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Scale without Conformal Invariance

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Beyond the Standard Model

based on

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arXiv:1110.1634 [hep-th] and work in progress

with Benjamín Grinstein and Andreas Stergiou

Scale and conformal invariance in two dimensions

Does scale invariance imply conformal invariance ?

- Polchinski following Zamolodchikov Polchinski (1988) & Zamolodchikov (1986)
  - Unitarity
  - Finiteness of EM tensor correlation functions

  $\Rightarrow$ Scale invariance implies conformal invariance from conservation of EM tensor

"Counter" examples

- Non-linear σ model Hull, Townsend (1986)
  - Non-existence of EM tensor two-point correlation functions
- Theory of elasticity Cardy, Riva (2005)
  - Non-reflection-positive

Scale invariance implies conformal invariance

Scale and conformal invariance in $d > 2$ dimensions

Does scale invariance imply conformal invariance ?

- No proof *à la* Polchinski
  - Conservation of EM tensor $\Rightarrow$ Not enough information

  $\Rightarrow$ Scale invariance does not necessarily imply conformal invariance

No relevant counterexamples

- AdS/CFT Kerr-AdS black holes in $d = 5, 7$ dimensions Awad, Johnson (1999)
  - Conformal invariance broken to scale invariance by black hole rotation
- Maxwell theory in $d \neq 4$ dimensions Jackiw, Pi (2011) & El-Showk, Nakayama, Rychkov (2011)
  - Free field theory
  - Scale invariance broken by interactions

Study of non-conformal scale-invariant QFTs

Scale invariance does not necessarily imply conformal invariance but no proper counterexamples $\Rightarrow$ Possible proof !

- Without proper counterexamples
  - Physical implications of non-conformal scale-invariant QFTs (correlation functions in non-conformal scale-invariant QFTs versus CFTs ?)
- With proper counterexamples
  - Scale invariance conditions weaker than conformal invariance conditions (plentiful examples ?)

Uncharted territory !

Outline

Preliminaries ($d > 2$)

- Dilatation current Polchinski (1988)
  - $\mathcal{D}^\mu(x) = x^\nu T_\nu{}^\mu(x) - V^\mu(x)$
  - $T_\nu{}^\mu(x)$ any symmetric EM tensor following from spacetime nature of scale transformations
  - $V^\mu(x)$ local operator (virial current) contributing to scale dimensions of fields
  - Freedom in choice of $T_\nu{}^\mu(x)$ compensated by freedom in choice of $V^\mu(x)$
- Scale invariance $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$

- Conformal current Polchinski (1988)
  - $\mathcal{C}^{\mu}_{\nu}(x) = v^{\nu}(x) T_{\nu}{}^{\mu}(x) - \partial \cdot v(x) V'^{\mu}(x) + \partial_{\nu}\partial \cdot v(x) L^{\nu\mu}(x)$
  - $T_{\nu}{}^{\mu}(x)$ any symmetric EM tensor following from spacetime nature of conformal transformations
  - $V'^{\mu}(x)$ local operator corresponding to ambiguity in choice of dilatation current
  - $L^{\nu\mu}(x)$ local symmetric operator correcting position dependence of scale factor
  - $\partial \cdot v(x)$ scale factor (general linear function of x^{μ})
  - Freedom in choice of $T_{\nu}{}^{\mu}(x)$ compensated by freedom in choice of $V'^{\mu}(x)$ and $L^{\nu\mu}(x)$
- Conformal invariance $\Rightarrow$ $T_{\mu}{}^{\mu}(x) = \partial_{\mu} V'^{\mu}(x) = \partial_{\mu}\partial_{\nu} L^{\nu\mu}(x)$
- Conformal invariance $\Rightarrow$ Existence of symmetric traceless energy-momentum tensor

Scale without conformal invariance

Non-conformal scale-invariant QFTs Polchinski (1988)

- Scale invariance $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$
- Conformal invariance $\Rightarrow T_\mu{}^\mu(x) = \partial_\mu \partial_\nu L^{\nu\mu}(x)$
- Scale without conformal invariance

