= \frac{1}{4} \end{aligned}$$

Similarly

$$\begin{aligned} (\Delta X_2)^2 &= \frac{1}{4} \\ \Delta X_1 \Delta X_2 &= \frac{1}{4} \end{aligned}$$

i.e. minimum uncertainty relation holds.

Coherent state is not a squeezed state.

Fock state:

$$\begin{aligned} (\Delta X_1)^2 &= \langle n | X_1^2 | n \rangle - (\langle n | X_1 | n \rangle)^2 \\ &= \frac{1}{4}(2n + 1) \end{aligned}$$

Similarly

$$(\Delta X_2)^2 = \frac{1}{4}(2n + 1)$$

This implies that Fock states  $|n\rangle$  are not squeezed states.

A coherent state with identical uncertainties in both  $X_1$  and  $X_2$  has a constant value for the variance of electric field. A squeezed state with reduced noise in  $X_1$  has reduced uncertainty in the amplitude at the expense of large uncertainty in the phase of electric field.

A squeezed state with reduced noise in  $X_2$  has reduced uncertainty in the phase at the expense of large uncertainty in the amplitude of the electric field. If we have a state such that  $\Delta X_1^2$  or  $\Delta X_2^2 < \frac{1}{4}$  then the state is a squeezed state. According to heisenberg uncertainty principle, vacuum is a circle. Area of the circle should be the same, otherwise it will violate Heisenberg uncertainty principle. In order to squeeze a coherent state we will define a squeezing operator.

### 2.1.1 Generation of squeezed state:

One example of generation of a squeezed state is the “Degenerate parametric process”. The two photon Hamiltonian can be written as

$$H = i\hbar(ga^{\dagger 2} - g^*a^2)$$

where  $g$  is the coupling constant. The state of the field is written as

$$|\Psi(t)\rangle = e^{(ga^{\dagger 2} - g^*a^2)} |0\rangle$$

and this leads to define the unitary squeezed operator.

### 2.1.2 Squeeze operator:

The squeezed states can be generated by using the unitary squeezed operator.

$$S(\zeta) = \exp\left(\frac{1}{2}\zeta^*a^2 - \frac{1}{2}\zeta a^{\dagger 2}\right) \quad (1)$$

this is a unitary operator i.e.

$$S^\dagger(\zeta) = S(-\zeta) = S^{-1}(\zeta)$$

where  $\zeta = re^{i\theta}$ . We want to find the values of  $\Delta X_1^2$  and  $\Delta X_2^2$  or ( $\Delta Y_1^2$  and  $\Delta Y_2^2$  for a rotated frame). For this we first find

$$S^\dagger(\zeta) \hat{a} S(\zeta) = e^{\frac{1}{2}(\zeta a^{\dagger 2} - \zeta^* a^2)} a e^{\frac{1}{2}(\zeta^* a^2 - \zeta a^{\dagger 2})} \quad (2)$$

using

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \frac{1}{3} [A, [A, [A, B]]] + \dots$$

Let us define

$$A = \frac{1}{2}\zeta a^{\dagger 2} - \frac{1}{2}\zeta^* a^2$$

and

$$B = \hat{a}$$

$B$  will commute with the second term of  $A$

$$[A, B] = \frac{1}{2}\zeta [a^{\dagger 2}, a] \quad (3)$$

using

$$\begin{aligned} [a, a^{\dagger n}] &= na^{\dagger(n-1)} \\ [a^\dagger, a^{n-1}] &= -na^{n-1} \end{aligned}$$

$$\implies [A, B] = \frac{1}{2}\zeta (-2a^\dagger) = -\zeta a^\dagger$$

Now we have to find

$$[A, [A, B]] = ?$$

Ist term of A i.e.  $\frac{1}{2}\zeta a^{\dagger 2}$  commutes with  $[A, B]$

$$\begin{aligned} [A, [A, B]] &= \left[ -\frac{1}{2}\zeta^* a^2, -\zeta a^\dagger \right] \\ &= \frac{1}{2}\zeta^* \zeta [a^2, a^\dagger] \\ &= \frac{|\zeta|^2}{2} (2a) \\ &= |\zeta|^2 a \end{aligned} \tag{4}$$

Now for

$$[A, [A, [A, B]]] = ?$$

the second term of A will commute with  $[A, [A, B]]$  therefore we have

$$\begin{aligned} [A, [A, [A, B]]] &= \left[ \frac{1}{2}\zeta a^{\dagger 2}, |\zeta|^2 a \right] \\ &= \frac{1}{2}\zeta |\zeta|^2 [a^{\dagger 2}, a] \\ &= \frac{1}{2}\zeta |\zeta|^2 [-2a^\dagger] \\ &= -\zeta |\zeta|^2 a^\dagger \end{aligned} \tag{5}$$

