Very hard

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Conformal bootstrap in $4 - \epsilon$ dimensions

In $D = 4 - \epsilon$ dimensions, perturbing the free scalar theory by $\lambda\phi^4$ (and finetuning the mass term), one can flow in the IR to a nontrivial CFT known as the *Wilson-Fisher fixed point*. All parameters of this CFT (dimensions, OPE coefficients) have power series expansions in ϵ around their free theory values. Although these series are only asymptotic, one can still try to resum them and continue to the physically relevant value $\epsilon = 1$ (corresponding to the critical point of the 3D Ising model). A lot of effort has been invested in computing these ϵ-*expansion* series.

Important unsolved problem: Can conformal symmetry be used to generate the ϵ-expansion series? [The existing techniques use normal Feynman diagram perturbation theory and do not take advantage of the conformal symmetry.]

Here we will do several modest exercises related to this problem.

1. Free theory spectrum. The free scalar theory consists of local operators built out of ϕ and its derivatives (modulo the relation $= 0$. From the CFT point of view, some of these operators are primaries (e.g ϕ and all its integer powers), while others are descendants (e.g. $\phi\partial_\mu\phi = \frac{1}{2}\partial_\mu(\phi^2)$). Write a program which enumerates all primaries made out of k ϕ's involving $\leqslant l_{\max}$ *non-contracted* derivatives (i.e. with spin equal the number of derivatives). Work in $D = 4$.

Hint. A general such operator will be a linear combination of terms of the schematic form

$$\partial^{n_1}\phi\,\partial^{n_2}\phi\ldots\partial^{n_k}\phi \tag{1}$$

where derivatives, which are by assumption non-contracted, must be symmetrized, and also traces must be subtracted to get a state of well-defined spin $l = \sum n_i$. In order to avoid these complications, let us look at the all-z component of such fields, i.e. when all derivatives are ∂_z, where $z = x_1 + ix_2$ is the complex coordinate in an arbitrarily chosen plane. Then there are also no traces to subtract.

Proceed recursively in l. For $l = 0$ we have one primary ϕ^k. Suppose we already found all primaries with $l - 1$ derivatives, N of them, and we want to see the new primaries which appear at order l. First we write down all candidates of the form (1) (use `IntegerPartitions`). Suppose there are M of them. Surely $M > N$ because we can obtain at least N states by acting with derivatives on already found primaries. We thus expect that $M - N$ linear combinations will be new primaries. These linear combinations can be found by imposing that their two-point functions with all previously found primaries be zero. (Compute the matrix of two-point functions and use `NullSpace`). The two point functions are evaluated using Wick's theorem. To simplify the problem further assume that the operators are inserted at two points x_1, x_2 with only z_i nonzero, while the orthogonal components $x_i^\perp = 0$ so that $(x_1 - x_2)^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$.

2. First-order correction. Each free theory operator of the form (1) will map to an operator at the Wilson-Fisher fixed point (although some operators which were primaries in the free theory will flow to operators which are descendants; more about it below). There is a simple way to compute the order-ϵ shift of the operator dimensions at the WF fixed point in terms of its dimension in the free scalar theory:[1]

$$[\mathcal{O}]_{WF} = [\mathcal{O}]_{free} + 2\frac{a(\mathcal{O})}{a(\phi^4)}\epsilon + O(\epsilon^2) \tag{2}$$

where $a(\mathcal{O})$ is the free theory OPE coefficients with which $\mathcal{O}$ appears in the OPE with the perturbing operator ϕ^4:

$$\phi^4 \times \mathcal{O} = \frac{a(\mathcal{O})}{|x|^4}\mathcal{O} + \dots \tag{3}$$

Notice that the dimension $[\mathcal{O}]_{free}$ has to be computed taking into account that $[\phi]_{free} = 1 - \epsilon/2$ in $D = 4 - \epsilon$ dimension, while within the accuracy of the formula (2) the coefficients $a(\mathcal{O})$ may be evaluated in $D = 4$.