  $\Rightarrow$ $T_\mu{}^\mu(x) = \partial_\mu V^\mu(x)$ where $V^\mu(x) \neq J^\mu(x) + \partial_\nu L^{\nu\mu}(x)$ with $\partial_\mu J^\mu(x) = 0$
- Constraints on possible virial current candidates
  - Gauge-invariant spatial integral
  - Fixed $d-1$ scale dimension in d spacetime dimensions
- No suitable virial current $\Rightarrow$ Scale invariance implies conformal invariance (examples: ϕ^p in $d = n - \epsilon$ for $(p, n) = (6, 3), (4, 4)$ and $(3, 6)$)

Virial current candidates

Most general classically scale-invariant renormalizable theory in $d = 4$ spacetime dimensions Jack, Osborn (1985)

$$
\begin{aligned}
\mathcal{L} = & -\mu^{-\epsilon} Z_A \frac{1}{4g_A^2} F^A_{\mu\nu} F^{A\mu\nu} + \tfrac{1}{2} Z^{\frac{1}{2}}_{ab} Z^{\frac{1}{2}}_{ac} D_\mu \phi_b D^\mu \phi_c \\
& + \tfrac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} \bar{\psi}_j i \bar{\sigma}^\mu D_\mu \psi_k - \tfrac{1}{2} Z^{\frac{1}{2}*}_{ij} Z^{\frac{1}{2}}_{ik} D_\mu \bar{\psi}_j i \bar{\sigma}^\mu \psi_k \\
& - \tfrac{1}{4!} \mu^\epsilon (\lambda Z^\lambda)_{abcd} \phi_a \phi_b \phi_c \phi_d \\
& - \tfrac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)_{a|ij} \phi_a \psi_i \psi_j - \tfrac{1}{2} \mu^{\frac{\epsilon}{2}} (y Z^y)^*_{a|ij} \phi_a \bar{\psi}_i \bar{\psi}_j
\end{aligned}
$$

- $A^A_\mu(x)$ gauge fields
- $\phi_a(x)$ real scalar fields
- $\psi^\alpha_i(x)$ Weyl fermions
- Dimensional regularization ($d = 4 - \epsilon$)

Virial current candidates and new improved EM tensor

- Virial current $V^{\mu}(x) = Q_{ab}\phi_a D^{\mu}\phi_b - P_{ij}\bar{\psi}_i i\bar{\sigma}^{\mu}\psi_j$
  - $Q_{ba} = -Q_{ab}$
  - $P^*_{ji} = -P_{ij}$

- New improved energy-momentum tensor $\Theta_{\nu}{}^{\mu}(x)$ Callan, Coleman, Jackiw (1970)
  - Finite
  - Not renormalized
  - Anomalous trace Robertson (1991)

$$
\begin{aligned}
\Theta_{\mu}{}^{\mu}(x) = & \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + \gamma_{aa'} D^2\phi_a\phi_{a'} \\
& - \gamma^*_{i'i}\bar{\psi}_i i\bar{\sigma}^{\mu} D_{\mu}\psi_{i'} + \gamma_{ii'} D_{\mu}\bar{\psi}_i i\bar{\sigma}^{\mu}\psi_{i'} \\
& - \frac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} \\
& \qquad - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d \\
& - \frac{1}{2}(\beta_{a|ij} - \gamma_{a'a}y_{a'|ij} - \gamma_{i'i}y_{a|i'j} - \gamma_{j'j}y_{a|ij'})\phi_a\psi_i\psi_j + \text{h.c.}
\end{aligned}
$$

- β-functions from vertex corrections and wavefunction renormalizations ($d = 4$ spacetime dimensions)
  - RG time $t = \ln(\mu_0/\mu)$

$$\beta_A = -\frac{dg_A}{dt} = \gamma_A g_A \quad \text{(no sum)}$$

$$\begin{aligned}\beta_{abcd} &= -\frac{d\lambda_{abcd}}{dt}\\ &= -(\lambda\gamma^{\lambda})_{abcd} + \gamma_{a'a}\lambda_{a'bcd} + \gamma_{b'b}\lambda_{ab'cd} + \gamma_{c'c}\lambda_{abc'd} + \gamma_{d'd}\lambda_{abcd'}\end{aligned}$$

$$\beta_{a|ij} = -\frac{dy_{a|ij}}{dt} = -(y\gamma^{y})_{a|ij} + \gamma_{a'a}y_{a'|ij} + \gamma_{i'i}y_{a|i'j} + \gamma_{j'j}y_{a|ij'}$$

