3. Conformal Blocks from Casimir differential equation

In the embedding formalism, each primary operator is promoted to an homogeneous field on the future light-cone of the origin of $\mathbb{M}^{d+2}$,

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta}\mathcal{O}(P) \ , \qquad P^2 = 0, \quad \lambda > 0 \ . \tag{34}$$

In this formalism, conformal transformations are just $SO(d+1,1)$ Lorentz transformations of Minkowski space $\mathbb{M}^{d+2}$. The conformal block decomposition can then be written as

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\mathcal{O}_3(P_3)\mathcal{O}_4(P_4)\rangle = \sum_k C_{12k}C_{k34}\, G^{(12)(34)}_{\Delta_k,l_k}(P_1,\dots,P_4) \tag{35}$$

where

$$G^{(12)(34)}_{\Delta,l}(P_1,\dots,P_4) = \frac{1}{P_{12}^{(\Delta_1+\Delta_2)/2}P_{34}^{(\Delta_3+\Delta_4)/2}}\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_{34}}{2}} g_{\Delta,l}(u,v) \ , \tag{36}$$

$P_{ij} = -2P_i \cdot P_j$ and u, v are conformal invariant cross ratios

$$u = \frac{P_{12}P_{34}}{P_{13}P_{24}} \ , \qquad v = \frac{P_{14}P_{23}}{P_{13}P_{24}} \ . \tag{37}$$

The conformal blocks are eigenfunctions of the conformal Casimir,

$$\frac{1}{2}(J_{1,AB}+J_{2,AB})(J_1^{AB}+J_2^{AB})G^{(12)(34)}_{\Delta,l}(P_1,\dots,P_4) = \mathcal{C}_{\Delta,l}\, G^{(12)(34)}_{\Delta,l}(P_1,\dots,P_4) \ , \tag{38}$$

with eigenvalue $\mathcal{C}_{\Delta,l} = \Delta(\Delta-d)+l(l+d-2)$, where

$$J_{AB} = i\left(P_A\frac{\partial}{\partial P^B} - P_B\frac{\partial}{\partial P^A}\right) \tag{39}$$

are the Lorentz generators in $\mathbb{M}^{d+2}$ with indices $A, B = 0, 1, \dots, d+1$.

a.* Show (using Mathematica) that (38) together with (36) is equivalent to

$$\mathcal{D}\, g_{\Delta,l}(u,v) = \frac{1}{2}\mathcal{C}_{\Delta,l}\, g_{\Delta,l}(u,v) \tag{40}$$

where

$$\mathcal{D} = (1-u-v)\frac{\partial}{\partial v}\left(v\frac{\partial}{\partial v}+a+b\right) + u\frac{\partial}{\partial u}\left(2u\frac{\partial}{\partial u}-d\right) \tag{41}$$

$$-(1+u-v)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+a\right)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+b\right) \tag{42}$$

and $a = (\Delta_2-\Delta_1)/2$ and $b = (\Delta_3-\Delta_4)/2$.

b. Transform to the coordinates z and $\bar{z}$ defined in (29) and obtain

$$\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\bar{z}} + (d-2)\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\frac{\partial}{\partial z} - (1-\bar{z})\frac{\partial}{\partial \bar{z}}\right) \tag{43}$$

with

$$\mathcal{D}_z = z^2(1-z)\frac{\partial^2}{\partial z^2} - (a+b+1)z^2\frac{\partial}{\partial z} - abz\ . \tag{44}$$

c. In two dimensions, the partial differential equation separates in two ordinary differential equations. Show that

$$g_{\Delta,l} = \frac{k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + k_{\Delta+l}(\bar{z})k_{\Delta-l}(z)}{2^l(1+\delta_{l,0})} \tag{45}$$

satisfies the boundary condition (32) if $k_\beta(z) \approx z^{\beta/2}$ for small z, and the Casimir differential equation if

$$\mathcal{D}_z k_\beta(z) = \frac{\beta}{2}\left(\frac{\beta}{2}+1\right)k_\beta(z)\ . \tag{46}$$

Conclude that

$$k_\beta(z) = z^{\beta/2}\ {}_2F_1\left(\frac{\beta}{2}+a, \frac{\beta}{2}+b, \beta, z\right)\ . \tag{47}$$

d. Check that

$$g_{\Delta,l} = \frac{z\bar{z}}{2^l(z-\bar{z})}\big(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta+l}(\bar{z})k_{\Delta-l-2}(z)\big) \tag{48}$$

satisfies both the differential equation and the boundary condition in $d=4$.

4. Stress tensor three-point function

The goal of this exercise is to determine how many independent tensor structures are available for the three point function of the stress-energy tensor in a conformal field theory. You should use the embedding formalism to encode the operator $T^{ab}(x)$ in a field $T(P,Z)$ obeying $T(\lambda P, \alpha Z + \beta P) = \lambda^d \alpha^2 T(P,Z)$. Then, the general solution compatible with conformal symmetry (and permutation symmetry and parity invariance) is

$$\langle T(P_1,Z_1)T(P_2,Z_2)T(P_3,Z_3)\rangle = \frac{a_1G_{000} + a_2G_{100} + a_3G_{110} + a_4G_{200} + a_5G_{111}}{(P_{12}P_{13}P_{23})^{\frac{d+2}{2}}} \tag{49}$$