For operators $\mathcal{O}$ of spin $l > 0$ the coefficient $a(\mathcal{O})$ has to be defined a bit more carefully. In this case rotation invariance allows terms in the OPE with the same $1/|x|^4$ leading singularity but with extra angular dependence:

$$\phi^4(x) \times \mathcal{O}_{zzz\ldots}(0) = \frac{\sum_{i=0}^{l} a_i(\mathcal{O})(z\bar{z}/x^2)^i}{|x|^4}\mathcal{O}_{zzz\ldots}(0) + \dots \tag{4}$$

Then $a(\mathcal{O})$ must be defined averaging every term in the RHS on S^3. This gives:

$$a(\mathcal{O}) = \sum_{i=0}^{l} \frac{a_i(\mathcal{O})}{i+1} \tag{5}$$

Write a program which computes $a(\mathcal{O})$ for any operator of the form (1). *Hint.* Work in the normalization $\langle\phi(x)\phi(0)\rangle = 1/|x|^2$ with same all$-z$ components as in Part 1, although now you have to take x general to keep track of the angular dependence. Start by doing two Wick contractions in $\phi^4(x) \times \mathcal{O}(0)$, picking out two ϕ's out of ϕ^4 and two out of $\mathcal{O}$ in all possible ways. Contracting with ϕ's carrying derivatives, those derivatives will be lost acting on the propagator. The $\phi^2(x)$ now has to be expanded around 0 to pick up the operator with the same number of derivatives as $\mathcal{O}$.

3. Multiplet recombination.[2]

(a) The field ϕ is interacting at the WF fixed point. Thus in WF we have $\partial^2\phi_{WF} \neq 0$ while in the free theory $\partial^2\phi_{free} = 0$. The state $\partial^2\phi_{WF}$ must originate from some free theory operator. The claim is that it originates from ϕ^3. In other words, ϕ^3 is a primary in free theory but becomes a descendant of ϕ in WF:

$$\partial^2\phi_{WF} \propto (\phi^3)_{WF} \tag{6}$$

[1] This formula is an easy consequence of conformal perturbation theory. See Ch.5 of Cardy's book "Scaling and renormalization is statistical physics" for the derivation.

[2] Leonardo Rastelli, Balt van Rees, personal communication.

One can say that the conformal multiplets of ϕ_{free} and $(\phi^3)_{free}$ recombine to form the conformal multiplet of ϕ_{WF}. Recombination is necessary since the multiplet of ϕ_{free} satisfies a shortening condition $\partial^2\phi_{free} = 0$ and contains fewer states than that of ϕ_{WF}, while the states cannot just appear or disappear.

Eq. (6) predicts that the dimensions of $(\phi^3)_{WF}$ and ϕ_{WF} differ by exactly 2. Let us check this to $O(\epsilon)$. In fact $[\phi]$ does not get correction at order-ϵ. (As we will see below, it does get a correction at $O(\epsilon^2)$). Using the code in Part 2, compute the correction to $[\phi^3]$ and check the prediction.

(b) The operator ϕ^2 gets order-ϵ correction. However, primary operators with non-contracted derivatives built out of two ϕ's don't. Compute the first few of these operators using the code of Part 1 and check that $O(\epsilon)$ corrections are zero using the code of Part 2.

(c) The spin $\ell = 2, 4, 6, \ldots$ operators considered in (b) in free theory are conserved spin l tensors satisfying the shortening condition $\partial \cdot J^{(l)} = 0$. Do they remain conserved at the WF fixed point? Similarly to the analysis in (a), conservation condition can be violated only if the multiplet of spin l, dimension $l+2$ current recombines with a multiplet of a spin $l-1$, dimension $l+3$ primary operator. By inspection, show that there is no candidate for $l = 2$ and thus the stress tensor remains conserved to all orders in ϵ. Use the Part 2 code to show that for $l = 4, 6$ there are candidates.

(d) In fact for $l = 4$ and higher the recombination *must* occur, since a Coleman-Mandula-like theorem (`1112.1016`) forbids higher-spin conserved currents for CFT in dimension $D \neq 2$. This implies that

$$\partial \cdot [J^{(l)}]_{WF} \propto [\tilde{J}^{(l-1)}]_{WF}, \qquad l \geqslant 4, \tag{7}$$

where $\tilde{J}$ is a candidate identified in (c). Since by (b) the $O(\epsilon)$ correction for $J^{(l)}$ vanishes, we get a prediction for the dimension of $\tilde{J}^{(l-1)}$. Check this prediction using the Part 2 code for $l = 4$.

