1 Conformal symmetry

Conformal transformations consist of translations, $x'_\mu = x_\mu + \epsilon_\mu$, Lorentz rotations $x'_\mu = x_\mu + \omega_{\mu\nu}x^\nu$, dilatations $x'_\mu = \lambda x_\mu$ and special conformal transformations

$$K_\mu : \quad x_\mu \to x'_\mu = \frac{x_\mu + a_\mu x^2}{1 + 2(a\cdot x) + a^2 x^2} \tag{1.1}$$

Problem 1: Examine the action of the special conformal transformations on the set of co-planar intersecting lines $x_i^\mu(t) = (t, \alpha_i + t\beta_i, 0, 0)$ with $-\infty < t < \infty$ and the parameter of transformation of the form $a^\mu = (a^1, a^2, 0, 0)$. Assign numerical values to the parameters α_i, β_i and a^i and verify that the special conformal transformations preserve the angle between the tangent vectors at the intersecting points.

Problem 2: Verify that the special conformal transformations can be realized as the following composition of inversions I and translations P

$$K_\mu = I\, P_\mu\, I\,, \qquad I: \quad x_\mu \mapsto \frac{x_\mu}{x^2}\,, \qquad P: \quad x_\mu \mapsto x_\mu + a_\mu \tag{1.2}$$

Problem 3: Verify that the distance between the two points $x_{ij}^2 \equiv (x_i - x_j)^2$ transforms under inversions according to

$$I[x_{ij}^2] = \frac{x_{ij}^2}{x_i^2 x_j^2} \tag{1.3}$$

Show that for any four points x_i their cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2} \tag{1.4}$$

are invariant under the conformal transformations $I[u] = I[v] = 0$.

Examine the correlation function of conformal primary scalar operators

$$G_n = \langle O_1(x_1) O_2(x_2) \dots O_n(x_n)\rangle \tag{1.5}$$

It is a function of the distances between n points, $G_n = G_n(x_{ij}^2)$ which transforms under inversions $x_i^\mu \to x_i^\mu / x_i^2$ as

$$I[G_n] = (x_1^2)^{\Delta_1}(x_2^2)^{\Delta_2}\dots(x_n^2)^{\Delta_n} G_n \tag{1.6}$$

with Δ_i being the scaling dimension of the operator O_i. For $n = 2$ this relation implies that two-point correlation function of operators with different scaling dimension vanishes

$$G_2 = \langle O_1(x_1) O_2(x_2) \rangle \sim \frac{\delta_{\Delta_1,\Delta_2}}{(x_{12}^2)^{\Delta_1}} \tag{1.7}$$

Problem 4: Consider a general expression for the three-point correlation function

$$G_3 = \frac{c_{123}}{(x_{12}^2)^{\alpha_1}(x_{23}^2)^{\alpha_2}(x_{31}^2)^{\alpha_3}} \tag{1.8}$$

and show that the relation (1.6) fixes the $\alpha-$parameters as

$$\alpha_1 = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)\,, \qquad \alpha_2 = \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)\,, \qquad \alpha_3 = \frac{1}{2}(\Delta_3 + \Delta_1 - \Delta_2) \tag{1.9}$$

Problem 5: Repeat the same analysis for the four-point correlation function

$$G_4 = \frac{c_{1234}}{(x_{12}^2)^{\alpha_1}(x_{13}^2)^{\alpha_2}(x_{14}^2)^{\alpha_3}(x_{23}^2)^{\alpha_4}(x_{24}^2)^{\alpha_5}(x_{34}^2)^{\alpha_6}} \tag{1.10}$$

and show that the relation (1.6) fixes G_4 up to an arbitrary function of conformal invariant cross ratios u and v. Show that in the special case of identical operators, $\Delta_1 = \ldots = \Delta_4 \equiv \Delta$, the general expression for G_4 is

$$G_4 = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta}} \mathcal{F}(u, v) \tag{1.11}$$

with $\mathcal{F}$ being an arbitrary function of cross ratios.