3 Correlation functions in $\mathcal{N}=4$ SYM

We shall consider the correlation function

$$G(1,\dots,n) = \langle \mathcal{O}(x_1,Y_1)\dots\mathcal{O}(x_n,Y_n)\rangle \tag{3.1}$$

of the operators $\mathcal{O}(x_i,Y_i)$ built from six real scalar fields Φ^I_{ij} (with $I=1,\dots,6$) in the $U(N_c)$ gauge group $(i,j=1,\dots,N_c^2)$

$$\mathcal{O}(x,Y) = Y_I\,Y_J\ \mathrm{tr}\left[\Phi^I(x)\Phi^J(x)\right] \tag{3.2}$$

Here Y_I is an auxiliary 6−dimensional null vector $(Y^2 \equiv Y_I Y_I = 0)$ and the scalar field is normalized as

$$\langle \Phi^{I_1}_{i_1 j_1}(x_1)\Phi^{I_2}_{i_2 j_2}(x_2)\rangle = \frac{1}{2}\delta^{I_1 I_2}\delta_{i_1 j_2}\delta_{j_1 i_2}\frac{1}{x_{12}^2} \tag{3.3}$$

Problem 1: Write a Mathematica code to compute 2-, 3- and 4-point correlation functions at Born level

$$\begin{aligned}
G^{(0)}(1,2) &= \frac{1}{2}N_c^2\frac{(y_{12}^2)^2}{(x_{12}^2)^2}\\
G^{(0)}(1,2,3) &= N_c^2\frac{y_{12}^2 y_{23}^2 y_{31}^2}{x_{12}^2 x_{23}^2 x_{31}^2}\\
G^{(0)}(1,2,3,4) &= \frac{N_c^4}{4}\left[\left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{34}^2}{x_{34}^2}\right)^2 + \left(\frac{y_{13}^2}{x_{13}^2}\frac{y_{24}^2}{x_{24}^2}\right)^2 + \left(\frac{y_{41}^2}{x_{41}^2}\frac{y_{23}^2}{x_{23}^2}\right)^2\right]\\
&\quad + N_c^2\left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{41}^2}{x_{41}^2} + \frac{y_{12}^2}{x_{12}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{13}^2}{x_{13}^2} + \frac{y_{13}^2}{x_{13}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{41}^2}{x_{41}^2}\right)
\end{aligned} \tag{3.4}$$

Here the notation is introduced for $x_{ij}^2 = (x_i - x_j)^2$ and $y_{ij}^2 = (Y_i\cdot Y_j)$.

The superconformal symmetry of the $\mathcal{N}=4$ SYM theory impose tight restrictions on the possible form of the loop corrections to the correlation function (3.1). $G(1,2)$ and $G(1,2,3)$ are protected from loop corrections. The loop corrections to $G(1,2,3,4)$ take the following factorized form (the so-called "partial non-renormalization"):

$$G_4 = G^{(0)}(1,2,3,4) + R(1,2,3,4)\times F(x_i;a) \qquad (\text{for } a = g_{\rm YM}^2 N_c/(8\pi^2))\,, \tag{3.5}$$

where $R(1,2,3,4)$ is a universal rational function of x_i- and y_i-coordinates at the four external points $1,2,3,4$:

$$\begin{aligned}
R(1,2,3,4) = 2N_c^2\Bigg[&\frac{y_{13}^4 y_{24}^4}{x_{13}^2 x_{24}^2} + \frac{y_{14}^4 y_{23}^4}{x_{14}^2 x_{23}^2} + \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2} + \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2}(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2)\\
&+\frac{y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2}(x_{14}^2 x_{23}^2 - x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2) + \frac{y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2}(x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2)\Bigg]
\end{aligned} \tag{3.6}$$

According to (3.5), the loop corrections to the four-point correlation function are determined by a single function with the perturbative expansion

$$F(x_i; a) = a\, F^{(1)}(x_i) + a^2 F^{(2)}(x_i) + O(a^3) \tag{3.7}$$

The goal of the next exercise is to determine the lowest order correction $F^{(1)}(x_i)$

Problem 2: Consider the correlation function $G(1,2,3,4)$, Eq. (3.1), for the $Y-$variables of the form

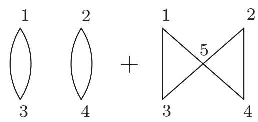
$$Y_1 = (1, i, 0, 0, 0, 0)\,, \qquad Y_2 = (0, 0, 1, i, 0, 0)\,, \qquad Y_3 = Y_1^*\,, \qquad Y_4 = Y_2^* \tag{3.8}$$

corresponding to the following choice of the operators (with $Z = \Phi^1 + i\Phi^2$ and $W = \Phi^3 + i\Phi^4$)

$$O(1) = \mathrm{tr}[Z^2(x_1)]\,, \quad O(2) = \mathrm{tr}[W^2(x_2)]\,, \quad O(3) = \mathrm{tr}[\overline{Z}^2(x_3)]\,, \quad O(4) = \mathrm{tr}[\overline{W}^2(x_4)] \tag{3.9}$$

Examine Feynman diagrams contributing to the correlation function to order a (with the interaction of the form $\sim a\,\mathrm{tr}\left([\Phi^I, \Phi^J][\Phi^I, \Phi^J]\right)$) and compare them with the general expression for $G(1,2,3,4)$, Eqs. (3.5) and (3.6),

$$G(1,2,3,4) =$$



$$= \frac{16N_c^2}{(x_{13}^2 x_{24}^2)^2}\left[\frac{N_c^2}{4} + 2a\, x_{13}^2 x_{24}^2 F^{(1)}(x_i) + O(a^2)\right] \tag{3.10}$$

to obtain

$$F^{(1)}(x_i) = -\frac{1}{4\pi^2}\int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{3.11}$$