where $P_{ij} = -2P_i\cdot P_j$ and

$$G_{000} = V_1^2V_2^2V_3^2 \tag{50}$$
$$G_{100} = H_{12}V_1V_2V_3^2 + \text{permutations} \tag{51}$$
$$G_{110} = H_{12}H_{13}V_2V_3 + \text{permutations} \tag{52}$$
$$G_{200} = H_{12}^2V_3^2 + \text{permutations} \tag{53}$$
$$G_{111} = H_{12}H_{13}H_{23} \tag{54}$$

with $V_1 = V_{1,23}$, $V_2 = V_{2,31}$ and $V_3 = V_{3,12}$. The coefficients a_k are further constrained by requiring conservation of the stress-energy tensor. This corresponds to the condition

$$\left[\left(h-1+Z\cdot\frac{\partial}{\partial Z}\right)\frac{\partial}{\partial Z}\cdot\frac{\partial}{\partial P} - \frac{1}{2}Z\cdot\frac{\partial}{\partial P}\frac{\partial^2}{\partial Z\cdot\partial Z}\right]\langle T(P,Z)T(P_2,Z_2)T(P_3,Z_3)\rangle = 0 \tag{55}$$

a.** Implement this condition in Mathematica and show that it is equivalent to

$$0 = a_1 - h(h+3)a_3 + 2h(h+5)a_4 - 4(h^2-1)a_5 \tag{56}$$
$$0 = a_2 - (h+1)a_3 + 4ha_4 - 2(h^2-1)a_5 \tag{57}$$

Tips

1. Define a scalar product to represent $Z_i \cdot Z_j$, $Z_i \cdot P_j$ and $P_i \cdot P_j$. You can use the function `CenterDot` and give it some useful properties like $Z_i \cdot Z_i = 0$.

2. Define a derivative operator $\frac{\partial}{\partial M^A}$ with respect to a vector with an open index (M^A could be Z_i^A or P_i^A). The basic rules that you need to give are $\frac{\partial}{\partial M^A} M_B = \eta_{AB}$ and $\frac{\partial}{\partial M^A} M \cdot Q = Q_A$.

3. Implement rules for index contraction.

4. After acting on the ansatz (49) with the differential operator that takes the divergence, as in (55), you will need to identify the independent building blocks in the result. produced by the action of the operator in (55). After performing all index contractions, you should be able to rewrite all $Z_i \cdot Z_j$ in terms of H_{ij} and all $Z_i \cdot P_j$ in terms of V_i.

b.* In 3 dimensions not all building blocks are independent. This follows from the fact that the 6 vectors Z_i and P_i can not be linearly independent in $3+2=5$ dimensions. Show that this reduces the number of independent tensor structures of the 3pt function of the stress-energy tensor from 3 to 2. *Hint*: show that the determinant of the 6×6 matrix of dot products $Z_i \cdot Z_j$, with $Z_i \to P_{i-3}$ for $i = 4, 5, 6$, is proportional to the numerator of (49) with $(a_1, a_2, a_3, a_4, a_5) = (4, 4, 2, 1, 2)$.

Summary of Embedding Space Formalism (from arXiv:1107.3554)

- **Embedding space**

  The natural habitat for conformal field theories is the light cone of the origin of $\mathbb{M}^{d+2}$. $SO(d+1,1)$ Lorentz transformations of the light rays generate conformal transformations. The usual flat physical space $\mathbb{R}^d$ can be obtained by projecting into the Poincaré section of the light cone

$$P_x = (P^+, P^-, P^a) = (1, x^2, x^a)\,. \tag{58}$$

- **Primary fields**

  Primary fields of dimension Δ and spin l are encoded into a field $F(P,Z)$, polynomial in the polarisation vector Z, such that

$$F(\lambda P; \alpha Z + \beta P) = \lambda^{-\Delta}\alpha^l F(P;Z)\,. \tag{59}$$

  The usual tensor form of the operator on $\mathbb{R}^d$ is obtained from

$$f_{a_1\ldots a_l}(x) = \frac{1}{l!(h-1)_l} D_{a_1}\cdots D_{a_l} F(P_x; Z_{\epsilon,x})\,, \tag{60}$$

  where $h = d/2$, D_a is the differential operator defined by

$$D_a = \left(h - 1 + \epsilon\cdot\frac{\partial}{\partial\epsilon}\right)\frac{\partial}{\partial\epsilon^a} - \frac{1}{2}\epsilon_a\frac{\partial^2}{\partial\epsilon\cdot\partial\epsilon}\,, \tag{61}$$

  and $Z_{\epsilon,x} = (0, 2x\cdot\epsilon, \epsilon^a)$.

- **Correlators**

  The most general form of the correlator

$$\langle F_1(P_1;Z_1)\cdots F_n(P_n;Z_n)\rangle\,, \tag{62}$$

  compatible with conformal invariance, is a linear combination of homogeneous polynomials of degree l_i in each Z_i, each constructed by multiplying the basic building blocks

$$V_{i,jk} = \frac{(Z_i\cdot P_j)(P_i\cdot P_k) - (Z_i\cdot P_k)(P_i\cdot P_j)}{(P_j\cdot P_k)}\,, \tag{63}$$

$$H_{ij} = -2\big[(Z_i\cdot Z_j)(P_i\cdot P_j) - (Z_i\cdot P_j)(Z_j\cdot P_i)\big]\,, \tag{64}$$

  The P_i dependence is then constrained by the scaling in (59).

- **Conserved fields**

  A spin l primary field of dimension $\Delta = d - 2 + l$ obeys the conservation equation

$$(\partial_P\cdot D)\,F(P,Z) = 0\,, \tag{65}$$

  where D is the differential operator defined in (61) with $\epsilon\to Z$. This condition generates additional constraints on the correlators of conserved fields that can be easily implemented.