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Solving the 3D Ising model with Conformal Bootstrap

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Solving 3D Ising w/ conf. bootstrap

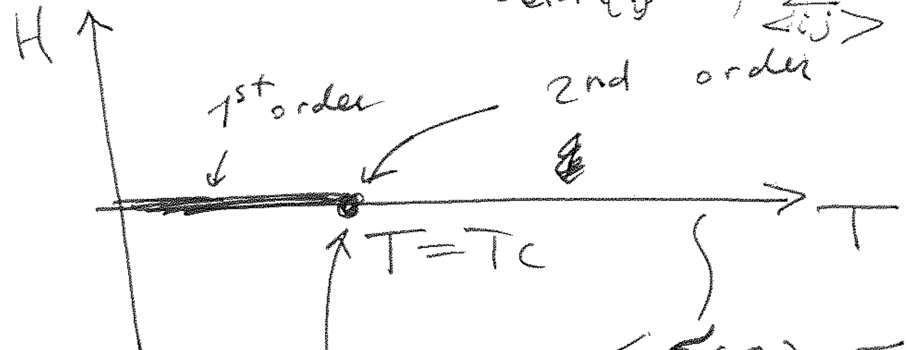
$QFT_{D\geq 3}$ is not in good shape.

Understand only weak coupling and large N (via AdS)

SUSY and integrability not a cure and limit the scope. ~~but not~~

3D Ising model @ $T=T_c$ should be the simplest ~~3D~~ CFT_3 to solve.

Uniaxial magnets $\quad H(\{s_i\}) = \beta \sum_{\langle ij\rangle} s_i s_j \quad s_i = \pm 1$



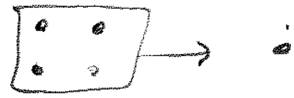
$\langle \sigma(0)\, \sigma(r)\rangle \sim e^{-r/\xi(T)}$

massive gap.

$\xi(T_c) \to \infty, \quad \langle \sigma(0)\, \sigma(r)\rangle \sim \dfrac{1}{|r|^{2\Delta_\sigma}}$

constant of Nature as π

scale invariance.

scale transf.
RG transformation 

$\mathcal{H}(\{s\}) \to \mathcal{H}(\{s'\})$.

scale inv $\Rightarrow$ fixed point hamiltonian.

very complicated not nearest-neighbor

(however, not very long-range

For example, one can show pretty easily that

$\mathcal{H}$ cannot contain terms

$$\iint dr\,dr' \frac{1}{|r-r'|^{\alpha}} s(r)\, s(r')$$

Then $\alpha \gtrsim 5$

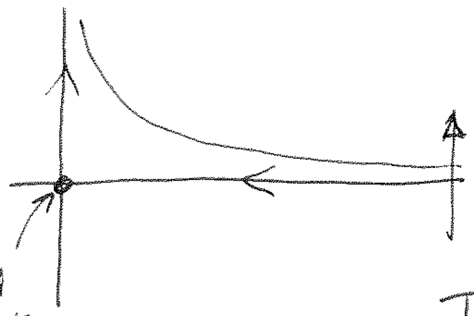
Hamiltonian is RG inv, local operators rescale. Can choose a basis of local ops. with Δ and l.

		$\mathbb{Z}_2$	Δ
$l=0$	σ	$-$	$0.5182(3)$
	σ'	$-$	$\gtrsim 4.5$
	ε	$+$	$1.413(1)$
	ε'	$+$	$3.84(4)$
	ε''	$+$	$4.67(11)$
	$T_{\mu\nu}$	$+$	3
	$C_{\mu\nu\lambda\sigma}$	$+$	$5.0208(12)$

$\sigma \approx s_i$

$\varepsilon \approx s_i s_j \quad (\langle ij \rangle) \qquad (\sigma \times \sigma \supset \varepsilon)$

$\varepsilon' \approx s_i s_j s_k s_l$ etc. $\qquad \sigma \times \sigma \supset \varepsilon + \varepsilon' + \varepsilon'' + \dots$



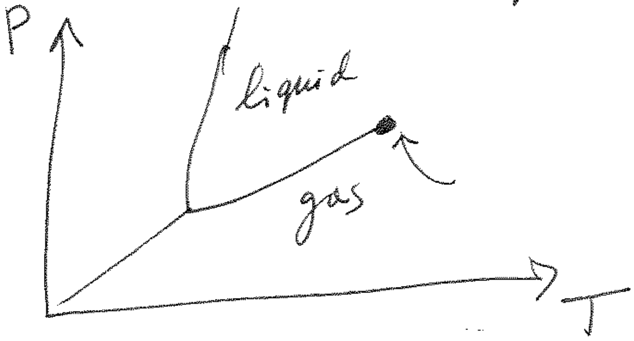
To get to the critical point we have to finetune **one** quantity (β)

