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Review of Electrodynamics and Electromagnetic waves in free space, Polarization

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REVIEW OF ELECTRODYNAMICS

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Layout

- Electrostatic : Revisited
- Magneto- static : Revisited
- Introduction to Maxwell’s equations
- Electrodynamics before Maxwell
- Maxwell’s correction to Ampere’s law
- General form of Maxwell’s equations
- Maxwell’s equations in vacuum
- Maxwell’s equations inside matter
- The Electromagnetic wave
- Energy and Momentum of Electromagnetic Waves
- Polarization of Light

Nomenclature

- E = Electric field
- D = Electric displacement
- B = Magnetic flux density
- H = Auxiliary field
- ρ= Charge density
- j = Current density
- μ_0 (permeability of free space) = $4\pi\times10^{-7}$T-m/A
- ε_0 (permittivity of free space) = 8.854×10^{-12}N-m^2/ C^2
- c (speed of light) = 2.99792458×10^8 m/s

Introduction

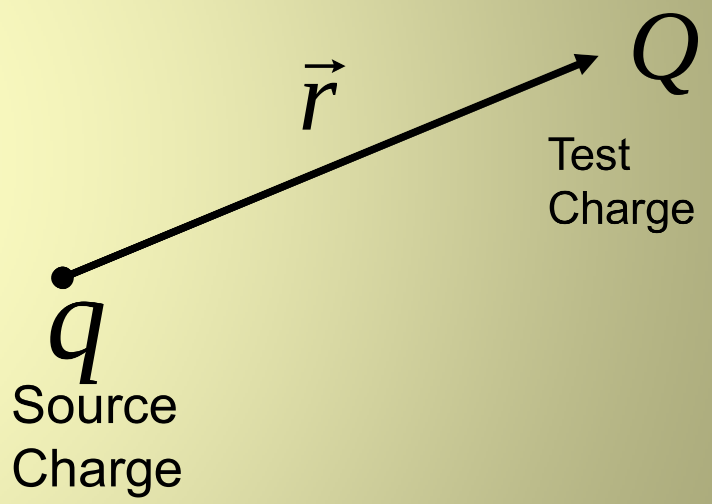
- **Electrostatics**
  - **Electrostatic field : Stationary charges produce electric fields that are constant in time. The theory of static charges is called electrostatics.**

Stationary charges ⟹ Constant Electric field;

Electrostatic :Revisited

Coulombs Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}$$



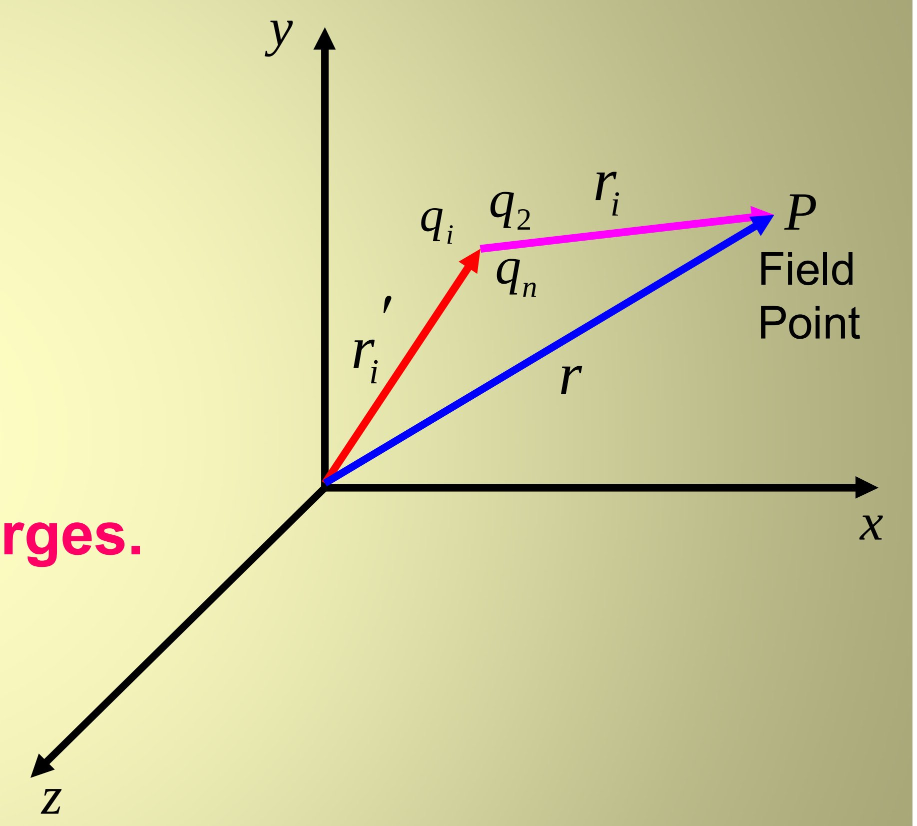
$$\varepsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{N-m^2}$$

Permittivity of free space

The Electric Field

$$\vec{F} = Q\vec{E}$$

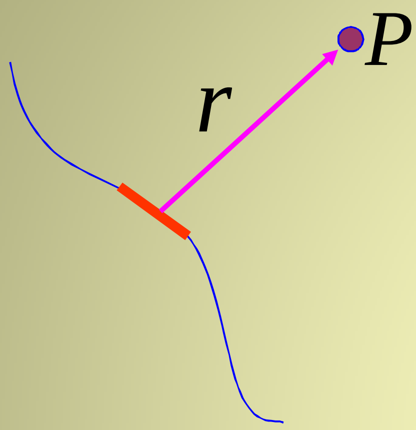
$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^{\,2}} \hat{r}_i$$



$\vec{E}$ - **the electric field of the source charges.**

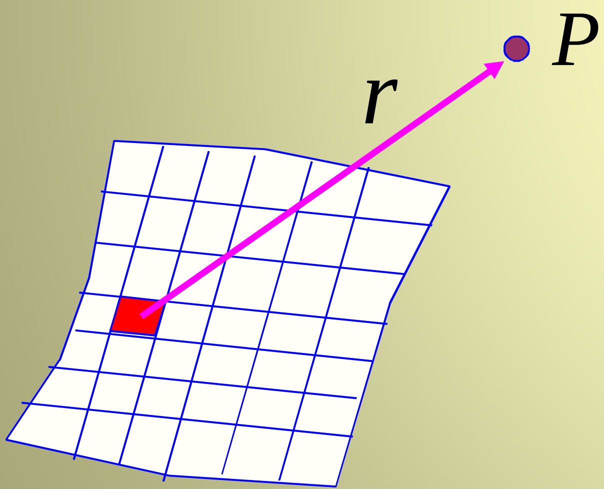
Physically E(P) Is force per unit charge exerted on a test charge placed at P.

