2 Conformal partial wave expansion

Consider a free complex scalar field $\phi(x)$ normalized in such a way that

$$\langle\bar\phi(x_1)\phi(x_2)\rangle = \frac{1}{x_{12}^2}\,, \qquad (x_{ij}\equiv x_i - x_j) \tag{2.1}$$

and define the following composite operators

$$\begin{aligned} O(x) &= \bar\phi\phi(x) \\ \mathcal{O}_S(x) &= \sum_{k=0}^{S} c_k\,(n\cdot\partial)^k\bar\phi(x)\,(n\cdot\partial)^{S-k}\phi(x) \end{aligned} \tag{2.2}$$

Here $(n\cdot\partial)\equiv n^\mu\partial/\partial x^\mu$ with n^μ being an auxiliary light-like vector $n^2=0$. The operator $\mathcal{O}_S(x)$ carries the Lorentz spin S and it is called twist-2 operator.[1] The expansion coefficients c_k will be determined from the requirement for $\mathcal{O}_S(x)$ to be a conformal primary operator. In that case, the correlation function $\langle O(x_1)O(x_2)\mathcal{O}_S(x_0)\rangle$ is fixed by the conformal symmetry

$$\langle O(x_1)O(x_2)\mathcal{O}_S(x_0)\rangle = \frac{1}{(x_{12}^2)^{(2\Delta_0-\Delta_S+S)/2}(x_{10}^2x_{20}^2)^{(\Delta_S-S)/2}}\left[\frac{2(n\cdot x_{20})}{x_{20}^2}-\frac{2(n\cdot x_{10})}{x_{10}^2}\right]^S + (x_1\leftrightarrow x_2) \tag{2.3}$$

where $\Delta_0=2$ and $\Delta_S=S+2$ are the scaling dimensions of the operators $O(x)$ and $\mathcal{O}_S(x)$, respectively, in a free theory and the last term takes into account the symmetry of the correlation function under exchange of the points x_1 and x_2.

The relation (2.3) can be simplified by choosing x_i to lie in a two-dimensional subspace defined by two light-like vectors n^μ and $\bar n^\mu$ satisfying $(n\bar n)=1/2$ and $n^2=\bar n^2=0$

$$x_i^\mu = x_i^+\bar n^\mu + x_i^- n^\mu\,, \qquad x_i^2 = x_i^+x_i^-\,, \qquad 2(n\cdot x_i)=x_i^+ \tag{2.4}$$

In this kinematics, the 3-point correlation function (2.3) takes factorized form

$$\langle O(x_1)O(x_2)\mathcal{O}_S(x_0)\rangle = \frac{1}{(x_{12}^+)^{\Delta_0}}\left(\frac{x_{12}^+}{x_{10}^+x_{20}^+}\right)^{(\Delta_S-S)/2}\times\frac{1}{(x_{12}^-)^{\Delta_0}}\left(\frac{x_{12}^-}{x_{10}^-x_{20}^-}\right)^{(\Delta_S+S)/2} + (x_1\leftrightarrow x_2) \tag{2.5}$$

with $\Delta_0=2$ and $\Delta_S=S+2$.

Problem 1: Evaluate the correlation function on the left-hand side of (2.5) by replacing the operator $\mathcal{O}_S(x_0)$ by its general expression (2.2) and applying (2.1). Match the result into the expression on the right-hand side of (2.5) and determine the expansion coefficients c_k for $0\le S\le 10$. Show that c_k satisfy the following relation

$$\sum_{k=0}^{S} c_k\,p_1^k\,p_2^{S-k} = \frac{1}{S!}(p_1+p_2)^S\,\mathrm{C}_S^{1/2}\left(\frac{p_1-p_2}{p_1+p_2}\right) \tag{2.6}$$

[1] Twist is defined as the difference between the scaling dimension and the Lorentz spin, twist $=\Delta_S - S$.

with $\mathrm{C}_S^{1/2}(x)$ being the Gegenbauer polynomial and p_1, p_2 being arbitrary.

Problem 2: Use the obtained expressions for the expansion coefficients c_k to show that the two-point correlation function of the operator $\mathcal{O}_S(x)$ has the form

$$\langle \mathcal{O}_S(x_1)\mathcal{O}_{S'}(x_2)\rangle = \delta_{S,S'}\frac{(2S)!}{(S!)^2}\frac{(2n\cdot x_{12})^{2S}}{(x_{12}^2)^{2+2S}} = \frac{(2S)!}{(S!)^2}\frac{\delta_{S,S'}}{(x_{12}^+)^2(x_{12}^-)^{2+2S}} \tag{2.7}$$

where the second relation holds in the restricted kinematics (2.4).

