Mathematica Summer School in Theoretical Physics - ICTP Trieste

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The level of difficulty of each question is proportional to the number of *.

1. Operator Product Expansion

The general form of the OPE of two scalar operators is

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+l}} \left[F^{(12k)}_{a_1\ldots a_l}(x,\partial_y)\,\mathcal{O}_k^{a_1\ldots a_l}(y)\right]_{y=0} \tag{1}$$

where the sum runs over all primary operators $\mathcal{O}_k$ with spin l and dimension Δ.

a. Show that scale invariance implies that

$$F^{(12k)}_{a_1\ldots a_l}\left(\lambda x, \lambda^{-1}\partial_y\right) = \lambda^l F^{(12k)}_{a_1\ldots a_l}(x,\partial_y) \tag{2}$$

b. Compute the three-point function of scalar primary operators,

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_3(w)\rangle = \frac{C_{123}}{|x|^{\Delta_1+\Delta_2-\Delta_3}|w|^{\Delta_3+\Delta_2-\Delta_1}|x-w|^{\Delta_1+\Delta_3-\Delta_2}}\ , \tag{3}$$

using the OPE above, and derive

$$\left[F^{(123)}(x,\partial_y)\left(1+\frac{y^2-2y\cdot w}{w^2}\right)^{-\Delta_3}\right]_{y=0} = \left(1+\frac{x^2-2x\cdot w}{w^2}\right)^{\frac{\Delta_2-\Delta_1-\Delta_3}{2}}\ . \tag{4}$$

c.* Write a Mathematica program that uses the last equation to compute the coefficients $a_{n,m}$ for $n+2m\le 10$ in the derivative expansion

$$F^{(123)}(x,\partial_y) = \sum_{n,m=0}^{\infty} a_{n,m}(x\cdot\partial_y)^n(x^2\partial_y^2)^m \tag{5}$$

Suggestion: choose $w^2=1$ in equation (4).

d.* Make a table of your results and try to guess an analytic formula for $a_{n,m}$. The function

$$\texttt{Pochhammer[t,k]} = (t)_k = \frac{\Gamma(t+k)}{\Gamma(t)} = t(t+1)\ldots(t+k-1) \tag{6}$$

will be very useful.

e.* In order to study the OPE terms that involve operators with non-zero spin it is convenient to introduce a polarization vector ϵ_a. The idea is that we can encode a symmetric traceless tensor in a harmonic polynomial. If we define

$$\mathcal{O}(x,\epsilon) = \epsilon^{a_1}\ldots\epsilon^{a_l}\mathcal{O}_{a_1\ldots a_l}(x) \tag{7}$$

we can recover the tensor from the polynomial using

$$\mathcal{O}_{a_1\dots a_l}(x) = \frac{1}{l!(h-1)_l} D_{a_1}\dots D_{a_l}\mathcal{O}(x,\epsilon) \tag{8}$$

where $2h$ is the dimension of (Euclidean) spacetime and

$$D_a = \left(h-1+\epsilon\cdot\frac{\partial}{\partial\epsilon}\right)\frac{\partial}{\partial\epsilon^a} - \frac{1}{2}\epsilon_a\frac{\partial^2}{\partial\epsilon\cdot\partial\epsilon}\ . \tag{9}$$

Show (using Mathematica) that

$$[D_a, D_b] = 0\ , \qquad D^2 \propto \epsilon^2\ , \qquad D_a\,\epsilon^2 = \epsilon^2\left(D_a + 2\frac{\partial}{\partial\epsilon^a}\right)\ . \tag{10}$$

These properties guarantee that the tensor (8) is symmetric and traceless and that we can set $\epsilon^2 = 0$ in $\mathcal{O}(x,\epsilon)$ (because D_a is an interior operator to this constraint).