The Electric Field: cont'd



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Line} \frac{\hat{r}}{r^2} \lambda dl$$

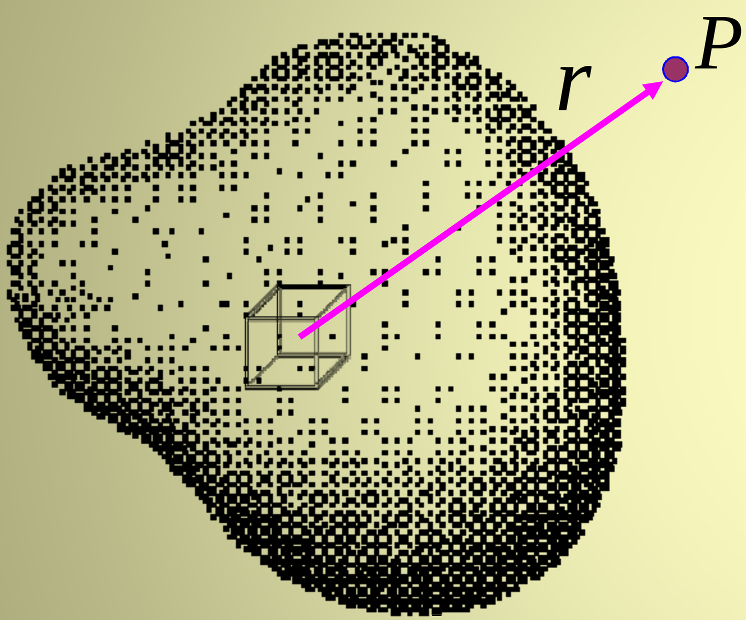
λ **is the line charge density**



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Surface} \frac{\hat{r}}{r^2} \sigma da$$

σ **is the surface charge density**

The Electric Field: cont'd



$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{Volume} \frac{\hat{r}}{r^2} \rho \, d\tau$$

ρ **is the volume charge density**

Electric Potential

The work done in moving a test charge Q in an electric field from point P_1 to P_2 with a constant speed.

$$W = Force \bullet dis\tan ce$$

$$W = -\int_{p_1}^{P_2} Q\vec{E} \bullet d\vec{l}$$

negative sign - work done is against the field.

For any distribution of fixed charges.

$$\oint \vec{E} \bullet d\vec{l} = 0$$

The electrostatic field is conservative

Electric Potential: cont'd

Stokes's Theorem gives

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{E} = -\vec{\nabla} V$$

where *V* is Scalar Potential

The work done in moving a charge Q from infinity to a point P_2 where potential is *V*

$$W = QV$$

V = Work per unit charge

= Volts = joules/Coulomb

Electric Potential : cont'd

Potential due to a single point charge q at origin

$$V = \int_r^\infty \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r}$$

$$F \propto \frac{1}{r^2}$$

$$E \propto \frac{1}{r^2}$$

$$V \propto \frac{1}{r}$$

Gauss's Law

$$\oint \vec{E} \bullet d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

Differential form of Gauss's Law

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$$

Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

Laplace's Equation

$$\nabla^2 V = 0$$

Electrostatic Fields in Matter

Matter: Solids, liquids, gases, metal, wood and glasses - behave differently in electric field.

Two Large Classes of Matter

(i) Conductors

(ii) Dielectric

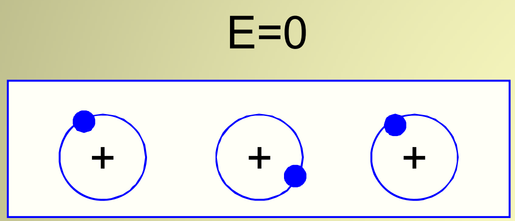
Conductors: Unlimited supply of free charges.

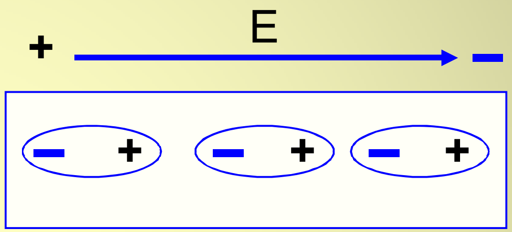
Dielectrics:

- Charges are attached to specific atoms or molecules- No free charges.
- Only possible motion - minute displacement of positive and negative charges in opposite direction.
- Large fields- pull the atom apart completely (ionizing it).

Polarization

A dielectric with charge displacements or induced dipole moment is said to be polarized.





Induced Dipole Moment $$p = \alpha E$$

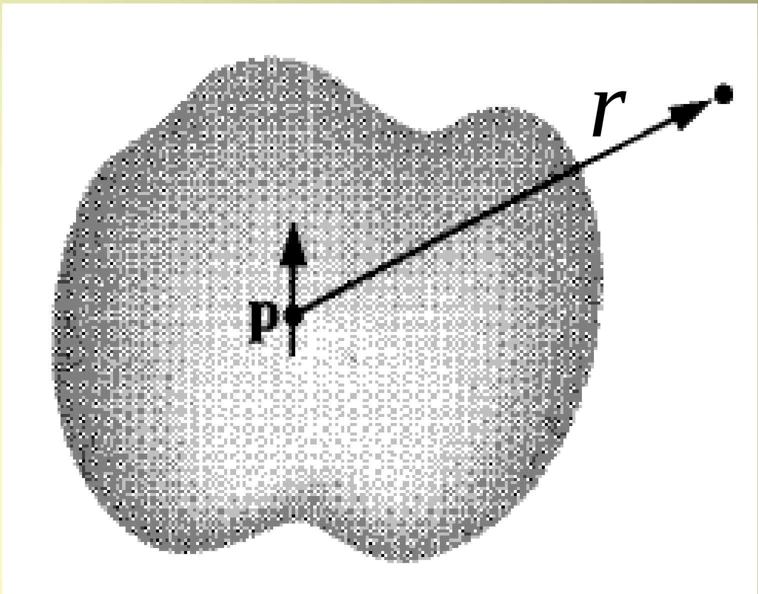
The constant of proportionality α is called the atomic polarizability