Experimentally $\boxed{\Delta_{\varepsilon'} > 3}$

$\boxed{\text{corrections to scaling}}$ $\Rightarrow$ one relevant $\mathbb{Z}_2$ inv. operator ε.

$$\boxed{\xi(T) \sim \frac{1}{|T - T_c|^{1/(3-\Delta_\varepsilon)}} \qquad T \to T_c.}$$

Why σ' irrel? Because Ising model also describes phase liquid-vapor transition in



in this case $\mathbb{Z}_2$ symmetry is emergent. To get to critical point one has to adjust coeff. of both σ and ε. But σ' must be irrelevant.

$T_{\mu\nu}$ stress tensor

$C_{\mu\nu\lambda\sigma}$ $\delta S \propto \int C_{1111} + C_{2222} + C_{3333}$

leading interactions which breaks rotation but preserves cubic lattice invariance.

How determined

•) experiment (Lab, MC simulations)

•) theory

→ high-T exp (β around $= 0$)

– $(4-\varepsilon)$ expansion $\varepsilon \to 0$

asymptotic, divergent

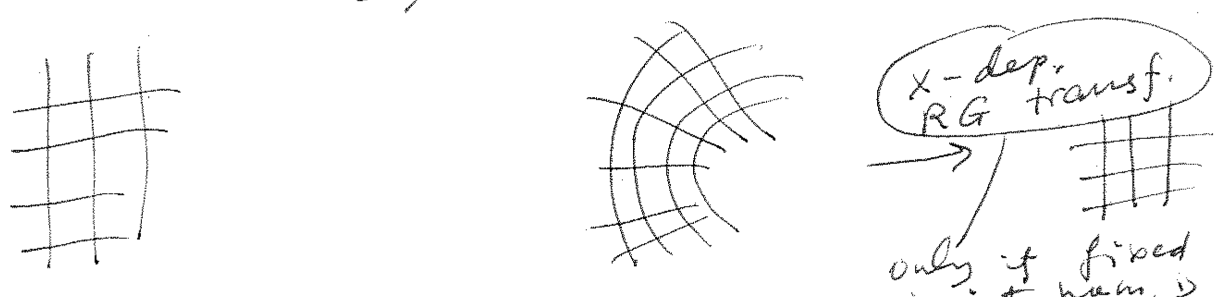
$(\partial\varphi)^2 + \lambda\varphi^4$

– Parisi do it directly in $D=3$ 5–6 loop calculations

– ERG ⇝ Asymptotic safety and other nonsense (for UV fixed point)

Conf. invariance (Polyakov 1870).

$x \to x'$ $g'_{\mu\nu} = \Omega(x)\, g_{\mu\nu}$



only if fixed point ham. is short-ranged.

Fields medal

Stas Smirnov 2D ~~2006~~ 2010 (5)

(but about interfaces, not correlators)

Conf. inv. $\rightarrow$ table.

- primaries, transform homogeneously & descendants.
- All listed are primaries.
- Positive anomalous dimensions

$\gamma = \Delta - \Delta_{free}$

$\Delta_{free}\,(\cancel{scalar}) = \frac{D-2}{2}$ ($1/2$ in $D=3$). (φ)

$\Delta_{free}\,(l) = D - 2 + l$ ($\varphi\,\partial_\mu \ldots \partial_\nu\,\varphi$).

$\boxed{\gamma_l \geq 0}$ (>0 in interacting theory except for currents and stress tensor)

True in Unitary theories

In Euclidean unitarity $\Leftrightarrow$ reflection positivity

$\bullet\ \varphi$

> 0.

$\bullet\ \varphi$ for $l \geq 1$ > 0 once indices are reflected.

For the Ising model this follows on the lattice from the functional integral.