Using these relations we get

$$\begin{aligned} S^\dagger(\zeta) \hat{a} S(\zeta) &= a - \zeta a^\dagger + \frac{1}{2!} |\zeta|^2 a - \frac{\zeta |\zeta|^2}{3!} a^\dagger + \dots \\ &= a \left[ 1 + \frac{1}{2!} |\zeta|^2 + \frac{1}{4!} |\zeta|^4 \dots \right] - a^\dagger \left[ \zeta + \frac{\zeta |\zeta|^2}{3!} + \dots \right] \end{aligned}$$

as

$$\zeta = r e^{i\theta} \implies |\zeta| = r$$

$$\begin{aligned} S^\dagger(\zeta) \hat{a} S(\zeta) &= a \left[ 1 + \frac{1}{2!} r^2 + \frac{1}{4!} r^4 + \dots \right] - a^\dagger \left[ r e^{i\theta} + \frac{r^3 e^{i\theta}}{3!} + \dots \right] \\ &= a \cosh r - a^\dagger e^{i\theta} \sinh r \end{aligned}$$

Similarly

$$S^\dagger(\zeta) \hat{a}^\dagger S(\zeta) = a^\dagger \cosh r - e^{-i\theta} a \sinh r$$



As we defined

$$\begin{aligned} X_1 &= \frac{1}{2}(a + a^\dagger) \\ X_2 &= \frac{1}{2}(a - a^\dagger) \\ \implies a &= X_1 + iX_2 \\ a^\dagger &= X_1 - iX_2 \end{aligned}$$

Rotate the axes by an amount  $\theta/2$ . Rotated axes  $Y_1$  and  $Y_2$  in terms of  $X_1$  and  $X_2$  are written as

$$\begin{aligned} Y_1 &= X_1 \cos \frac{\theta}{2} + X_2 \sin \frac{\theta}{2} \\ Y_2 &= -X_1 \sin \frac{\theta}{2} + X_2 \cos \frac{\theta}{2} \\ Y_1 + iY_2 &= X_1 \left( \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right) + X_2 \left( \sin \frac{\theta}{2} + i \cos \frac{\theta}{2} \right) \\ &= X_1 e^{-i\theta/2} + iX_2 \left( -i \sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right) \\ &= (X_1 + iX_2) e^{-i\theta/2} \end{aligned}$$

is the rotated complex amplitude at an angle  $\theta/2$ .

$$\begin{aligned} S^\dagger(\zeta) (X_1 + iX_2) S(\zeta) &= S^\dagger(\zeta) a S(\zeta) \\ S^\dagger(\zeta) (X_1 + iX_2) S(\zeta) &= a \cosh r - a^\dagger e^{i\theta} \sinh r \end{aligned}$$

For rotated frame we have

$$\begin{aligned} S^\dagger(\zeta) (X_1 + iX_2) e^{-i\theta/2} S(\zeta) &= a \cosh r e^{-i\theta/2} - a^\dagger e^{i\theta/2} \sinh r \\ S^\dagger(\zeta) (Y_1 + iY_2) S(\zeta) &= a e^{-i\theta/2} \left( \frac{e^r + e^{-r}}{2} \right) - a^\dagger e^{i\theta/2} \left( \frac{e^r - e^{-r}}{2} \right) \\ &= \left( a e^{-i\theta/2} - a^\dagger e^{i\theta/2} \right) \frac{e^r}{2} \\ &\quad + \left( a e^{-i\theta/2} + a^\dagger e^{i\theta/2} \right) \frac{e^{-r}}{2} \end{aligned}$$

As

$$\begin{aligned} Y_1 &= X_1 \cos \frac{\theta}{2} + X_2 \sin \frac{\theta}{2} \\ &= \frac{1}{2}(a + a^\dagger) \cos \frac{\theta}{2} + \frac{1}{2i}(a - a^\dagger) \sin \frac{\theta}{2} \\ &= \frac{1}{2}a \left( \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right) + \frac{1}{2}a^\dagger \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \\ &= \frac{1}{2} \left( a e^{-i\theta/2} + a^\dagger e^{i\theta/2} \right) \end{aligned} \tag{a}$$

$$\implies S^\dagger(\zeta)(Y_1 + iY_2)S(\zeta) = Y_1 e^{-r} + iY_2 e^r$$

where

$$iY_2 = \frac{1}{2} \left( a e^{-i\theta/2} - a^\dagger e^{i\theta/2} \right) \quad (\text{b})$$

The squeezed operator attenuates one component of the (rotated) complex amplitude and it amplifies the other component. The degree of attenuation and amplification is determined by  $r = |\zeta|$  which is called squeeze factor. The squeezed state  $|\alpha, \zeta\rangle$  is obtained by first squeezing the vacuum and then displacing it.

$$|\alpha, \zeta\rangle = D(\alpha)S(\zeta)|0\rangle$$

where  $\alpha^2$  is the intensity of the state,  $\theta$  is the orientation of the squeezing axis and  $r$  the degree of squeezing. The reverse order of  $D(\alpha)$  and  $S(\zeta)$  in the above equation is also possible. This results in the so called two photon correlated state. A coherent state is generated by linear terms in  $a$  and  $a^\dagger$  in the exponent

$$D(\alpha)|0\rangle = |\alpha\rangle$$

where

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

the squeezed coherent state requires quadratic terms.