P ≡ dipole moment per unit volume

The Field of a Polarized Object

Potential of single dipole p is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\bullet\vec{\mathrm{p}}}{\mathrm{r}^2}$$



$$V = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\vec{\mathrm{P}}\bullet\hat{r}}{\mathrm{r}^2}d\tau$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface}\frac{1}{r}\vec{P}\bullet da - \int_{volume}\frac{1}{r}\left(\vec{\nabla}\bullet\vec{P}\right)d\tau\right]$$

Potential due to dipoles in the dielectric

The Field of a Polarized Object: cont'd

$$\sigma_b = \vec{P} \bullet \hat{n}$$

Bound charges at surface

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

Bound charges in volume

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{surface} \frac{1}{r}\sigma_b \, da + \int_{volume} \frac{1}{r}\rho_b \, d\tau\right]$$

The total field is field due to bound charges plus due to free charges

Gauss's law in Dielectric

- Effect of polarization is to produce accumulations of bound charges.
- The total charge density

$$\rho = \rho_f + \rho_b$$

$$\oint \vec{D} \bullet d\vec{a} = Q_{fenc}$$

From Gauss's law

Q_{fenc} -Free charges enclosed

$$\varepsilon_0 \vec{\nabla} \bullet \vec{E} = \rho = \rho_b + \rho_f$$

Displacement vector

$$\vec{\nabla} \bullet \vec{D} = \rho_f$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

Magnetostatics : Revisited

- **Magnetostatics**
  - **Steady current produce magnetic fields that are constant in time. The theory of constant current is called magnetostatics.**

Steady currents → Constant Magnetic field;

Magnetic Forces

Lorentz Force

$$\vec{F} = q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$$

- **The magnetic force on a segment of current carrying wire is**

$$F_{mag} = \int \left(\vec{I} \times \vec{B}\right) dl$$

$$F_{mag} = \int I\left(d\vec{l} \times \vec{B}\right)$$

Equation of Continuity

The current crossing a surface s can be written as

$$I = \int_s \vec{J} \bullet d\vec{a} = \int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau$$

$$\int_v \left(\vec{\nabla} \bullet \vec{J}\right) d\tau = -\frac{d}{dt}\int \rho d\tau = -\int \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Charge is conserved whatever flows out must come at the expense of that remaining inside - outward flow decreases the charge left in v

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

This is called equation of continuity

Equation of Continuity 1

In Magnetostatic steady currents flow in the wire and its magnitude I must be the same along the line- otherwise charge would be pilling up some where and current can not be maintained indefinitely.

$$\frac{\partial \rho}{\partial t} = 0$$

In Magnetostatic and equation of continuity

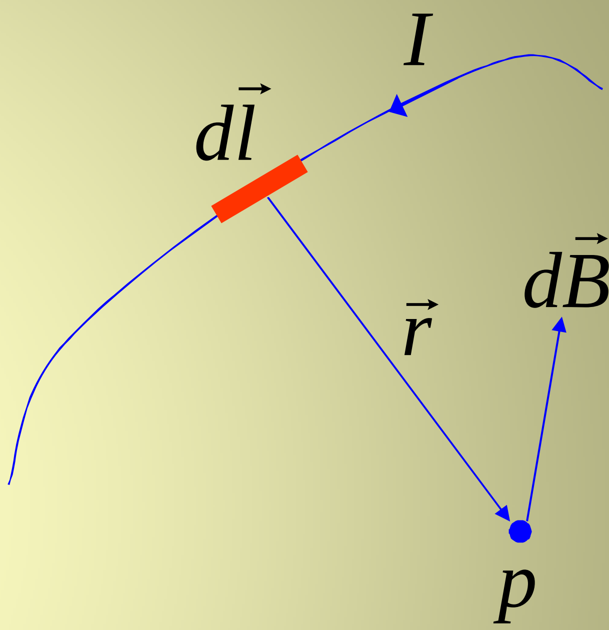
$$\vec{\nabla} \bullet \vec{J} = 0$$

Steady Currents: The flow of charges that has been going on forever - never increasing - never decreasing.

Magnetostatic and Current Distributions

Biot and Savart Law

$$\vec{B}(p) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \vec{r}}{|\vec{r}|^3} dl$$



dl is an element of length.

$\vec{r}$ vector from source to point p.

μ_0 Permeability of free space.

Unit of B = N/Am = Tesla (T)

Biot and Savart Law for Surface and Volume Currents

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\times\vec{r}}{|\vec{r}|^3}\,da$$

For Surface Currents

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\vec{r}}{|\vec{r}|^3}\,d\tau$$