- Divergence of dilatation current

$$
\begin{aligned}
\partial_\mu \mathcal{D}^\mu(x) = & \frac{\beta_A}{2g_A^3} F^A_{\mu\nu} F^{A\mu\nu} + (\gamma_{aa'} + Q_{aa'}) D^2 \phi_a \phi_{a'} \\
& - (\gamma^*_{i'i} + P^*_{i'i}) \bar{\psi}_i i \bar{\sigma}^\mu D_\mu \psi_{i'} + (\gamma_{ii'} + P_{ii'}) D_\mu \bar{\psi}_i i \bar{\sigma}^\mu \psi_{i'} \\
& - \frac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} \\
& \qquad - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d \\
& - \frac{1}{2}(\beta_{a|ij} - \gamma_{a'a} y_{a'|ij} - \gamma_{i'i} y_{a|i'j} - \gamma_{j'j} y_{a|ij'})\phi_a \psi_i \psi_j + \text{h.c.}
\end{aligned}
$$

- Conserved dilatation current $\partial_\mu \mathcal{D}^\mu(x) = 0$ (up to EOMs)

$$
\begin{aligned}
\beta_A &= 0 \\
\beta_{abcd} &= -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'} \\
\beta_{a|ij} &= -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a|i'j} - P_{j'j} y_{a|ij'}
\end{aligned}
$$

- Conserved conformal current $\partial_\mu \mathcal{C}^\mu_\nu(x) = 0$ (up to EOMs)

$$
\beta_A = \beta_{abcd} = \beta_{a|ij} = 0
$$

Interlude: Current conservation

- Divergence of current $J^\mu(x)$ without use of EOMs Collins (1984)

$$\partial_\mu J^\mu(x) = \Delta_{\rm EOM} + \Delta_{\rm Classical} + \Delta_{\rm Anomaly}$$

- Green's function of elementary fields with current $J^\mu(x)$ and Ward identity

  ✔ $\Delta_{\rm EOM} \Rightarrow$ Expected contact terms from Ward identity
  ✗ $\Delta_{\rm Classical} \Rightarrow$ Usual non-anomalous classical violation
  ✗ $\Delta_{\rm Anomaly} \Rightarrow$ Possible anomalous violation in divergent Green's function

- Example: $\mathcal{L} = -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} + \bar\psi_i(i\gamma^\mu D_\mu\delta_{ij} - M_{ij})\psi_j$
  - Vector current $J_V^{\mu a}(x) = \bar\psi\gamma^\mu t^a\psi$ with $\Delta_{\rm EOM} \neq 0$, $\Delta_{\rm Classical} = i\bar\psi[M, t^a]\psi$ and $\Delta_{\rm Anomaly} = 0$
  - Axial current $J_A^{\mu a}(x) = \bar\psi\frac{1}{2}[\gamma^\mu, \gamma^5]t^a\psi$ with $\Delta_{\rm EOM} \neq 0$, $\Delta_{\rm Classical} = i\bar\psi\gamma^5\{M, t^a\}\psi$ and $\Delta_{\rm Anomaly} = \frac{1}{2}\bar\psi\{\gamma^\mu, \gamma^5\}t^a D_\mu\psi - \frac{1}{2}D_\mu\bar\psi\{\gamma^\mu, \gamma^5\}t^a\psi$

Virial current and unitarity bounds

- New improved energy-momentum tensor ⇒ Finite and not renormalized Callan, Coleman, Jackiw (1970)
- Operators related to EOMs ⇒ Finite and not renormalized Politzer (1980) & Robertson (1991)
- Virial current ⇒ Finite and not renormalized
  - Unconserved current with scale dimension exactly 3
- Unitarity bounds for conformal versus scale-invariant QFTs Grinstein, Intriligator, Rothstein (2008)
- Non-trivial virial current ⇒ Non-conformal scale-invariant QFTs

