Heavy-Heavy-Light C_{123} in the Frolov-Tseytlin Limit (easy)

Pen and paper part

In this exercise we will study a particular class of string solutions characterized to be point-like in AdS with global time $t = \kappa\tau$, but non trivial motion in the sphere. In the sphere we describe the motion by using three complex coordinates

$$\mathbf{U}(\sigma, \tau) = (U_1, U_2, U_3) \tag{1}$$

constrained by $U \cdot \bar{U} = 1$. The Polyakov action is given by

$$S = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} (\partial_a \bar{\mathbf{U}} \cdot \partial^a \mathbf{U} - \Lambda(\bar{\mathbf{U}} \cdot \mathbf{U} - 1)) \tag{2}$$

- Show that

$$\bar{\mathbf{U}} \cdot \partial^a \partial_a \mathbf{U} = -(\partial_a \bar{\mathbf{U}} \cdot \partial^a \mathbf{U}) \tag{3}$$

- Show that the equations of motion are given by

$$\partial^a \partial_a \mathbf{U} = -(\partial_a \bar{\mathbf{U}} \cdot \partial^a \mathbf{U}) \mathbf{U} \tag{4}$$

- We also have to impose the Virasoro constraints that read

$$(\partial_\tau \bar{\mathbf{U}} \pm \partial_\sigma \bar{\mathbf{U}}) \cdot (\partial_\tau \mathbf{U} \pm \partial_\sigma \mathbf{U}) = \kappa^2 \tag{5}$$

  Where do they come from?

- We are interested in a particular subclass of solutions of the form

$$\mathbf{U}(\sigma, \tau) = e^{i\kappa\tau} \mathbf{u}(\sigma, \tau) \tag{6}$$

  in the limit where

$$\kappa \to \infty \quad \text{with} \quad \kappa \partial_\tau \mathbf{u}, \partial_\sigma \mathbf{u} \;\; \text{held fixed} \tag{7}$$

  Of course $\bar{\mathbf{u}} \cdot \mathbf{u} = 1$. Show that the Virasoro constraints give the following conditions

$$\partial_\sigma \bar{\mathbf{u}}.\partial_\sigma \mathbf{u} = 2i\kappa \bar{\mathbf{u}} \cdot \partial_\tau \mathbf{u}. \tag{8}$$

  and

$$\bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} = 0. \tag{9}$$

- Show that in this limit the equations of motion reduce to the first order equations

$$2i\kappa\partial_\tau \mathbf{u} = \partial_\sigma^2 \mathbf{u} + 2\mathbf{u}(\partial_\sigma \bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u}) \tag{10}$$

- The transformation $U_a \to e^{i\phi} U_a$ is a symmetry of the original action (2). Show that the Noether charge (angular momentum) associated to this symmetry is

$$J_a = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\kappa \bar{u}_a u_a - i\bar{u}_a \partial_\tau u_a) \tag{11}$$

  We also have that the energy is related to κ by $E = \sqrt{\lambda}\kappa$.

The final formula for the ratio $r = \frac{C_{123}^{\bullet\bullet\circ}}{C_{123}^{\circ\circ\circ}}$ is

$$r = \frac{C_{123}^{\bullet\bullet\circ}}{C_{123}^{\circ\circ\circ}} = \frac{1}{v_1^{j_1}\, v_2^{j_2}\, \bar{v}_3^{j_3}} \int\limits_0^{2\pi} \frac{d\sigma}{2\pi}\, u_1^{j_1}\, u_2^{j_2}\, \bar{u}_3^{j_3} \Bigg|_{\tau_e=0} , \qquad v_a = \sqrt{\frac{J_a}{J}} \,, \tag{12}$$

where we are taking $\mathcal{O}_3$ to be the string dual to the small BPS operator

$$\mathcal{O}_3 = \mathcal{N} \left(\text{Tr} \left(X^{j_1} Y^{j_2} Z^{j_3} \right) + \text{all possible permutations} \right) , \tag{13}$$

Now we can go to Mathematica. We will find a particular solution of the string equations of motion in this limit which goes by the name of Folded String. They we will plug it into (12) to obtain a prediction for the structure constants. You may turn the page.

Heavy-Heavy-Light C_{123} in the Frolov-Tseytlin limit

Mathematica Part (easy)

Jacobi Functions

• Plot the functions **JacobiDN[p x,q]** and **JacobiSN[p x,q]** for **x** $\in [0,2\pi]$ for a few values of **p** and **q** (take **q** between 0 and 1)

• What happens for **p=2 EllipticK[q]/**π ?

• Find the value of **a** for which **JacobiDN[*p*, *x*, *q*]^2 + *a* JacobiSN[*p*, *x*, *q*]^2 = 1**

• Simplify **JacobiAmplitude[2 EllipticK[q],q]** and **JacobiAmplitude[4 EllipticK[q],q]** . Hint: evaluate it numerically for several **q**'s.

Ansatz

We will now consider an ansatz for a closed string solution in the sphere:

$$\mathtt{u1} = e^{i\,\omega 1\,\frac{\tau}{\kappa}}\sqrt{\mathtt{a}}\ \mathtt{JacobiSN}\left[\frac{2\,\mathtt{EllipticK[q]}}{\pi}\,\sigma,\ \mathtt{q}\right];$$

$$\mathtt{u2} = 0;$$

$$\mathtt{u3} = e^{i\,\omega 3\,\frac{\tau}{\kappa}}\ \mathtt{JacobiDN}\left[\frac{2\,\mathtt{EllipticK[q]}}{\pi}\,\sigma,\ \mathtt{q}\right];$$

• Note that we picked the particular choice of **p** appearing the previous section. Why is this the only reasonable choice?

• What value should we take for **a**? (hint: we are describing strings in the sphere...)

• Plug the ansatz into the equations of motion (10) and find ω**1** and ω**3.** The results can be FullSimplified.

• Check Virasoro (8) and (9).

Relation between q,κ and α,J

• We define $J_1 = \alpha\, J$ and $J_3 = (1-\alpha)\, J$. We have (11). Show that to leading order in large **J** we have $\alpha = 1 - \frac{\mathtt{EllipticE[q]}}{\mathtt{EllipticK[q]}}$ and $\kappa = J + \frac{1}{\pi^2 J}\,2\,\mathtt{EllipticK[q]}\,(\mathtt{EllipticE[q]} + (-1+q)\,\mathtt{EllipticK[q]})$. Hint: use the last point of the Jacobi Functions part of the exercise.

• Find (numerically) the value of **q** for α**=1/4.**

• **(Hard (*skip at first and come back at the end if everything else works*))**

Compute the energy for that value of **q.** Compare with the energy of the two cut solutions found in the exercise about solving Bethe roots.

The structure constants

• We can now simply plug our solutions into the FT limit of the HHL structure constant formulae by Zarembo and Costa et al, see (12) in the first part of this exercise. Take $j_1 = j_3 = 2$ and $j_2 = 0$ and compare your result with equations (31) and (32) of arXiv:1104.5501.

There is another exercise on this school where you will find exactly the same number from a totally different point of view; namely from a weak coupling computation where one solves Bethe equations and works with some nice determinants...