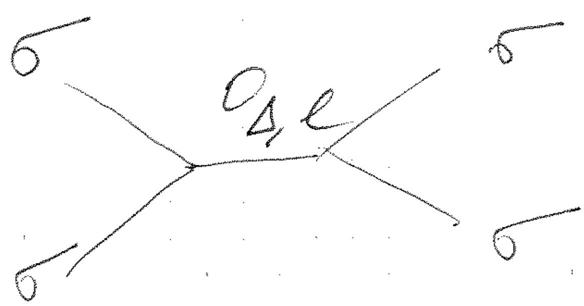
Check of conf. inv: measure on the lattice

$$\langle \sigma(x)\, \sigma(y)\, \varepsilon(0) \rangle = \frac{C}{|x|^{\Delta_\varepsilon} |y|^{\Delta_\varepsilon} |x-y|^{2\Delta_\sigma - \Delta_\varepsilon}}$$

Not done.

Our goal: determine dimensions of ~~all~~ fields by CFT methods (without having to flow)

$$= \frac{g(u,v)}{x_{12}^{2\Delta_\sigma} x_{34}^{2\Delta_\sigma}}$$

$\sum'$ 

$\boxed{\text{El-Showk, Paulos, Poland, Simmons-Duffin, + S.R.}}$

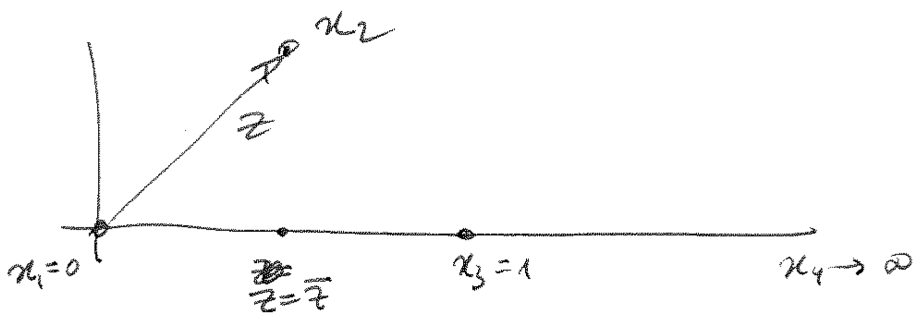
$$g(u,v) = \sum C^2_{\Delta,\ell}\, g_{\Delta,\ell}(u,v).$$

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta_\sigma} g(v,u)$$

Conformal blocks in 3D are

- in principle ~~known~~ fixed functions

But not known in closed form in 3D. $\Rightarrow$ slow progress.



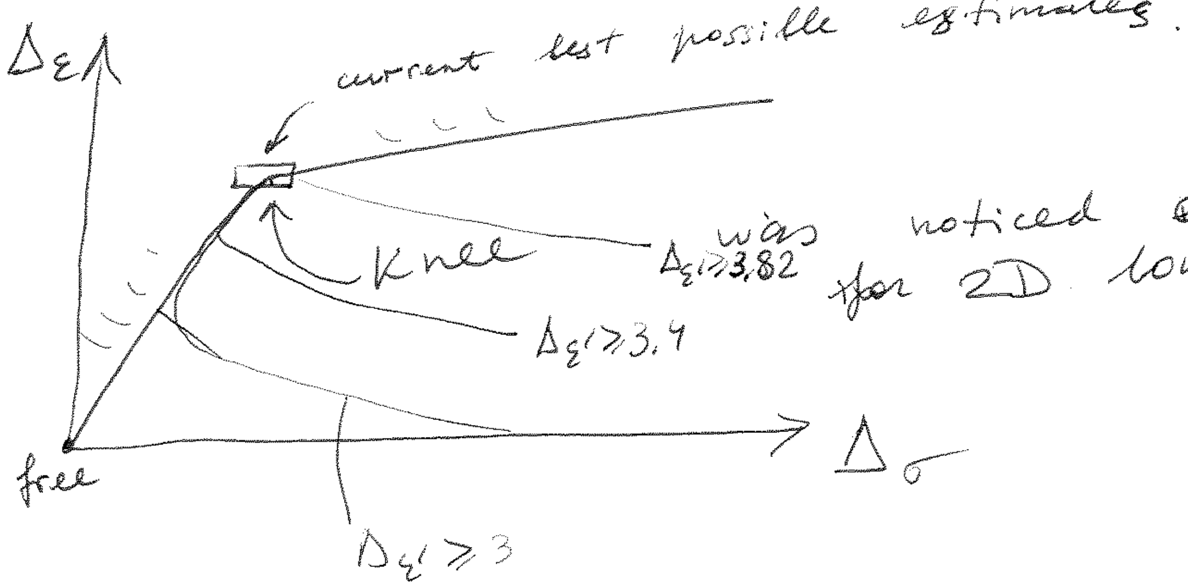
— On $z=\bar{z}$ ${}_3F_2$ $\ell=0$, $\ell=1$

higher ℓ — recursion relations (Dolan-Osborn 2011)

— ~~for~~ $z\neq\bar{z}$ $g_{\Delta,\ell}$ satisfy 2nd order partial diff. equation (Casimir diff. equation) $\Rightarrow$ can be power expanded by Cauchy-Kowalewski theorem.