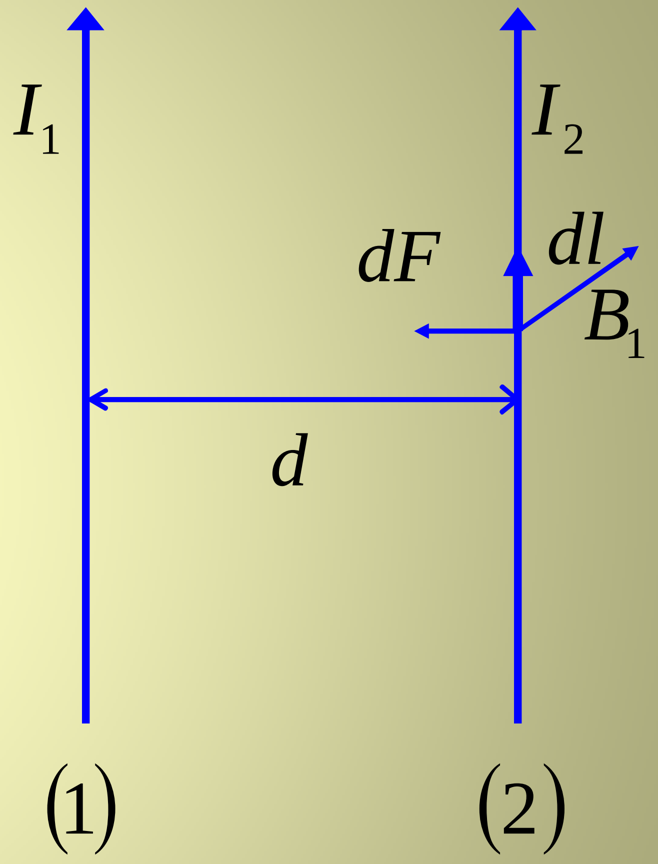
For Volume Currents

Force between two parallel wires

The magnetic field at (2) due to current I_1 is

$$B_1 = \frac{\mu_0 I_1}{2\pi d}$$

Points inside



Magnetic force law

$$dF = \int I_2 \left(d\vec{l}_2 \times \vec{B}_1 \right)$$

$$dF = \int I_2 \left(d\vec{l}_2 \times \frac{\mu_0 I_1}{2\pi d} \hat{k} \right)$$

Force between two parallel wires

$$dF = \frac{\mu_0 I_1 I_2}{2\pi d} dl_2$$

The total force is infinite but force per unit length is

$$\frac{dF}{dl_2} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

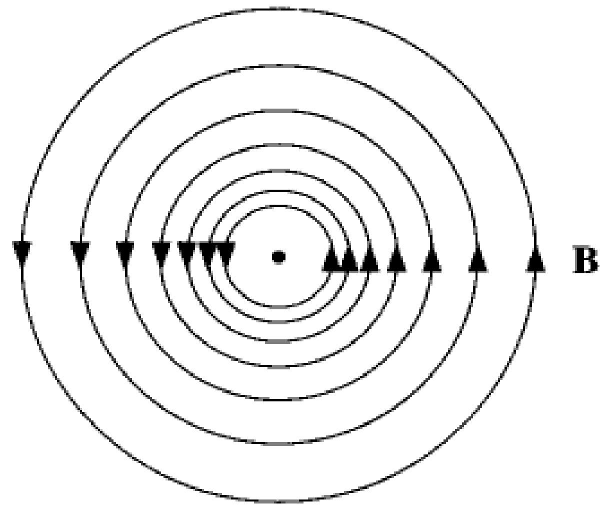
If currents are anti-parallel the force is repulsive.

Straight line currents

The integral of B around a circular path of radius s, centered at the wire is

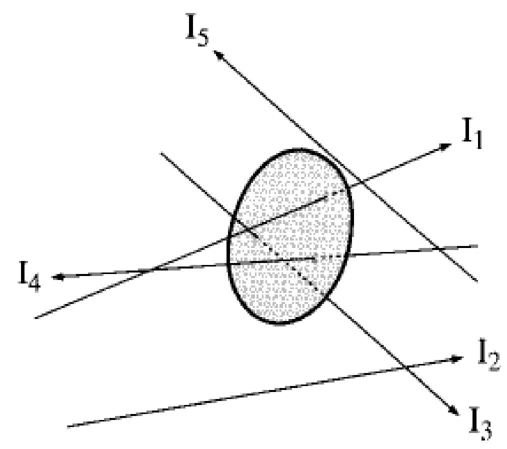
$$\oint \vec{B} \bullet d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \mu_0 I$$

The current is out of the page



For bundle of straight wires. Wire that passes through loop contributes only.

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$



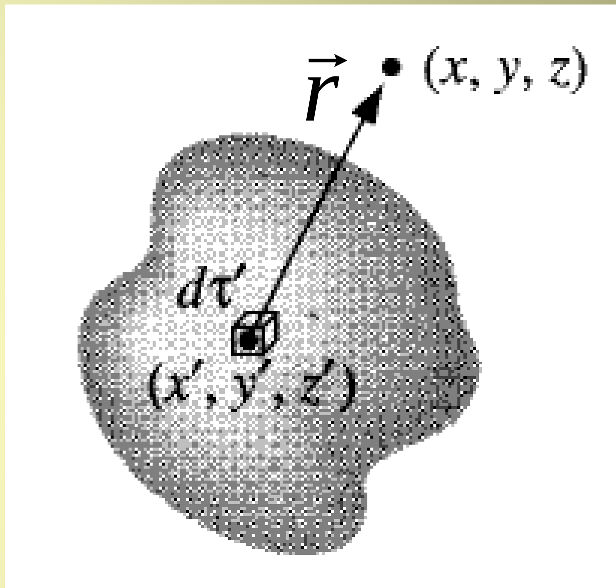
Applying Stokes' theorem

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Divergence and Curl of B

Biot-Savart law for the general case of a volume current reads

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(r') \times \vec{r}}{r^3}\, d\tau'$$



$\mathbf{B}$ is a function of (x, y, z),

$\mathbf{J}$ is a function of (x', y', z'),

$$\vec{r} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

$$\vec{\nabla} \bullet \vec{B} = 0$$ **and** $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Ampere's Law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Ampere's law

Integral form of Ampere's law

Using Stokes' theorem

$$\int \left(\vec{\nabla} \times \vec{B}\right) \bullet d\vec{a} = \oint \vec{B} \bullet d\vec{l} = \mu_0 \int \vec{J} \bullet d\vec{a}$$

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 I_{enc}$$

Vector Potential

The basic differential law of Magnetostatics

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

B is curl of some vector field called vector potential $A(P)$

$$\vec{B}(P) = \vec{\nabla} \times \vec{A}(P)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 J$$

Coulomb's gauge

$$\vec{\nabla} \bullet \vec{A} = 0$$

$$\nabla^2 A = -\mu_0 J$$

Magnetostatic Field in Matter

- Magnetic fields- due to electrical charges in motion.
- Examine a magnet on atomic scale we would find tiny currents.
- Two reasons for atomic currents.
  - Electrons orbiting around nuclei.
  - Electrons spinning on their axes.
- Current loops form magnetic dipoles - they cancel each other due to random orientation of the atoms.
- Under an applied Magnetic field- a net alignment of - magnetic dipole occurs- and medium becomes magnetically polarized or magnetized

Magnetization

If m is the average magnetic dipole moment per unit atom and N is the number of atoms per unit volume, the magnetization is define as

$$\vec{M} = N\vec{m}$$

$$\vec{m} = I\vec{a} = Am^2$$

or

$$m = Md\tau$$

$$M = \frac{Am^2}{m^3} = \frac{A}{m}$$

Magnetic Materials

Paramagnetic Materials

The materials having magnetization parallel to B are called paramagnets.