RG flows along scale-invariant trajectories

Scale-invariant solution $(g_A, \lambda_{abcd}, y_{a|ij}) \Rightarrow$ RG trajectory

$$\begin{aligned}
\bar{g}_A(t) &= g_A \\
\bar{\lambda}_{abcd}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{b'b}(t)\widehat{Z}_{c'c}(t)\widehat{Z}_{d'd}(t)\lambda_{a'b'c'd'} \\
\bar{y}_{a|ij}(t) &= \widehat{Z}_{a'a}(t)\widehat{Z}_{i'i}(t)\widehat{Z}_{j'j}(t)y_{a'|i'j'}
\end{aligned}$$

$$\begin{aligned}
\widehat{Z}_{aa'}(t) &= \left(e^{Qt}\right)_{aa'} \\
\widehat{Z}_{ii'}(t) &= \left(e^{Pt}\right)_{ii'}
\end{aligned}$$

- $(\bar{g}_A(t,g,\lambda,y), \bar{\lambda}_{abcd}(t,g,\lambda,y), \bar{y}_{a|ij}(t,g,\lambda,y))$ also scale-invariant solution
- Q_{ab} and P_{ij} constant along RG trajectory
- $\widehat{Z}_{ab}(t)$ orthogonal and $\widehat{Z}_{ij}(t)$ unitary $\Rightarrow$ Always non-vanishing β-functions along scale-invariant trajectory

Scale invariance and recurrent behaviors

RG flows along scale-invariant trajectories $\Rightarrow$ Recurrent behaviors !

Lorenz (1963,1964), Wilson (1971) & Kogut, Wilson (1974)

- Virial current $\Rightarrow$ Transformation in symmetry group of kinetic terms ($SO(N_S) \times U(N_F)$)
  - Q_{ab} antisymmetric and P_{ij} antihermitian $\Rightarrow$ Purely imaginary eigenvalues
  - $\widehat{Z}_{ab}(t)$ and $\widehat{Z}_{ij}(t)$ in $SO(N_S) \times U(N_F)$

$\Rightarrow$ Periodic (limit cycle) or quasi-periodic (ergodicity) scale-invariant trajectories

Recurrent behaviors

Intuition from $\mathcal{D}^\mu(x) = x^\nu \Theta_\nu{}^\mu(x) - V^\mu(x)$

- RG flow $\Rightarrow$ Generated by scale transformation $(x^\nu \Theta_\nu{}^\mu(x))$
- RG flow $\Rightarrow$ Related to virial current through conservation of dilatation current
- Virial current $\Rightarrow$ Generates internal transformation of the fields
  - Internal transformation in compact group $SO(N_S) \times U(N_F)$

  $\Rightarrow$ Rotate back to or close to identity
- RG flow return back to or close to identity $\Rightarrow$ Recurrent behavior

Scale-invariant trajectories ?

RG flows $\sim$ Field redefinitions $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- RG-time-dependent field redefinitions $\Rightarrow$ Generates RG flows
  Wegner (1974) & Latorre, Morris (2001)
  - RG-time-dependent field redefinitions $\Rightarrow$ All exact RG flows (Wilson, Wegner, Polchinski, etc.)

β-function operators $\sim$ Redundant operators $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- Wavefunction renormalization operators $\Rightarrow$ Redundant operators
  - Redundant β-function operators necessary for scale invariance

Non-conformal scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows (recurrent behaviors)

Scale invariance, gradient flows and *a*-theorem

- Gradient flow

$$\beta_i(g) = -\frac{dg_i}{dt} = G_{ij}(g)\frac{\partial c(g)}{\partial g_j}$$

  - G_{ij} positive-definite metric
  - Potential $c(g)$ function of couplings

- Potential $c(g)$ monotonically decreasing along RG trajectory

$$\frac{dc(g(t))}{dt} = -G^{ij}(g)\beta_i\beta_j \leq 0$$

  - Recurrent behaviors (scale-invariant trajectories) $\not\Leftrightarrow$ Gradient flows (scale implies conformal invariance) Wallace, Zia (1975)

