1 The conformal group

1.1 The algebra

The conformal group in d dimension is generated by $SO(d)$ rotation generators $M_{ij} = -M_{ji}$, translations P_i, special conformal transformations K_i and the dilatations generator D. The non-zero commutators take the form

$$\begin{aligned}
&[M_{ab}, M_{cd}] = i\left(\delta_{ac}M_{bd} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc} + \delta_{bd}M_{ac}\right) \\
&[M_{ab}, P_c] = i\left(\delta_{ac}P_b + \delta_{bc}P_a\right) \qquad [M_{ab}, K_c] = i\left(\delta_{ac}K_b + \delta_{bc}K_a\right) \\
&[D, P_a] = P_a \qquad [D, K_a] = -K_a \\
&[K_a, P_b] = 2\delta_{ab}D + c_1 M_{ab}
\end{aligned} \tag{1}$$

Fix the constant c_1 by the Jacobi identity which should be true for any generators t_1, t_2, t_3:

$$[[t_1, t_2], t_3] + [[t_2, t_3], t_1] + [[t_3, t_1], t_2] = 0 \tag{2}$$

Implement the commutation relations for the conformal group in Mathematica. If possible, use $d = 6$. Smaller choices such as $d = 3$ or $d = 4$ are also acceptable here and in the rest of the exercise.

1.2 Representations

The local operators of a CFT form representations of the conformal group. A highest weight representation of the conformal group consists of a primary operator $O_{\Delta,R}(0)$ sitting at the origin, or the corresponding state on the sphere S^{d-1} $|\Delta, R\rangle$, together with its descendants.

The lowers weight states $|\Delta, R\rangle$ are annihilated by the lowering operators and have eigenvalue Δ under dilatations

$$K_i|\Delta, R\rangle = 0 \qquad D|\Delta, R\rangle = \Delta|\Delta, R\rangle \tag{3}$$

Furthermore, they belong to some irreducible unitary representation R of the rotation group $SO(d)$. We can denote a basis in R as $|\Delta, R, s\rangle$ and the $SO(d)$ generators in R as $\left(m^R_{ij}\right)^s_{\ t}$.

Implement some highest weight representations for the conformal group in Mathematica. You should be able to act with the algebra generators on a generic linear combination of descendants. You should at least implement the representations corresponding to highest weight vectors in a scalar and vector representations R of $SO(d)$.

Extra: *implement the representation for a spinor R.*

In order to work with spinors, it is useful to identify $SO(6)$ with $SU(4)$, i.e. label the rotation generators as M^A_B, with A, B running from 1 to 4 and $\sum_A M^A_A = 0$. An $SO(6)$ vector becomes an antisymmetric matrix $P_{AB} = -P_{BA}$, etc. The usual metric δ_{ij} becomes the totally antisymmetric tensor ϵ^{ABCD}. Then the spinors simply carry a single fundamental or anti-fundamental $SU(4)$ index.

1.3 Unitarity bounds and short representations

A highest weight representation will be unitary if it has no descendants of negative norm. The hermitean inner product in the highest weight representation is defined by the rule $P_a^\dagger = K_a$:

$$(P_a P_b \cdots |\Delta, R, n\rangle)^\dagger \, P_c P_d \cdots |\Delta, R, m\rangle = \langle \Delta, R, n| \cdots K_b K_a P_c P_d \cdots |\Delta, R, m\rangle \tag{4}$$

The right hand side is computed by applying repeatedly the commutation relations of the conformal group.

Use your Mathematica implementation to figure out for which range of conformal dimensions your highest weight representations are unitary, by computing the matrix of inner products at any given level (number of Ps) up to some large value. It should become soon clear that only the first few levels are important to answer this question

At the boundary of the allowed region, the highest weight representations have null vectors, i.e. descendants of norm zero. You should show that the first null vector to appear in the representation is actually a primary, i.e. is annihilated by all the K_i. Viceversa a descendant which is also a primary will have zero norm. In this case, a unitary representation is obtained by setting to zero these null vectors and all their descendants, which are also null. The result is a short representation of the conformal group.

Use your Mathematica implementation to look for the short representations with highest weights scalar and vector (and spinor if you coded it). You should recognize that these short representations correspond to free fields and conserved currents respectively.

1.4 Towards conformal blocks

A primary operator inserted at a point x^i near the origin can be described by the state $|\Delta, R; x\rangle = e^{x^i P_i}|\Delta, R\rangle$. This state satisfies deformed constraints

$$\begin{aligned} &K_i|\Delta, R; x\rangle = \mathcal{K}_i^{\Delta,R}(x)|\Delta, R; x\rangle \qquad D|\Delta, R; x\rangle = \mathcal{D}^{\Delta,R}(x)|\Delta, R; x\rangle \\ &M_{ij}|\Delta, R; x\rangle = \mathcal{M}_{ij}^{\Delta,R}(x)|\Delta, R; x\rangle \end{aligned} \tag{5}$$

for appropriate differential operators $\mathcal{K}_i(x), \mathcal{D}(x), \mathcal{M}_{ij}(x)$.