Check that, for unit vectors x and y, we have

$$(x\cdot D)^l(\epsilon\cdot y)^l = 2^{-l}(l!)^2 C_l^{h-1}(x\cdot y) \tag{11}$$

where $C_l^{h-1}(t) =$ `GegenbauerC[l,h-1,t]` is the Gegenbauer polynomial.

f. In this formalism, the OPE can be written as

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+l}}\left[F^{(12k)}\left(x,\partial_y,D\right)\mathcal{O}_k(y,\epsilon)\right]_{y=0} \tag{12}$$

where

$$F^{(12k)}\left(\lambda x,\lambda^{-1}\partial_y,\alpha D\right) = (\lambda\alpha)^l F^{(12k)}\left(x,\partial_y,D\right)\ . \tag{13}$$

Compute the three-point function of two scalar primary operators with a spin l operator,

$$\langle\mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_k(w,\epsilon')\rangle = C_{12k}\frac{\left(\epsilon'\cdot w\,(x-w)^2 - \epsilon'\cdot(w-x)\,w^2\right)^l}{|x|^{\Delta_1+\Delta_2-\Delta+l}|w|^{\Delta+\Delta_2-\Delta_1+l}|x-w|^{\Delta_1+\Delta-\Delta_2+l}}\ , \tag{14}$$

using the OPE (12) and the two-point function

$$\langle\mathcal{O}_k(y,\epsilon)\mathcal{O}_k(w,\epsilon')\rangle = \frac{\left(\epsilon\cdot\epsilon'\,(y-w)^2 - 2\epsilon\cdot(y-w)\,\epsilon'\cdot(y-w)\right)^l}{(y-w)^{2(\Delta+l)}} \tag{15}$$

and derive

$$\left[F^{(12k)}\left(x,\partial_y,D\right)\frac{\left(\epsilon\cdot\epsilon' - 2\frac{\epsilon\cdot(y-w)\,\epsilon'\cdot(y-w)}{1-2y\cdot w+y^2}\right)^l}{(1-2y\cdot w+y^2)^{\Delta}}\right]_{y=0} = \frac{\left(\epsilon'\cdot x + \epsilon'\cdot w\,(x^2-2w\cdot x)\right)^l}{(1-2x\cdot w+x^2)^{\frac{\Delta_1+\Delta-\Delta_2+l}{2}}} \tag{16}$$

where we have chosen $w^2 = 1$.

g.** Write a Mathematica program that uses the last equation to compute the coefficients $a_{n,m}$ and $b_{n,m}$ for $n + 2m \leq 4$ in the derivative expansion of the spin 1 case,

$$F^{(12k)}(x, \partial_y, D) = \sum_{n,m=0}^{\infty} \left[a_{n,m}\, x \cdot D + b_{n,m}\, x^2\, \partial_y \cdot D\right] (x \cdot \partial_y)^n (x^2 \partial_y^2)^m \ . \tag{17}$$

h. You can also study the case of general spin using the expansion

$$F^{(12k)}(x, \partial_y, D) = \sum_{n,m=0}^{\infty} \sum_{q=0}^{l} a_{n,m,q}\, (x \cdot D)^{l-q} (x^2\, \partial_y \cdot D)^q (x \cdot \partial_y)^n (x^2 \partial_y^2)^m \ . \tag{18}$$

Show that the leading term in the OPE gives

$$a_{0,0,0} = \frac{1}{l!(h-1)_l} \ . \tag{19}$$

or equivalently

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+l}} \left[x_{a_1} \dots x_{a_l} \mathcal{O}_k^{a_1 \dots a_l}(0) + \dots\right] \tag{20}$$

2. Conformal Blocks from OPE

The four-point function of scalar primary operators can be expanded using the OPE (1). This leads to the conformal block decomposition

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_k C_{12k} C_{k34}\, G^{(12)(34)}_{\Delta_k, l_k}(x_1, \dots, x_4) \tag{21}$$

where

$$G^{(12)(34)}_{\Delta,l}(x_1, \dots, x_4) = \frac{F^{(12k)}(x_{12}, \partial_{x_2}, D)\, \langle \mathcal{O}_k(x_2, \epsilon)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle}{|x_{12}|^{\Delta_1+\Delta_2-\Delta+l} C_{k34}} \tag{22}$$

$$= \frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}} \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}} g_{\Delta,l}(u,v) \tag{23}$$