- *a*-theorem Barnes, Intriligator, Wecht, Wright (2004)
  - RG flow $\Rightarrow$ Irreversible process (integrating out DOFs)
  - $c(g) \sim$ measure of number of massless DOFs
  - *a*-theorem $\Rightarrow$ weak ($c_{IR} < c_{UV}$), stronger ($\frac{dc}{dt} \leq 0$), ~~strongest~~ (RG flows as gradient flows)

Why dilatation generators generate dilatations

Dilatation generators do not generate dilatations in non-scale-invariant QFTs Coleman, Jackiw (1971)

- Quantum anomalies at low orders
  - Anomalous dimensions

  ⇒ Possible to absorb into redefinition of scale dimensions of fields

  ✔ Preserve scale invariance

- Quantum anomalies at high orders
  - β-functions

  ⇒ Not possible to absorb

  ✗ Break scale invariance

Why dilatation generators generate dilatations in non-conformal scale-invariant QFTs ?

- β-functions on scale-invariant trajectories
  - Both vertex correction and wavefunction renormalization contributions
  - Very specific form for vertex correction contribution
  - Equivalent in form to wavefunction renormalization contribution (redundant operators)

  $\Rightarrow$ Also possible to absorb into redefinition of scale dimensions of fields

  ✔ Preserve scale invariance !

Ward identity for scale invariance

Callan-Symanzik equation for effective action Callan (1970) & Symanzik (1970)

$$\left[M\frac{\partial}{\partial M} + \beta_i \frac{\partial}{\partial g_i} + \gamma_j^{\ i} \int d^4x\, \varphi_i(x) \frac{\delta}{\delta \varphi_j(x)}\right] \Gamma[\varphi(x), g, M] = 0$$

- In non-scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✗ β-functions
- In CFTs
  - ✔ Anomalous dimensions
  - ✔ Vanishing β-functions

$$\left[M\frac{\partial}{\partial M} + (\gamma_j^{\ i} + Q_j^{\ i}) \int d^4x\, \varphi_i(x) \frac{\delta}{\delta \varphi_j(x)}\right] \Gamma[\varphi(x), g, M] = 0$$

- In non-conformal scale-invariant QFTs
  - ✔ Anomalous dimensions
  - ✔ β-functions (redundant operators)

Poincaré algebra augmented with dilatation charge

- β-functions on scale-invariant trajectories
  - Quantum-mechanical generation of scale dimensions
  - Appropriate scale dimensions required by virial current
  - $\Rightarrow$ Conserved dilatation current $\mathcal{D}^\mu(x)$
- Poincaré algebra with dilatation charge $D = \int d^3x\, \mathcal{D}^0(x)$

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho}) \\
[M_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\
[D, P_\mu] &= -iP_\mu
\end{aligned}
$$

- Algebra action on fields $\mathcal{O}_I(x)$

$$
\begin{aligned}
[M_{\mu\nu}, \mathcal{O}_I(x)] &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu + \Sigma_{\mu\nu})\mathcal{O}_I(x) \\
[P_\mu, \mathcal{O}_I(x)] &= -i\partial_\mu\mathcal{O}_I(x) \\
[D, \mathcal{O}_I(x)] &= -i(x\cdot\partial + \Delta)\mathcal{O}_I(x)
\end{aligned}
$$

- New classical scale dimensions of fields due to virial current

$$[D, \phi_a(x)] = -i(x \cdot \partial + 1)\phi_a(x) - iQ_{ab}\phi_b(x)$$
$$[D, \psi_i(x)] = -i(x \cdot \partial + \tfrac{3}{2})\psi_i(x) - iP_{ij}\psi_j(x)$$

  - How do non-conformal scale-invariant QFTs know about new scale dimensions ?

  $\Rightarrow$ Generated by β-functions !

- Quantum-mechanical scale dimensions of fields

$$\Delta_{ab} = \delta_{ab} + \gamma_{ab} + Q_{ab}$$
$$\Delta_{ij} = \tfrac{3}{2}\delta_{ij} + \gamma_{ij} + P_{ij}$$

Scale-invariant trajectories ??