Compute the differential operators, by hand or using your Mathematica implementation to represent $|\Delta, R; x\rangle$.

In an OPE, we consider two local operators, say $O_{\Delta_1,R_1}(0)$ at the origin and $O_{\Delta_2,R_2}(x)$ near the origin, and look at the state they construct on a sphere surrounding them. We want to look at the projection $|x\rangle$ of that state on the space of descendants of a given conformal primary $|\Delta, R\rangle$. The state is characterized by the equations

$$\begin{aligned} &K_i|x\rangle = \mathcal{K}_i^{\Delta_2,R_2}(x)|x\rangle \qquad D|x\rangle = \left[\Delta_1 + \mathcal{D}^{\Delta_2,R_2}(x)\right]|x\rangle \\ &M_{ij}|x\rangle = \left[m_{ij}^{R_1} + \mathcal{M}_{ij}^{\Delta_2,R_2}(x)\right]|x\rangle \end{aligned} \tag{6}$$

It should be clear that these equations can be solved recursively level-by-level, starting from

$$|x\rangle = \frac{C_{a_1\cdots a_n;m}x^{a_1}\cdots x^{a_n}}{|x|^{\Delta_1+\Delta_2-\Delta+n}}|\Delta, R, m\rangle + \cdots \tag{7}$$

where the structure constants $C_{a_1\cdots a_n;m}$ are constrained by rotation invariance.

Compute $|x\rangle$ up to a reasonable level with Mathematica for a few choices of R_1, R_2, R. Include at least the case where all operators are scalars, and the cases where one is a vector and the others scalars. Are the answers unique or are there degrees of freedom?

A conformal block for a four-point function is then simply defined as an inner product $\langle x|y\rangle$ where $\langle x|$ encodes two operators, say O_{Δ_4,R_4} at infinity and O_{Δ_3,R_3} at $x/|x|^2$, and $|y\rangle$ the other two, say $O_{\Delta_1,R_1}(0)$ at the origin and O_{Δ_2,R_2} at y.

Compute $\langle x|y\rangle$ up to some reasonable power of x, y with your Mathematica implementation, for a conformal block of 4 scalar operators in a scalar and vector channel. Can you teach Mathematica to recognize the result as a linear combination of hypergeometric functions? Repeat the exercise for four spinors if you coded that.

1.5 Conformal blocks and short representations

If the state $|\Delta, R\rangle$ belongs to a short representation, it may be impossible to solve the linear equations for $|x\rangle$, unless the external operators satisfy some constraints.

Can you see this for the 4 scalar operators in the scalar and vector channels? Can you give this a simple physical explanation?

On the other hand, if one of the external operator, say O_{Δ_2,R_2}, belongs to a short representation, one needs to impose by hand that null descendants of O_{Δ_2,R_2} decouple from the answer. In turns, this can constrain the allowed operators running in the internal channel, or reduce the degrees of freedom of the answer.

Test this for the case where O_{Δ_2,R_2} is a free scalar field.

2 Extras: the $OSp(8*|4)$ superconformal algebra

The maximal superconformal algebra in six dimensions combines the conformal group, 16 supercharges Q^s_A, 16 superconformal charges S^A_s and the $Sp(4)$ R-symmetry rotation generators L_{st}. Schematically, the new relations are

$$\begin{aligned}
&[P,Q]=0 \qquad [Q,Q]=P \quad [M,Q]=Q \qquad [D,Q]=\frac{1}{2}Q \qquad [L,Q]=Q\\
&[S,Q]=M+D+L \qquad [K,Q]=S\\
&[P,S]=Q \qquad [M,S]=S \qquad [D,S]=-\frac{1}{2}S \qquad [L,S]=S\\
&[S,S]=K \qquad [K,S]=0
\end{aligned} \tag{8}$$

Implement the commutation relations and fix the constants with Jacobi identities

The highest weight representations are still built starting from a primary $|\Delta, R, r\rangle$, annihilated by S and K, with descendants under the action of Q and P. Now we need to pick a representation R of $SU(4)$ and r of $Sp(4)$.

Implement some highest weight representations. An important example has scalar R and r corresponding to a traceless antisymmetric tensor of $Sp(4)$, i.e. the vector representation of $SO(5) = Sp(4)$. Find the null vectors for $\Delta = 4$, which corresponds to a very important short representation, which includes the stress tensor.

Ultimately, you can use this code to compute the superconformal blocks for a four point function of this short operator.