The operator product expansion allows us to expand product of operators at short distances as

$$O(x_1)O(x_2) \overset{x_{12}^2\to 0}{\sim} \frac{1}{(x_{12}^2)^2} + \frac{1}{x_{12}^2}\sum_S P_S(x_{12},\partial_{x_2})\mathcal{O}_S(x_2) + \dots \tag{2.8}$$

where the first term on the right-hand side describes the contribution of the identity operator and dots denote contribution of terms subdominant for $x_{12}^2 \to 0$. In the second term on the right-hand side of (2.8) the sum runs over an infinite number of twist-2 operators. It involves a series $P_S(x_{12},\partial_{x_2})$ whose form can be fixed from the requirement that substitution of (2.8) into the left-hand side of (2.3) should yield the expected form of three-point correlation function

$$\langle O(x_1)O(x_2)\mathcal{O}_S(x_0)\rangle \overset{x_{12}^2\to 0}{\sim} \frac{1}{x_{12}^2}P_S(x_{12},\partial_{x_2})\langle \mathcal{O}_S(x_0)\mathcal{O}_S(x_2)\rangle \tag{2.9}$$

Problem 3: Examine (2.9) in the restricted kinematics (2.4), in which case $x_{12}^2 = x_{12}^+x_{12}^-$ and the limit $x_{12}^2 \to 0$ is understood as $x_{12}^+ \to 0$ with $x_{12}^- =$ fixed. Use the ansatz

$$P_S(x_{12},\partial_{x_2}) = (x_{12}^-)^S\sum_{k\geq 0} a_k (x_{12}^-)^k(\partial_{x_2^-})^k \tag{2.10}$$

and replace the two- and three-point correlation functions by their explicit expressions, Eqs. (2.7) and (2.5), respectively, to evaluate the coefficients a_k and, then, to obtain

$$\begin{aligned} P_S(x_{12},\partial_{x_2}) &= (x_{12}^-)^S\frac{(S!)^2}{(2S)!}\,{}_1F_1\left(\begin{matrix} S+1 \\ 2S+2\end{matrix}\,\middle|\, x_{12}^-\,\partial_{x_2^-}\right) \\ &= (x_{12}^-)^S(2S+1)\int_0^1 d\sigma(\sigma(1-\sigma))^S\,\mathrm{e}^{\sigma x_{12}^-\partial_{x_2^-}} \end{aligned} \tag{2.11}$$

The last relation allows us to rewrite the OPE in a nonlocal form

$$O(x_1)O(x_2) \overset{x_{12}^+\to 0}{\sim} \frac{1}{(x_{12}^2)^2} + \frac{2}{x_{12}^2}\sum_{S=0,2,4,\dots}(x_{12}^-)^S(2S+1)\int_0^1 d\sigma(\sigma(1-\sigma))^S\mathcal{O}_S(x_2+\sigma x_{12}) + \dots \tag{2.12}$$

where the sum runs over nonnegative even spin S and dots denote terms subleading as $x_{12}^2 \to 0$. Verify the correctness of this relation by evaluating the expectation value of the both sides with $\mathcal{O}_S(0)$.

Consider a four-point correlation function

$$G_4 = \langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle \tag{2.13}$$

In a free theory, it is given by the product of scalar propagators

$$G_4 = \frac{1}{(x_{12}^2 x_{34}^2)^2} + \frac{2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{41}^2} + \text{perm(2,3,4)} \tag{2.14}$$

where the last term denotes cyclic permutations of the points x_2, x_3 and x_4.

Problem 4: Verify that the four-point function (2.14) has the correct conformal properties (1.6) with $\Delta_i = 2$. Show that G_4 admits the following representation

$$G_4 = \frac{1}{(x_{12}^2 x_{34}^2)^2}\mathcal{F}(u,v)\,, \tag{2.15}$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2}$ are conformal cross-ratios and

$$\mathcal{F}(u,v) = \frac{(u+v+uv)^2}{v^2} \tag{2.16}$$

(Hint: to simplify the calculation take the limit $x_4 \to \infty$).

We deduce from the last two relations that G_4 has the following asymptotic behavior at short distances $x_{12}^2 \to 0$

$$G_4 \overset{x_{12}^2 \to 0}{\sim} \frac{1}{(x_{12}^2 x_{34}^2)^2}\left[1 + 2u\frac{1+v}{v} + O(u^2)\right] \tag{2.17}$$

The goal of the next exercise is to reproduce the same asymptotic behavior from the conformal OPE, Eq. (2.12).

Problem 5: Examine the correlation function (2.13) in the limit $x_{12}^2 \to 0$ in the restricted kinematics (2.4), that is for $x_{12}^+ \to 0$ with x_{12}^- fixed. Apply the OPE (2.12) together with (2.5) and expand G_4 over the contributions of conformal operators, the so-called conformal partial wave expansion

$$G_4 \overset{x_{12}^+ \to 0}{\sim} \frac{1}{(x_{12}^2 x_{34}^2)^2}\left[\mathcal{F}_I + \sum_{S=0,2,4,\dots} \mathcal{F}_S(u,v) + O(u^2)\right] \tag{2.18}$$

Here the conformal blocks $\mathcal{F}_I$ and $\mathcal{F}_S$ correspond to the identity operator and to twist-2 operator $\mathcal{O}_S$, respectively,

$$\mathcal{F}_I = 1\,, \qquad \mathcal{F}_S = 4u(1-v)^S \frac{(S!)^2}{(2S)!}\, {}_2F_1\left(\begin{matrix} S+1, S+1 \\ 2S+2 \end{matrix} \middle| 1-v \right) \tag{2.19}$$

and the conformal cross-ratios have the following form

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \to 0\,, \qquad v = \frac{x_{23}^2 x_{41}^2}{x_{13}^2 x_{24}^2} = \frac{x_{23}^- x_{14}^-}{x_{13}^- x_{24}^-} \tag{2.20}$$

Verify the identity

$$\sum_{S=0,2,4,\dots} \mathcal{F}_S = 2u\frac{1+v}{v} \tag{2.21}$$

and reproduce (2.17).