β-functions $\sim$ Anomalous dimensions $\Rightarrow$ Scale-invariant trajectories or fixed points ?

- Shift β-functions away $\Rightarrow$ Scheme change
  - ✗ Non-conformal scale-invariant QFTs with traceless EM tensor
- Shift β-functions away $\Rightarrow$ Global shift
  - ✗ Conformal fixed points become conformal trajectories

Non-conformal scale-invariant QFTs $\Rightarrow$ Non-trivial RG flows

Non-conformal scale-invariant correlation functions

- Scalar fields $\mathcal{O}_I(x)$ with scale dimensions Δ_I

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\rangle = \frac{g_{IJ}}{(x_1-x_2)^{\Delta_I+\Delta_J}}$$

$$\langle \mathcal{O}_I(x_1)\mathcal{O}_J(x_2)\mathcal{O}_K(x_3)\rangle = \sum_{\substack{\delta_1+\delta_2+\delta_3= \\ \Delta_I+\Delta_J+\Delta_K}} \frac{c_{IJK}^{\delta_1\delta_2\delta_3}}{(x_1-x_2)^{\delta_1}(x_2-x_3)^{\delta_2}(x_3-x_1)^{\delta_3}}$$

  - Non-vanishing two-point functions with $\Delta_I \neq \Delta_J$ contrary to CFTs

- Two-point correlation functions of fundamental real scalar fields

$$\langle \phi_a(x)\phi_b(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\, G^{\phi}\, (x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$

  - G^{ϕ} constant real symmetric matrix

- Two-point correlation functions of scalar operators $\mathcal{O}_a(x)$

$$\langle \mathcal{O}_a(x)\mathcal{O}_b(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\, G\,(x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$
$$= i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\left[(-p^2-i\epsilon)^{\frac{\Delta}{2}-1}\,\widetilde{G}\,(-p^2-i\epsilon)^{\frac{\Delta^T}{2}-1}\right]_{ab}$$

  - G (and $\widetilde{G}$) constant real symmetric matrices

- Two-point correlation functions of vector operators $\mathcal{O}_a^\mu(x)$

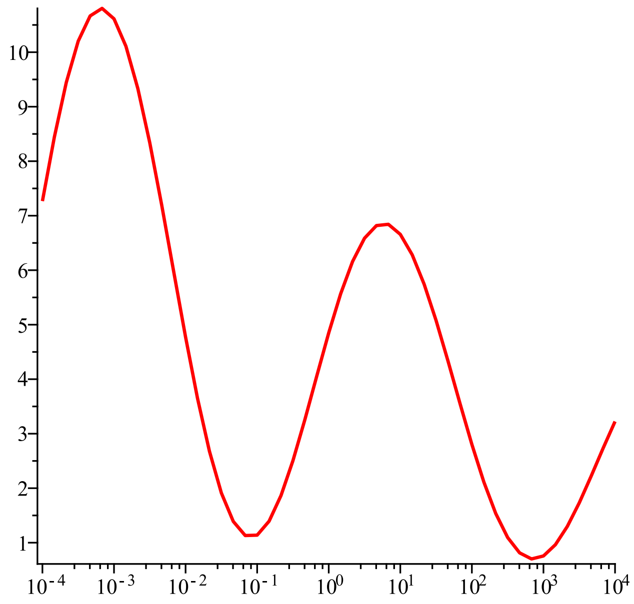
$$\langle \mathcal{O}_a^\mu(x)\mathcal{O}_b^\nu(0)\rangle = \left[(x^2)^{-\frac{\Delta}{2}}\left(g^{\mu\nu}A+\frac{x^\mu x^\nu}{x^2}B\right)(x^2)^{-\frac{\Delta^T}{2}}\right]_{ab}$$
$$= -i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\left[(-p^2-i\epsilon)^{\frac{\Delta}{2}-1}\left(g^{\mu\nu}\widetilde{A}+\frac{p^\mu p^\nu}{p^2}\widetilde{B}\right)(-p^2-i\epsilon)^{\frac{\Delta^T}{2}-1}\right]_{ab}$$