Diamagnetic Materials

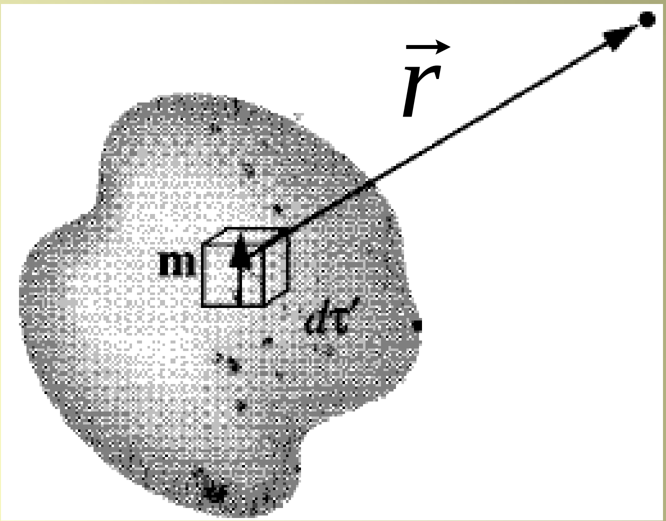
The elementary moment are not permanent but are induced according to Faraday's law of induction. In these materials magnetization is opposite to B.

Ferromagnetic Materials

Have large magnetization due to electron spin. Elementary moments are aligned in form of groups called domain

The Field of Magnetized Object

Using the vector potential of current loop



$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{M}\times\hat{n}}{r}da + \frac{\mu_0}{4\pi}\int\frac{\vec{\nabla}\times\vec{M}}{r}d\tau$$

$$\vec{K_b} = \vec{M}\times\hat{n}$$ **Bound Surface Current**

$$\vec{J_b} = \vec{\nabla}\times\vec{M}$$ **Bound Volume Current**

Ampere's Law in Magnetized Material

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{J} = \vec{J}_b + \vec{J}_f$$

$$\frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) = \vec{J}_b + \vec{J}_f = \vec{J}_f + \left(\vec{\nabla} \times \vec{M}\right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f$$

where

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

Integral form

$$\oint \vec{H} \bullet d\vec{l} = I_{fenc}$$

Faraday's Law of Induction

- Faraday's Law - a changing - magnetic flux through circuit induces an electromotive force around the circuit.

$$\varepsilon = \oint \vec{E} \bullet \vec{dl} = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \vec{B} \bullet \vec{da}$$

ϵ – Induced emf

E – Induced electric field intensity

Differential form of Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Faraday's Law of Induction

Induced Electric field intensity in terms of vector potential

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$$

For steady currents

$$\vec{E} = -\vec{\nabla} V$$

V – Scalar potential

Induced emf in a system moving in a changing magnetic field

$$\varepsilon = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)$$

Maxwell's Equations

Introduction to Maxwell's Equation

- In electrodynamics Maxwell's equations are a set of four equations, that describes the behavior of both the electric and magnetic fields as well as their interaction with matter
- Maxwell's four equations express
  - How electric charges produce electric field (Gauss's law)
  - The absence of magnetic monopoles
  - How currents and changing electric fields produces magnetic fields (Ampere's law)
  - How changing magnetic fields produces electric fields (Faraday's law of induction)

Electrodynamics Before Maxwell

Gauss's Law

$$(i)\vec{\nabla}\bullet\vec{E}=\frac{\rho}{\varepsilon_o}$$

No name

$$(ii)\vec{\nabla}\bullet\vec{B}=0$$

Faraday's Law

$$(iii)\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

Ampere's Law

$$(iv)\vec{\nabla}\times\vec{B}=\mu_o\vec{J}$$

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

Electrodynamics Before Maxwell (Cont'd)

Apply divergence to (iii)

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\bullet\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{B}\right)$$

The left hand side is zero, because divergence of a curl is zero.

The right hand side is zero because $\vec{\nabla}\bullet\vec{B}=0.$

Apply divergence to (iv)

$$\vec{\nabla}\bullet\left(\vec{\nabla}\times\vec{B}\right)=\mu_o\left(\vec{\nabla}\bullet\vec{J}\right)$$

Electrodynamics Before Maxwell (Cont'd)

- The left hand side is zero, because divergence of a curl is zero.
- The right hand side is zero for steady currents i.e.,

$$\vec{\nabla} \bullet \vec{J} = 0$$

- In electrodynamics from conservation of charge

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0$$

ρ is constant at any point in space which is wrong.

Maxwell's Correction to Ampere's Law

Consider Gauss's Law

$$\vec{\nabla} \bullet \varepsilon_o \vec{E} = \rho$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \bullet \varepsilon_o \vec{E}\right) = \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \vec{\nabla} \bullet \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial \vec{D}}{\partial t} = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Displacement current

This result along with Ampere's law and the conservation of charge equation suggest that there are actually two sources of magnetic field.
The current density and displacement current.

