Hard

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Analytic toy model for conformal bootstrap

The conformal bootstrap equation for the correlator of four identical scalars ϕ of dimension $[\phi] = d$ reads:

$$F_{0,0}(z,\bar{z}) + \sum p_{\Delta,l} F_{\Delta,l}(z,\bar{z}) = 0, \qquad p_{\Delta,l} \geqslant 0 \tag{1}$$

where

$$F_{0,0}(z,\bar{z}) \equiv (1-z)^d (1-\bar{z})^d - z^d \bar{z}^d \tag{2}$$

$$F_{\Delta,l}(z,\bar{z}) \equiv (1-z)^d (1-\bar{z})^d G_{\Delta,l}(z,\bar{z}) - z^d \bar{z}^d G_{\Delta,l}(1-z, 1-\bar{z}) \tag{3}$$

The sum in (1) is over conformal blocks of conformal primaries of dimension Δ and spin l. Conformal blocks are complicated objects, but we know that they have an expansion of the form:

$$G_{\Delta,l}(z,\bar{z}) = \sum_{n=2,4,6\ldots} \sum_{l'} b_{n,l'} H_{\Delta+n,l'}(z,\bar{z}) \tag{4}$$

where

$$H_{\Delta',l'}(z,\bar{z}) = r^{\Delta'} Q_{l'}(\cos\alpha), \qquad re^{i\alpha} = \rho(z) = \frac{z}{(1+\sqrt{1-z})^2} \tag{5}$$

are contributions of descendants. Here $Q_l(x)$ are the Gegenbauer polynomials $C_l^{(D/2-1)}(x)$

$$C_l^{(a)}(x) \leftrightarrow \texttt{GegenbauerC[l,a,x]} \tag{6}$$

For $D = 2$ we have $a = 0$ and the regularized Gegenbauer polynomial must be used. In practice this means that in this case:

$$Q_l(x) = \cos(l \arccos x) \tag{7}$$

The coefficients $b_{n,l'}$ in (4) are complicated functions of Δ,l and the spacetime dimension. They are all positive (assuming that the theory is unitary so that the primaries satisfy the unitarity bounds). Substituting (4) into (1) we get a simpler equation:

$$F_{0,0}(z,\bar{z}) + \sum q_{\Delta,l} H_{\Delta,l}(z,\bar{z}) = 0, \qquad q_{\Delta,l} \geqslant 0 \tag{8}$$

where the sum is now over all states, primaries or descendants. The coefficients are again positive. It is important that only even spins are present in (8), as in (1).

Passing from Eq. (1) to Eq. (8) we discarded a part of the information available in conformal theory (namely, that the coefficients of descendants are proportional to those of primaries with fixed coefficients). But we got a simpler equation since all involved functions are now elementary.

Problem. For dimensions d very close to the unitarity bound $D/2-1$, $2 \leqslant D \leqslant 4$, show that Eq. (8) does not have a solution unless a scalar operator of dimension below a certain bound $\Delta_{\max}(D)$ appears in the OPE $\phi \times \phi$. [The bound also depends on d but for simplicity consider the $d \to (D/2-1)^+$ limit.]

You will have to follow the following steps:

1. (Analytical with the help of Mathematica)

(a) Taylor-expand Eq. (8) to third order in z and $\bar{z}$ around the point $z = \bar{z} = 1/2$. Evaluate the three independent expansion coefficients as explicit functions of d, Δ, l, D. You can choose the derivatives $\partial_z, \partial_z^3, \partial_z^2\partial_{\bar{z}}$ as a basis. You will need to derive a formula for the ratio

$$k = \left.\frac{[C_l^{(a)}]'(x)}{C_l^{(a)}(x)}\right|_{x=1} \tag{9}$$

as a function of l and a. This can be derived from the generating function relation:

$$(1-2xt+t^2)^{-a} = \sum_{l=0}^{\infty} C_l^{(a)}(x)t^l \tag{10}$$

The actual value of $C_l^{(a)}(1) > 0$ can also be computed but will not be needed since it (like other positive factors) can be absorbed redefining $q_{\Delta,l}$.

(b) Eliminate one of the three equations, bringing the remaining two into the homogeneous form. A convenient choice is to eliminate the ∂_z equation. Taking linear combinations of the remaining equations and rescaling the vectors bring the system to the form:

$$\sum q_{\Delta,l}\vec{h}_{\Delta,l} = (0,0), \qquad q_{\Delta,l} \geqslant 0, \sum q_{\Delta,l} > 0. \tag{11}$$

[The last condition just expresses the fact that not all coefficients must be zero, eliminating the trivial solution which does not satisfy the eliminated equation.] with

$$\vec{h}_{\Delta,l} = \begin{pmatrix} 1 + a_1(d)\Delta^{-1} + a_2(d)\Delta^{-2} \\ \Delta^{-1} + b_1(d,k)\Delta^{-2} + b_2(d,k)\Delta^{-3} \end{pmatrix} \tag{12}$$

where the coefficients a_i and b_i can be found explicitly. Notice that the dependence on spin and D will enter only through k.

3. (Numerical with the help of Mathematica) For $D = 2, 3, 4$, explore how the vectors $\vec{h}_{\Delta,l}$ move in the plane while you vary Δ, l respecting the D-dimensional unitarity bounds:

$$\Delta \geqslant l + D - 2 \ (l \geqslant 1), \qquad \Delta \geqslant D/2 - 1 \ (l = 0) \tag{13}$$

Recall that only even spins need to be examined. In particular examine how the slope of the vectors varies. Study the convex hull (i.e. a set of all linear combinations with nonnegative coefficients) of the subset of vectors

$$\{\vec{h}_{\Delta,\ell} : \Delta \geqslant \Delta_*(l=0), l + D - 2(l = 2, 4, 6\ldots)\} \tag{14}$$

as a function of Δ_*. You will find that for small Δ_* the convex hull is $\mathbb{R}^2$, while for large Δ_* the convex hull will be a sector of $\mathbb{R}^2$ of angle $< \pi$. In the first case Eq. (11) has solutions, while in the second one it does not. The bound $\Delta_{\max}$ that we are computing is the boundary between the two cases.

4. (Partly numerical, partly analytical) Convince yourself that the actual value of $\Delta_{\max}$ can be determined by looking only at $\vec{h}_{\Delta,0}$ (as a function of Δ) and the stress tensor vector $\vec{h}_{D,2}$. This is because it so happens that the rest of the vectors lie in the convex hull of this smaller set.

Then find the actual bound $\Delta_{\max}(D)$, as a solution of a quadratic equation with D-dependent coefficients. Plot the bound. You should see it interpolating smoothly between ≈ 0.7 for $D = 2$ and ≈ 3.6 for $D = 4$.