### 2.1.3 Quadrature variance:

Now we will find  $\Delta Y_1^2$  and  $\Delta Y_2^2$ , in order to find the  $\Delta Y_1$  we need to find  $\langle a \rangle$ ,  $\langle a^\dagger \rangle$ ,  $\langle a^2 \rangle$ ,  $\langle a^{\dagger 2} \rangle$  and  $\langle a^\dagger a \rangle$ . We will find these one by one

$$\begin{aligned} \langle a \rangle &= \langle \alpha, \zeta | a | \alpha, \zeta \rangle \\ &= \langle 0 | D^\dagger(\alpha) S^\dagger(\zeta) a S(\zeta) D(\alpha) | 0 \rangle \\ &= \langle \alpha | (a \cosh r - a^\dagger e^{i\theta} \sinh r) | \alpha \rangle \\ &= \langle \alpha | a | \alpha \rangle \cosh r - \langle \alpha | a^\dagger | \alpha \rangle e^{i\theta} \sinh r \\ &= \alpha \cosh r - \alpha^* e^{i\theta} \sinh r \end{aligned}$$

$$\begin{aligned} \langle a^2 \rangle &= \langle 0 | D^\dagger(\alpha) S^\dagger(\zeta) a^2 S(\zeta) D(\alpha) | 0 \rangle \\ &= \langle 0 | D^\dagger(\alpha) S^\dagger(\zeta) a S(\zeta) S^\dagger(\zeta) a S(\zeta) D(\alpha) | 0 \rangle \\ &= \langle \alpha | (a \cosh r - a^\dagger e^{i\theta} \sinh r)^2 | \alpha \rangle \\ &= \langle \alpha | (a^2 \cosh^2 r + e^{2i\theta} a^{\dagger 2} \sinh^2 r - a a^\dagger e^{i\theta} \cosh r \sinh r \\ &\quad - a^\dagger a e^{i\theta} \cosh r \sinh r) | \alpha \rangle \\ &= \alpha^2 \cosh^2 r + e^{2i\theta} \alpha^{*2} \sinh^2 r - 2|\alpha|^2 e^{i\theta} \cosh r \sinh r \\ &\quad - e^{i\theta} \cosh r \sinh r \\ &= \langle (a^\dagger)^2 \rangle^* \end{aligned}$$

Now

$$\begin{aligned}
\langle a^\dagger a \rangle &= \langle \alpha | S^\dagger(\zeta) a^\dagger S(\zeta) S^\dagger(\zeta) a S(\zeta) | \alpha \rangle \\
&= \langle \alpha | (a^\dagger \cosh r - e^{-i\theta} a \sinh r) (a \cosh r - a^\dagger e^{i\theta} \sinh r) | \alpha \rangle \\
&= |\alpha|^2 (\cosh^2 r + \sinh^2 r) - (\alpha^*)^2 e^{i\theta} \sinh r \cosh r \\
&\quad - \alpha^2 e^{-i\theta} \cosh r \sinh r + \sinh^2 r
\end{aligned}$$

As from Equ(a) and (b)

$$\begin{aligned}
Y_1 &= \frac{1}{2} (ae^{-i\theta/2} + a^\dagger e^{i\theta/2}) \\
Y_2 &= \frac{1}{2i} (ae^{-i\theta/2} - a^\dagger e^{i\theta/2})
\end{aligned}$$

$$\begin{aligned}
\langle Y_1 \rangle &= \frac{1}{2} [\langle a \rangle e^{-i\theta/2} + \langle a^\dagger \rangle e^{i\theta/2}] \\
&= \frac{1}{2} (\alpha \cosh r - e^{-i\theta} \alpha^* \sinh r) e^{-i\theta/2} + \frac{1}{2} (\alpha^* \cosh r - e^{-i\theta} \alpha \sinh r) e^{i\theta/2}
\end{aligned}$$

$$\langle Y_1^2 \rangle = \frac{1}{4} [\langle a^2 \rangle e^{-i\theta} + \langle a^{\dagger 2} \rangle e^{i\theta} + 2\langle a^\dagger a \rangle + 1]$$

Now

$$\begin{aligned}
(\Delta Y_1)^2 &= \langle Y_1^2 \rangle - \langle Y_1 \rangle^2 \\
&= \frac{1}{4} e^{-2r}
\end{aligned}$$

and

$$(\Delta Y_2)^2 = \frac{1}{4} e^{2r}$$

$$\implies \Delta Y_1 \Delta Y_2 = \frac{1}{4}$$

A squeezed coherent state is therefore an ideal squeezed state. In the complex amplitude plane the coherent state error circle is squeezed into an error ellipse of the same area. The principle axes of the ellipse lie along the  $Y_1$  and  $Y_2$  axes, and the principle radii are  $\Delta Y_1$  and  $\Delta Y_2$ .