Here, $\Delta_{ij} = \Delta_i - \Delta_j$ and u, v are conformal invariant cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ . \tag{24}$$

a.* Use the expansion (5) of the scalar OPE to compute the first terms of the double series expansion of the scalar conformal block

$$g_{\Delta,0}(u,v) = u^{\frac{\Delta}{2}} \sum_{p,q=0}^{\infty} b_{p,q} u^p (1-v)^q \ . \tag{25}$$

Suggestion: choose $x_4 \to \infty$ and $x_{13}^2 = 1$ to show that

$$g_{\Delta,0}(x_{12}^2, 1 - 2x_{12} \cdot x_{13} + x_{12}^2) = |x_{12}|^{\Delta} F^{(12k)}\left(x_{12}, \partial_{x_2}\right) |x_{23}|^{-\Delta-\Delta_{34}} \tag{26}$$

and

$$\sum_{p,q=0}^{\infty} b_{p,q} x^{2p} (2x \cdot w - x^2)^q = \left[F^{(12k)}\left(x, \partial_y\right) |y|^{-\Delta-\Delta_{34}}\right]_{y=w-x} \tag{27}$$

where we have written $x_{12} = x$ and $x_{13} = w$. Then, expand at small x to determine the coefficients $b_{p,q}$ for $q + 2p \leq 6$. Can you guess the general formula?

b. In the non-zero spin case, choose $x_4 \to \infty$ and $x_{13}^2 = 1$ to show that

$$g_{\Delta,l}(x_{12}^2, 1 - 2x_{12} \cdot x_{13} + x_{12}^2) = |x_{12}|^{\Delta-l} F^{(12k)}\left(x_{12}, \partial_{x_2}, D\right) \frac{(\epsilon \cdot x_{23})^l}{x_{23}^{\Delta+\Delta_{34}+l}} \tag{28}$$

c. It is convenient to parametrize the cross ratios by

$$u = z\bar{z} \ , \qquad v = (1-z)(1-\bar{z}) \ , \tag{29}$$

where z and $\bar{z}$ are independent variables. Show that for the choice $x_4 \to \infty$ and $x_{13}^2 = 1$ in Euclidean space, we have $z = |z|e^{i\theta}$ and $\bar{z} = |z|e^{-i\theta}$ with $|z|^2 = x_{12}^2$ and θ the angle between the vectors x_{12} and x_{13}.

d. Use the leading order term in the OPE

$$F^{(12k)}\left(x, \partial_y, D\right) = \frac{1}{l!(h-1)_l}(x \cdot D)^l + \dots \tag{30}$$

to derive the small $|z|$ behaviour of the conformal block

$$g_{\Delta,l} \approx \frac{|x_{12}|^{\Delta-l}}{l!(h-1)_l}(x_{12} \cdot D)^l (\epsilon \cdot x_{13})^l = \frac{l!}{2^l (h-1)_l} |z|^{\Delta} C_l^{h-1}(\cos\theta) \tag{31}$$

where $C_l^{h-1}(\cos\theta)$ is the Gegenbauer polynomial. Notice that this limit is particularly simple in two and four dimensions

$$g_{\Delta,l} \approx \frac{1}{2^l}|z|^{\Delta} \frac{e^{il\theta} + e^{-il\theta}}{1 + \delta_{l,0}} \ , \qquad d = 2 \ , \tag{32}$$

$$g_{\Delta,l} \approx \frac{1}{2^l}|z|^{\Delta} \frac{e^{i(l+1)\theta} - e^{-i(l+1)\theta}}{e^{i\theta} - e^{-i\theta}} \ , \qquad d = 4 \ . \tag{33}$$

Note that the result in $d = 2$ is defined as the limit $d \to 2$ of the expression in general dimension.