  - A and B (and $\widetilde{A}$ and $\widetilde{B}$) constant real symmetric matrices

Coupled QFT/SIT where $\mathcal{L} \supset g_a \chi \mathcal{O}_a + \mathrm{h.c.}$ with external source χ and scalar operator $\mathcal{O}_a$

$$\mathcal{M} = g_a g_b |\chi|^2 \left[(-p^2 - i\epsilon)^{\frac{\Delta}{2}-1} \widetilde{G} \, (-p^2 - i\epsilon)^{\frac{\Delta^T}{2}-1} \right]_{ab}$$

$$\mathrm{Im}\,\mathcal{M}_{\mathrm{fwd}} = g_a g_b |\chi|^2 \left[(p^2)^{\frac{\Delta}{2}-1} \left\{ \cos\left[\left(1-\tfrac{\Delta}{2}\right)\pi\right] \widetilde{G} \sin\left[\left(1-\frac{\Delta^T}{2}\right)\pi\right] + \sin\left[\left(1-\tfrac{\Delta}{2}\right)\pi\right] \widetilde{G} \cos\left[\left(1-\frac{\Delta^T}{2}\right)\pi\right] \right\} (p^2)^{\frac{\Delta^T}{2}-1} \right]_{ab} \theta(p^0)\theta(p^2)$$



Systematic approach

Scale-invariant trajectories at weak coupling

$$g_A = \sum_{n\geq 1} g_A^{(n)} \epsilon^{n-\frac{1}{2}} \qquad \lambda_{abcd} = \sum_{n\geq 1} \lambda_{abcd}^{(n)} \epsilon^{n} \qquad y_{a|ij} = \sum_{n\geq 1} y_{a|ij}^{(n)} \epsilon^{n-\frac{1}{2}}$$

$$Q_{ab} = \sum_{n\geq 3} Q_{ab}^{(n)} \epsilon^{n} \qquad P_{ij} = \sum_{n\geq 3} P_{ij}^{(n)} \epsilon^{n}$$

- ϵ small parameter
  - Obvious choice in $d = 4 - \epsilon$
  - One-loop gauge coupling β-function coefficient in $d = 4$ Banks, Zaks (1982)
- Form of expansions determined by β-functions
  - For coupling constants $\Rightarrow$ Lowest-order terms in β-functions (would-be conformal fixed points)
  - For virial current $\Rightarrow$ Higher-order terms in β-functions due to Polchinski–Dorigoni–Rychkov argument and gradient flow interpretation

Examples

Physical $d = 4$ case

- No proper example yet $\Rightarrow$ Maybe none ?
  - Technically difficult to generate three-loop β-functions
- $SU(2)$ gauge theory with two real scalars (singlet) and two active flavors of Weyl fermions (fundamental)
  - Unbounded-from-below scalar potential
  - Three-loop β-functions necessary to conclude

Unphysical $d = 4 - \epsilon$ case

- Two real scalars and two Weyl fermions
  - Limit cycle (bounded-from-below scalar potential, CP conservation, vacuum at origin of field space)
  - Unsatisfactory unless condensed matter example in $\epsilon \to 1$ limit (universality class ?)