4. Second-order correction to the dimension of ϕ.

We know that

$$[\phi]_{WF} = [\phi]_{free} + A\epsilon^2 + \ldots. \qquad [\phi^2]_{WF} = [\phi^2]_{free} + \epsilon/3 + B\epsilon^2 + \ldots, \tag{8}$$

where $O(\epsilon)$ corrections are computed by the method described in Part 2. Computation of A and B is more complicated - with usual field theory techniques it requires a two-loop calculation. Here we will see how conformal symmetry can be used to determine A.

Let's denote $\phi_{WF} = \mathcal{O}_1$, $(\phi^2)_{WF} = \mathcal{O}_2$. Consider the four point function

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_1(x_4)\rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{12}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{-\frac{1}{2}\Delta_{12}} \frac{g(u,v)}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}}, \tag{9}$$

fixed by conformal symmetry to have this form ($\Delta_{12} \equiv \Delta_1 - \Delta_2$). The functions of the cross ratios has an expansion into conformal blocks of the operators appearing in the OPE $\mathcal{O}_1 \times \mathcal{O}_2$:

$$g(u,v) = \sum (C_{\Delta,l})^2 G_{\Delta_{12};\Delta,l}(u.v) \tag{10}$$

As indicated, they blocks will also depend on Δ_{12}, see below.

Now consider the OPEs. In free theory we have (check this!)

$$\phi \times (\phi^2)_{norm} = \frac{1}{\sqrt{2}}\phi + \frac{\sqrt{3}}{2}(\phi^3)_{norm} + \dots \tag{11}$$

where the subscript *norm* means that the field has been normalized so that its two-point function is $1/|x|^{2\Delta}$:

$$(\phi^2)_{norm} \equiv \frac{1}{\sqrt{2}}\phi^2, \qquad (\phi^3)_{norm} \equiv \frac{1}{\sqrt{6}}\phi^3 \tag{12}$$

Couplings between the states have to vary continuously with ϵ, so that we expect that at the WF fixed point:

$$\mathcal{O}_1 \times \mathcal{O}_2 = (\frac{1}{\sqrt{2}} + O(\epsilon))\mathcal{O}_1 + \dots \tag{13}$$

As we discussed in Part 3(a), at the WF fixed point the ϕ^3 is not a primary but a descendant of $\mathcal{O}_1$. So its coupling in the OPE $\mathcal{O}_1 \times \mathcal{O}_2$ will be fixed by conformal symmetry. To extract this coupling consider the known expression for the scalar conformal block in D dimensions (`hep-th/0011040`)

$$G_{\Delta_{12};\Delta,l=0}(u,v) = u^{\Delta/2} \sum_{m,n=0}^{\infty} \frac{\left[\left(\frac{\Delta+\Delta_{12}}{2}\right)_m \left(\frac{\Delta-\Delta_{12}}{2}\right)_{m+n}\right]^2}{m!\,n!(\Delta+1-\frac{D}{2})_m(\Delta)_{2m+n}} u^m(1-v)^n, \tag{14}$$

where $(x)_n$ is the Pochhammer symbol. Notice that for generic Δ_{12} this conformal block has a singularity at $\Delta = \Delta_{free}$, which is not a problem since the free scalar must be decoupled from the rest of the theory, hence have zero OPE coefficient. If however $\Delta_{12} = -\Delta_{free}$ then the limit $\Delta \to \Delta_{free}$ is finite and defines the free scalar conformal block in the OPE $\phi \times \phi^2$.

Now let us substitute the WF dimensions into this expression. It turns out that:

$$\lim_{\epsilon\to 0} G^{D=4-\epsilon}_{\Delta_{12};\Delta_1,0}(u,v) = G^{D=1}_{-1;1,0} + const.G^{D=1}_{-1;3,0} \tag{15}$$

This expression means that, as expected, in the limit $D \to 4$ the conformal block of $\mathcal{O}_1$ splits into the sum of conformal blocks of ϕ and ϕ^3. Write a program to check this and find the *const.*, which turns out to depend on A but not on B. Since the relative couplings of ϕ and ϕ^3 are expected to vary continuously, we expect *const.* $= 3/2$ from (11). Use this to show that $A = 1/108$.