Maxwell's Correction to Ampere's Law (Cont'd)

Amperes law with Maxwell's correction

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

General Form of Maxwell's Equations

Differential Form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Integral Form

$$\oint_S \vec{E} \bullet d\vec{a} = \frac{1}{\varepsilon_o} \int_V \rho dV$$

$$\oint_S \vec{B} \bullet d\vec{a} = 0$$

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \bullet d\vec{a}$$

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_o I_{enc} + \mu_o \varepsilon_o \frac{d}{dt} \oint_S \vec{E} \bullet d\vec{a}$$

Maxwell's Equations in vacuum

- The vacuum is a linear, homogeneous, isotropic and dispersion less medium
- Since there is no current or electric charge is present in the vacuum, hence Maxwell's equations reads as
- These equations have a simple solution in terms of traveling sinusoidal waves, with the electric and magnetic fields direction orthogonal to each other and to the direction of travel

$$\vec{\nabla} \bullet \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations Inside Matter

Maxwell's equations are modified for polarized and magnetized materials. For linear materials the polarization *P* and magnetization *M* is given by

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$$\vec{M} = \chi_m \vec{H}$$

And the *D* and *B* fields are related to *E* and *H* by

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = \left(1 + \chi_e\right)\varepsilon_o \vec{E} = \varepsilon \vec{E}$$

$$\vec{B} = \mu_o \left(\vec{H} + \vec{M}\right) = \left(1 + \chi_m\right)\mu_o \vec{H} = \mu \vec{H}$$

Where χ_e is the electric susceptibility of material,

χ_m is the magnetic susceptibility of material.

Maxwell's Equations Inside Matter (Cont'd)

- For polarized materials we have bound charges in addition to free charges

$$\sigma_b = \vec{P} \bullet \hat{n}$$

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

- For magnetized materials we have bound currents

$$\overrightarrow{K_b} = \overrightarrow{M} \times \hat{n}$$

$$\overrightarrow{J_b} = \vec{\nabla} \times \overrightarrow{M}$$

Maxwell's Equations Inside Matter (Cont'd)

- In electrodynamics any change in the electric polarization involves a flow of bound charges resulting in polarization current J_P

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

Polarization current density is due to linear motion of charge when the Electric polarization changes

$$\text{Total charge density}$$

$$\rho_t = \rho_f + \rho_b$$

$$\text{Total current density}$$

$$J_t = J_f + J_b + J_p$$

Maxwell's Equations Inside Matter (Cont'd)

- Maxwell's equations inside matter are written as

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho_t}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_f + \mu_o \vec{J}_p + \mu_o \vec{J}_b + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_o} = \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_o} - \vec{M} \right) = \vec{J}_f + \frac{\partial}{\partial t}\left(\varepsilon_o \vec{E} + \vec{P} \right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial}{\partial t} \vec{D}$$

Maxwell's Equations Inside Matter (Cont'd)

- In non-dispersive, isotropic media ε and μ are time-independent scalars, and Maxwell's equations reduces to

$$\vec{\nabla} \bullet \varepsilon \vec{E} = \rho$$

$$\vec{\nabla} \bullet \mu \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Maxwell’s Equations Inside Matter (Cont’d)

- In uniform (homogeneous) medium ε and μ are independent of position, hence Maxwell’s equations reads as

$$\vec{\nabla}\bullet\vec{D}=\rho_f \qquad \oint_S \vec{D}\bullet d\vec{a}=Q_{f\ enc}$$

$$\vec{\nabla}\bullet\vec{H}=0 \qquad \oint_S \vec{H}\bullet d\vec{a}=0$$

$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \qquad \oint_C \vec{E}\bullet d\vec{l}=-\mu\frac{d}{dt}\int_S \vec{H}\bullet d\vec{a}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\varepsilon\frac{\partial\vec{E}}{\partial t} \qquad \oint_C \vec{H}\bullet d\vec{l}=I_{f\ enc}+\frac{d}{dt}\int_S \vec{D}\bullet d\vec{a}$$

Generally, ε and μ can be rank-2 tensor (3X3 matrices) describing bi-refringent anisotropic materials.

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Maxwell's equations in integral form:

Gauss' Law:

$$\int_v \vec{\nabla}\bullet\vec{E}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \rho^E_{Tot}(\vec{r},t)\,d\tau' = \frac{1}{\varepsilon_0}\int_v \left(\rho^E_{free}(\vec{r},t) + \rho^E_{bound}(\vec{r},t)\right)d\tau'$$

$$= \oint_S \vec{E}(\vec{r},t)\bullet d\vec{a} = \frac{1}{\varepsilon_0} Q^{enclosed}_{ToT}(t) = \frac{1}{\varepsilon_0}\left(Q^{enclosed}_{free}(t) + Q^{enclosed}_{bound}(t)\right)$$

$$\oint_S \vec{D}(\vec{r},t)\,d\vec{a} = Q^{enclosed}_{free}(t)$$

$$\oint_S \vec{\mathrm{P}}(\vec{r},t)\bullet d\vec{a} \equiv -Q^{enclosed}_{bound}(t)$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Auxiliary Relation: $\vec{D}(\vec{r},t) = \varepsilon_o \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$

$$\rho_{Bound}(\vec{r},t) \equiv -\vec{\nabla} \bullet \vec{P}(\vec{r},t) \qquad \sigma_{Bound}(\vec{r},t) \equiv \vec{P}(\vec{r},t) \bullet \hat{n}\Big|_{intf}$$

No Magnetic Monopoles: $\int_v \vec{\nabla} \bullet \vec{B}(\vec{r},t)\, d\tau' = \oint_S \vec{B}(\vec{r},t) \bullet d\vec{a} = 0$

Faraday's Law:

$$\int_S \vec{\nabla} \times \vec{E}(\vec{r},t) \bullet d\vec{a} = \oint_C \vec{E}(\vec{r},t) \bullet d\vec{\ell} = -\int_S \frac{\partial \vec{B}(\vec{r},t)}{\partial t} \bullet d\vec{a} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \bullet d\vec{a}\right]$$

$$emf \;\; \varepsilon(t) \equiv \oint_C \vec{E}(\vec{r},t) \bullet d\vec{\ell} = -\frac{d}{dt}\left[\int_S \vec{B}(\vec{r},t) \bullet d\vec{a}\right] = -\frac{d\Phi_M^{enclosed}(t)}{dt}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Ampere's Law:

$$\int_S \vec{\nabla}\times\vec{B}(\vec{r},t)\bullet d\vec{a} = \oint_C \vec{B}(\vec{r},t)\bullet d\vec{\ell} = \mu_0\int_S\left(\vec{J}_{ToT}(\vec{r},t)+\vec{J}_D(\vec{r},t)\right)\bullet d\vec{a}$$

$$= \oint_C \vec{B}(\vec{r},t)\bullet d\vec{\ell} = \mu_0\left(I_{ToT}^{encl}(t)+I_D^{encl}(t)\right) = \mu_0\left(I_{free}^{encl}(t)+I_{bound}^{m\ encl}(t)+I_{P_{bound}}^{encl}(t)+I_D^{encl}(t)\right)$$