To any order ℓ in the coupling, the function $F^{(\ell)}(x_i)$ has the following properties. It admits the representation

$$F^{(\ell)}(x_i) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{\ell!\,(-4\pi^2)^\ell}\int d^4x_5 \dots d^4x_{4+\ell}\, f^{(\ell)}(x_1, \dots, x_{4+\ell}) \tag{3.12}$$

where $f^{(\ell)}$ is a rational function depending on the four external coordinates $x_1, \dots, x_4$ and the ℓ additional (internal) coordinates $x_5, \dots, x_{4+\ell}$ giving the positions of the interaction vertices. It has the following form

$$f^{(\ell)}(x_1, \dots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1, \dots, x_{4+\ell})}{\prod_{1\le i<j\le 4+\ell} x_{ij}^2} \tag{3.13}$$

where the denominator contains the product of all distances between the $(4+\ell)$ points. The function $F^{(\ell)}(x_i)$ should satisfy the following two conditions:

- It should transform under inversions $I[x_i^\mu] = x_i^\mu / x_i^2$ as

$$I\left[F^{(\ell)}(x_1,x_2,x_3,x_4)\right] = x_1^2x_2^2x_3^2x_4^2\, F^{(\ell)}(x_1,x_2,x_3,x_4) \tag{3.14}$$

- Its *integrand* $f^{(\ell)}(x_1,\dots,x_{4+\ell})$ should be symmetric under the exchange of any pair of points (both external and internal)

$$f^{(\ell)}(\dots,x_i,\dots,x_j,\dots) = f^{(\ell)}(\dots,x_j,\dots,x_i,\dots) \tag{3.15}$$

Problem 3:

Show that the above two conditions imply that

(a) $P^{(\ell)}$ should also be invariant under the $S_{4+\ell}$ permutations of $x_1,\dots,x_{4+\ell}$

(b) $P^{(\ell)}$ should transform under inversions as

$$I[P^{(\ell)}(x_1,\dots,x_{4+\ell})] = (x_1^2\dots x_{4+\ell}^2)^{-\ell+1}P^{(\ell)}(x_1,\dots,x_{4+\ell}) \tag{3.16}$$

(c) $P^{(\ell)}$ should be a homogeneous polynomial in x_{ij}^2 of degree $(\ell-1)(\ell+4)/2$

Verify that at one ($\ell=1$) and two ($\ell=2$) loops the most general expression for $P^{(\ell)}$ is

$$P^{(1)} = 1\,, \qquad P^{(2)} = \frac{c^{(2)}}{48}\sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_5\sigma_6} = c^{(2)}\left[x_{12}^2x_{34}^2x_{56}^2+\dots\right] \tag{3.17}$$

with $c^{(2}$ arbitrary constant.

Substitute (3.17) into (3.12) and (3.13) and obtain the following result

$$\begin{aligned} F^{(1)} &= g(1,2,3,4)\,,\\ F^{(2)}/c^{(2)} &= h(1,2;3,4)+h(3,4;1,2)+h(2,3;1,4)+h(1,4;2,3)\\ &\quad + h(1,3;2,4)+h(2,4;1,3)+\frac{1}{2}\left(x_{12}^2x_{34}^2+x_{13}^2x_{24}^2+x_{14}^2x_{23}^2\right)[g(1,2,3,4)]^2\,, \end{aligned} \tag{3.18}$$

where $g-$ and $h-$ integrals have the form

$$\begin{aligned} g(1,2,3,4) &= -\frac{1}{4\pi^2}\int\frac{d^4x_5}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}\,,\\ h(1,2;3,4) &= \frac{x_{34}^2}{(4\pi^2)^2}\int\frac{d^4x_5\,d^4x_6}{(x_{15}^2x_{35}^2x_{45}^2)x_{56}^2(x_{26}^2x_{36}^2x_{46}^2)}\,. \end{aligned} \tag{3.19}$$

The correlation function $G(1,2,3,4)$ is related in planar $\mathcal{N}=4$ SYM to (the square of) the scattering amplitude A_4 in the light-cone limit x_{12}^2, x_{23}^2, x_{34}^2, $x_{41}^2 \to 0$ through the duality relation

$$\lim_{x_{i,i+1}^2\to 0} \frac{G(1,2,3,4)}{G^{(0)}(1,2,3,4)} \sim [A(p_1,p_2,p_3,p_4)]^2 \bigg|_{p_i=x_i-x_{i+1}}, \tag{3.20}$$

or equivalently

$$\lim_{x_{i,i+1}^2\to 0} \left(1+2x_{13}^2x_{24}^2\sum_\ell a^\ell F^{(\ell)}\right) = \left(1+\sum_\ell a^\ell M^{(\ell)}\right)^2 . \tag{3.21}$$

Here $M^{(\ell)}$ defines $\ell-$loop correction to *planar* four-gluon scattering amplitude $A_4 \sim 1+\sum_\ell a^\ell M^{(\ell)}$ and it is given by the sum of *connected* (nonfactorizable) planar integrals.

Problem 4:

(a) Use the duality relation (3.21) together with (3.18) to show that the condition of absence of (disconnected) $[g(1,2,3,4)]^2$ integral inside $M^{(2)}$ fixes uniquely the normalization coefficient in (3.18)

$$c^{(2)} = 1 \tag{3.22}$$

(b) Use the duality relation (3.21) to compute two-loop corrections to the 4-particle scattering amplitude

$$\begin{aligned} M^{(1)} &= x_{13}^2x_{24}^2 g(1,2,3,4)\,, \\ M^{(2)} &= x_{13}^2x_{24}^2\,[h(1,3;2,4)+h(2,4;1,3)]\,. \end{aligned} \tag{3.23}$$