$d = 4 - \epsilon$ example with two real scalars and two Weyl fermions

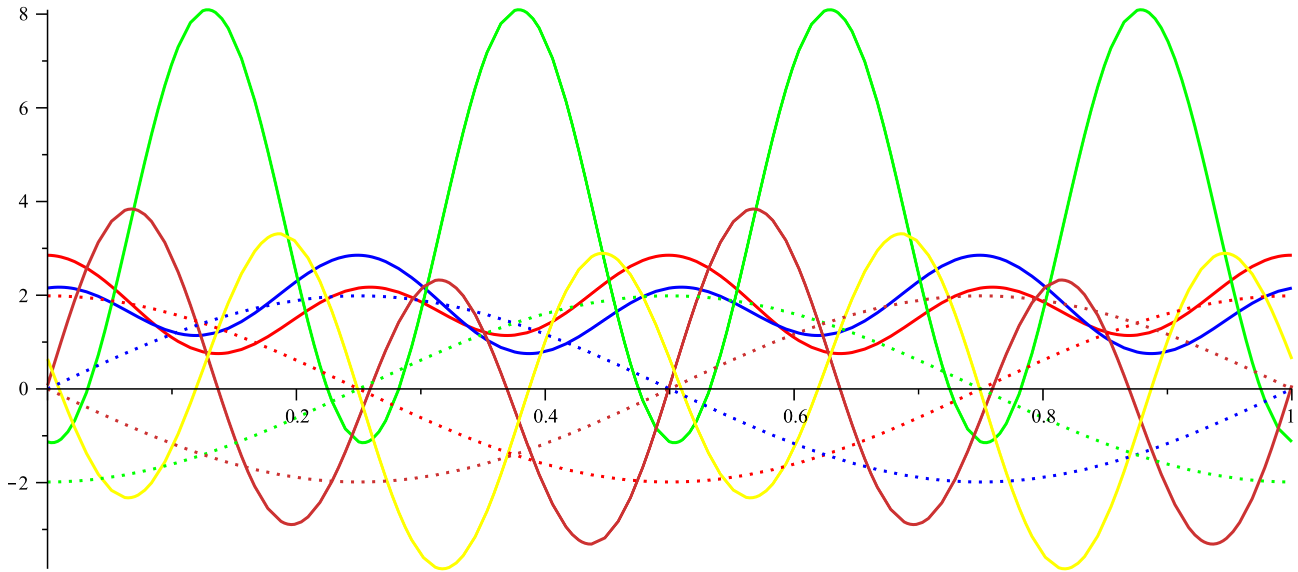
$$V = \tfrac{1}{24}\lambda_1\phi_1^4 + \tfrac{1}{24}\lambda_2\phi_2^4 + \tfrac{1}{4}\lambda_3\phi_1^2\phi_2^2 + \tfrac{1}{6}\lambda_4\phi_1^3\phi_2 + \tfrac{1}{6}\lambda_5\phi_1\phi_2^3$$
$$+ \left(\tfrac{1}{2}y_1\phi_1\psi_1\psi_1 + \tfrac{1}{2}y_2\phi_2\psi_1\psi_1 + \tfrac{1}{2}y_3\phi_1\psi_2\psi_2 + \tfrac{1}{2}y_4\phi_2\psi_2\psi_2 + \text{h.c.}\right)$$

$$Q = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} ip_1 & p_3 + ip_4 \\ -p_3 + ip_4 & ip_2 \end{pmatrix}$$

$$\lambda_1 = \frac{8(7087+357\sqrt{52953})}{102885}\pi^2\epsilon \qquad \lambda_2 = \frac{64(6346+9\sqrt{52953})}{102885}\pi^2\epsilon \qquad \lambda_3 = -\frac{272(\sqrt{52953}-57)}{102885}\pi^2\epsilon$$

$$\lambda_4 = \frac{32\sqrt{\frac{17}{19}(757-3\sqrt{52953})}}{5415}\pi^2\epsilon \qquad \lambda_5 = \frac{272\sqrt{\frac{17}{19}(757-3\sqrt{52953})}}{5415}\pi^2\epsilon \qquad y_1 = 2\sqrt{\frac{2}{5}}\pi\sqrt{\epsilon}$$

$$y_3 = -y_1 \qquad q_1 = \frac{\sqrt{\frac{17}{19}(757-3\sqrt{52953})}}{108300}\epsilon^3 \qquad p_4 = \text{undetermined}$$



Features and future work

Physics of non-conformal scale-invariant QFTs

- Less constrained than CFTs
- β-functions $\sim$ Anomalous dimensions
- Rare RG flows (recurrent behaviors)
  - RG flows $\neq$ Gradient flows
  - Strongest version of a-theorem violated
- Phenomenological applications
  - Cyclic unparticle physics JFF, Grinstein, Stergiou (2011)

Future work

- Generic $d = 4$ examples (with β-functions at higher order) ?