Easy

Slava Rychkov

Decomposition of conformal blocks into contributions of descendants

As we discussed, conformal block of a spin l dimension Δ primary exchanged between scalar external fields must have the following expansion:

$$G_{\Delta,l}(z,\bar{z}) = \sum_{n=0}^{\infty}\sum_{l'} a_{n,l'} r^{\Delta+n} Q_{l'}(\cos\alpha), \qquad re^{i\alpha} = z, \tag{1}$$

where $Q_l(x)$ are the Gegenbauer polynomials $C_l^{(D/2-1)}(x)$

$$C_l^{(a)}(x) \leftrightarrow \texttt{GegenbauerC[l,a,x]} \tag{2}$$

For $D = 2$ we have $a = 0$ and the regularized Gegenbauer polynomial must be used. In practice this means that in this case:

$$Q_l(x) = \cos(l \arccos x) \tag{3}$$

Each term in the sum (1) is interpreted as a contribution of a spin l' descendant at level n. Each such descendant is obtained by acting on the primary by n derivatives, hence $l' \leqslant l + n$. Unitarity implies that the coefficients must be positive: $a_{n,l'} \geqslant 0$.

In $D = 2$ and $D = 4$ conformal blocks for external scalars have been found explicitly by Dolan and Osborn (`hep-th/0011040`). In case of all external dimensions equal, they are given by (in a convenient normalization):

$$\begin{aligned} G_{\Delta,l}^{D=2}(z,\bar{z}) &= \frac{1}{2}\left[k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})\right], \\ G_{\Delta,l}^{D=4}(z,\bar{z}) &= \frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right], \end{aligned} \tag{4}$$

where

$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1\left(\tfrac{1}{2}\beta, \tfrac{1}{2}\beta; \beta; x\right). \tag{5}$$

Part 1. Write a code which expands the Dolan-Osborn blocks (4) into the series of the form (1) and check the positivity of coefficients at the first few levels (say for $n \leqslant 6$). Work at fixed primary spin, say $l = 0$ and/or $l = 2$, but keep the dimension Δ as a free parameter, so that $a_{n,l'}$ will be functions of Δ.

As we also discussed, conformal blocks allow a similar but slightly different expansion of the form:

$$G_{\Delta,l}(z,\bar{z}) = \sum_{n=0}^{\infty}\sum_{l'} b_{n,l'} r^{\Delta+n} Q_{l'}(\cos\alpha), \qquad re^{i\alpha} = \rho(z) = \frac{z}{(1+\sqrt{1-z})^2}, \tag{6}$$

The individual terms are again contributions of descendants, but in a different geometry in which the operators are inserted symmetrically with respect to the origin. In particular the coefficients should still be positive: $b_{n,l'} \geqslant 0$.

Part 2. Same as Part 1 but for the coefficients $b_{n,l'}$. Apart from positivity, you should be able to observe the following properties of these coefficients:

(a) only even-order descendants enter with nonzero coefficients. This is a consequence of having symmetric operator insertions: one needs an even number of derivatives to couple the descendant consistently with the exchange symmetry. In particular the first subleading term in the expansion has $n = 2$ (while for (1) it is $n = 1$).

(b) the coefficients $b_{n,l'}$ are bounded as a function of Δ (while $a_{n,l'}$ grow with Δ making the expansion poorly convergent at large Δ).

Because of (a)+(b), the first term of the expansion (6) provides a good approximation to the full conformal block which has accuracy $O(\rho(z)^2)$ independently of the value of Δ, large or small. At the crossing-symmetric point $z = 1/2$ we have $\rho(1/2)^2 \approx 3\%$.

Formally both (a) and (b) follow from the relation:

$$k_\beta(4\rho^2/(1+\rho)^2) = (4\rho)^{\beta/2} {}_2F_1\left(\tfrac{1}{2}\beta, \tfrac{1}{2}; \tfrac{1}{2}\beta + \tfrac{1}{2}; \rho^2\right) , \tag{7}$$

by a quadratic transformation for the hypergeometric function.