Old methods can be generalized.

What do we see??

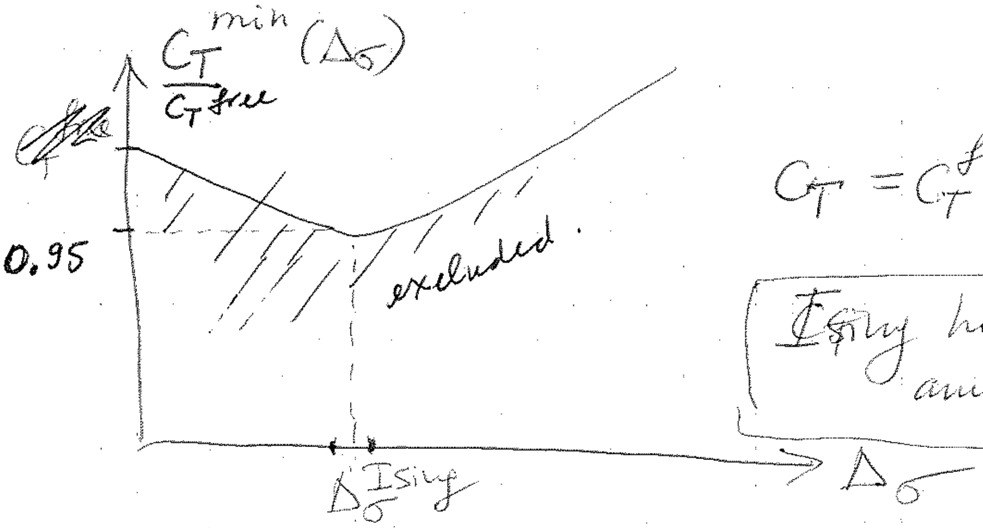


was noticed earlier for 2D bounds.

Varying ε' dimension the allowed region shrinks. This was also noticed before for
$\Delta_{\varepsilon'} \gtrsim 3.82$ is excluded.
(compare with $3.84(4)$) estimate).

Finally central charge C_T

$$\langle T_{\mu\nu} T_{\lambda\sigma} \rangle \sim C_T \text{ (tensor)}$$



$$C_T = C_T^{free}\big(\underbrace{1 - \tfrac{5}{324}\varepsilon^2}_{0.984} + O(\varepsilon^3)\big)$$

Ising has minimal C_T among all 3D CFTs ?

Conclusion

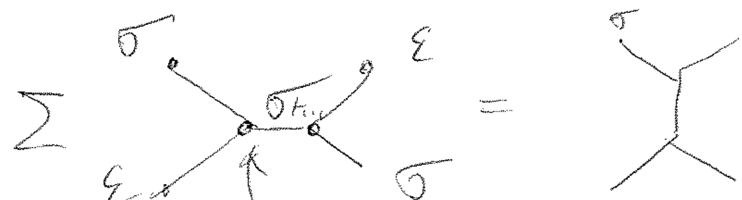
1) for the first time ever, 3D Ising CFT is shown amenable to the analysis by conformal bootstrap (Polyakov's dream since 1974)

2) Approximate Results of other method lie on the boundary of region that we find.
$\Rightarrow$ proves that assumption of conf. inv. is consistent.

Obvious next steps

Add other correlators to the game.

$\langle \sigma \varepsilon \sigma \varepsilon \rangle$ & $\langle \varepsilon \varepsilon \varepsilon \varepsilon \rangle$



same opt coeff. as in $\sigma \times \sigma \supset \varepsilon$.

coeff. is fixed. in terms of C_T

Such multiple correlator analysis has never before been accomplished.

Our hope is that once these new constraints are added, allowed regions will shrink significantly.

We do not yet know how to efficiently calculate 3D conf. blocks for unequal dimensions. But the problem does not seem unsurmountable. So we hope that we will turn the page in the very near future.