Auxiliary Relation:

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{B}(\vec{r},t)-\vec{\mathrm{M}}(\vec{r},t)$$

$$\vec{J}_{bound}^{m}(\vec{r},t) \equiv \vec{\nabla}\times\vec{\mathrm{M}}(\vec{r},t)$$

$$\vec{K}_{bound}^{m}(\vec{r},t) \equiv \vec{\mathrm{M}}(\vec{r},t)\times\hat{n}\Big|_{intf}$$

$$\vec{J}_{P_{bound}}(\vec{r},t) \equiv \frac{\partial\vec{P}(\vec{r},t)}{\partial t}$$

$$\rho_m^{Bound}(\vec{r},t) \equiv -\vec{\nabla}\bullet\vec{\mathrm{M}}(\vec{r},t)$$

$$\sigma_m^{Bound}(\vec{r},t) \equiv \vec{\mathrm{M}}(\vec{r},t)\bullet\hat{n}\Big|_{intf}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

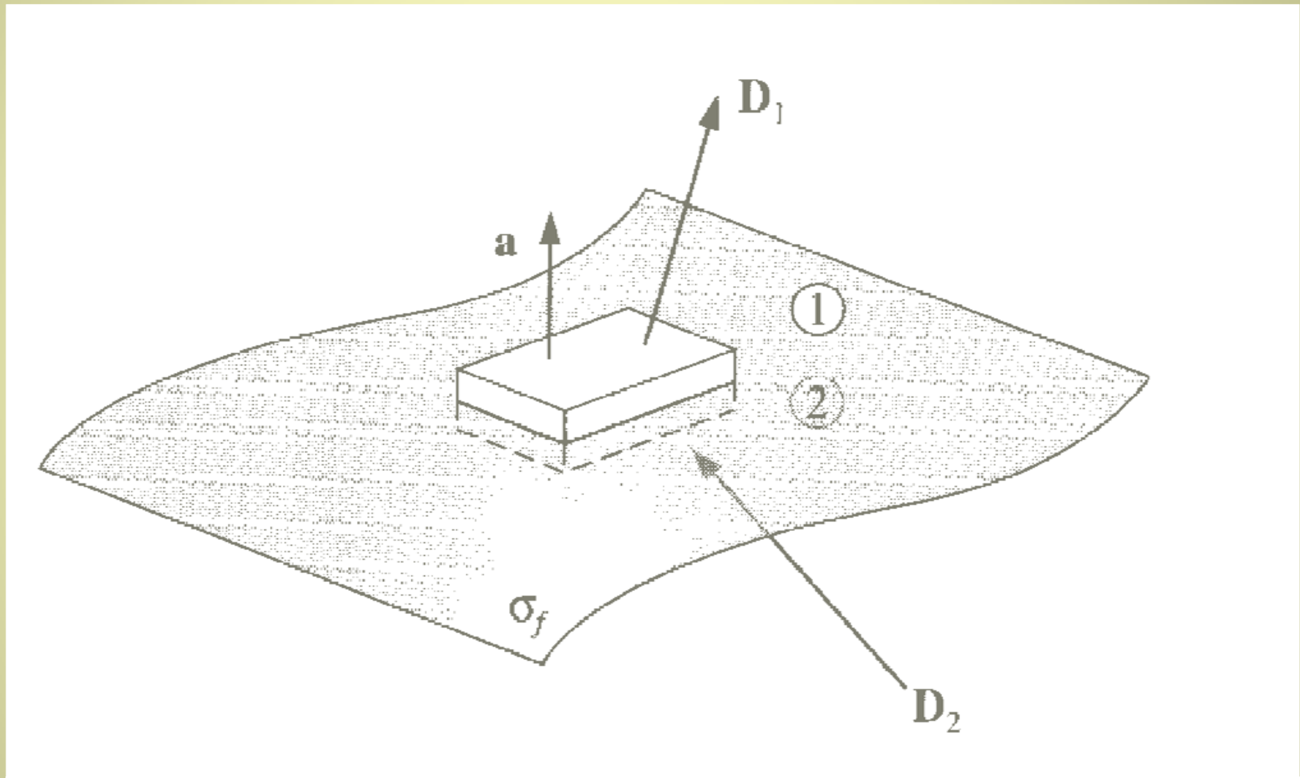
$$\boxed{\int_S \vec{\nabla}\times\vec{H}(\vec{r},t)\bullet d\vec{a} = \oint_C \vec{H}(\vec{r},t)\bullet d\vec{\ell} = I_{free}^{enclosed}(t) + \int_S \frac{\partial\vec{D}(\vec{r},t)}{\partial t}\bullet d\vec{a} = I_{free}^{enclosed}(t) + \frac{d}{dt}\left[\int_S \vec{D}(\vec{r},t)\bullet d\vec{a}\right]}$$

1) Apply the integral form of Gauss' Law at a dielectric interface/boundary using infinitesimally thin Gaussian pillbox extending slightly into dielectric material on either side of interface:

$$\oint_S \vec{E}\bullet d\vec{a} = \frac{1}{\varepsilon_0}Q_{TOT}^{enclosed} = \frac{1}{\varepsilon_0}Q_{free}^{enclosed} + \frac{1}{\varepsilon_0}Q_{bound}^{enclosed} = \frac{1}{\varepsilon_0}\oint_S \sigma_{free}\,da + \frac{1}{\varepsilon_0}\oint_S \sigma_{bound}\,da$$

Gives: $\boxed{\underset{above}{E_2^{\perp}} - \underset{below}{E_1^{\perp}} = \frac{1}{\varepsilon_0}\sigma_{TOT} = \frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)}$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter



Maxwell's Equations and Boundary Conditions at Interfaces in Matter

The positive direction is from medium 2 (below) to medium 1 (above)

$$\oint_S \vec{D}\bullet d\vec{a} = Q_{free}^{enclosed} = \oint_S \sigma_{free} da \quad \Rightarrow \quad \boxed{D^{\perp}_{\underset{above}{2}} - D^{\perp}_{\underset{below}{1}} = \sigma_{free}}$$ (at interface)

Likewise: $$\oint_S \vec{P}\bullet d\vec{a} = Q_{bound}^{enclosed} = -\oint_S \sigma_{bound} da \Rightarrow \boxed{P^{\perp}_{\underset{above}{2}} - P^{\perp}_{\underset{below}{1}} = \sigma_{bound}}$$ (at interface)

$$\vec{E} \equiv -\vec{\nabla} V$$

Since: $$\boxed{\left.\left(\frac{\partial V_2^{above}}{\partial n} - \frac{\partial V_1^{below}}{\partial n}\right)\right|_{interface} = -\frac{1}{\varepsilon_0}\sigma_{ToT} = -\frac{1}{\varepsilon_0}\left(\sigma_{free} + \sigma_{bound}\right)}$$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{D} = \varepsilon\vec{E} = -\varepsilon\vec{\nabla}V$$

Since: $$\left.\left(\varepsilon_2 \frac{\partial V_2^{above}}{\partial n} - \varepsilon_1 \frac{\partial V_1^{below}}{\partial n}\right)\right|_{interface} = -\sigma_{free}$$ (at interface)

Similarly, for $\int_v \vec{\nabla}\bullet\vec{B}\,d\tau' = \oint_S \vec{B}\bullet d\vec{a} = 0$ (no magnetic monopoles), then at an interface:

$$\vec{B}_2^{above}\bullet\vec{a} - \vec{B}_1^{above}\bullet\vec{a} = 0 \Rightarrow \boxed{B^{\perp}_{2\,above} - B^{\perp}_{1\,below} = 0} \text{ or: } \boxed{B^{\perp}_{2\,above} = B^{\perp}_{1\,below}}$$ (at interface)

Since: $$\vec{H} = \left(\frac{1}{\mu_o}\right)\vec{B} - \vec{M}$$ Then: $$\vec{B} = \mu_o\left(\vec{H} + \vec{M}\right)$$

$$\oint_S \vec{B}\bullet d\vec{a} = \mu_o \oint_S \left(\vec{H} + \vec{M}\right)\bullet d\vec{a} = 0$$ or: $$\oint_S \vec{H}\bullet d\vec{a} = -\oint_S \vec{M}\bullet d\vec{a}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Then: $\vec{H}_2^{above} \bullet \vec{a} - \vec{H}_1^{below} \bullet \vec{a} = -\left(\vec{M}_2^{above} \bullet \vec{a} - \vec{M}_1^{below} \bullet \vec{a} \right)$ (at interface)

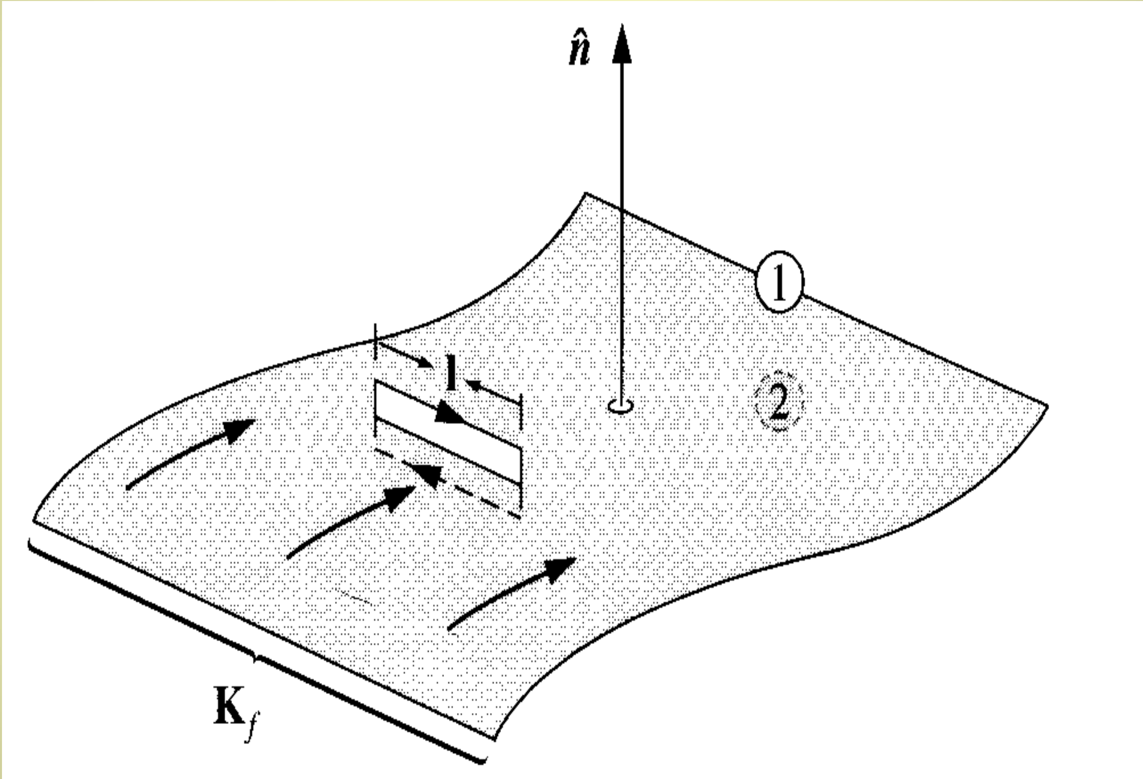
Or: $$\left(H_{2\,above}^{\perp} - H_{1\,below}^{\perp} \right) = -\left(M_{2\,above}^{\perp} - M_{1\,below}^{\perp} \right) = -\sigma_{magnetic}^{bound}$$ (at interface)

Effective bound magnetic charge at interface

3) For Faraday's Law: EMF, $\varepsilon = \oint_C \vec{E} \bullet d\vec{\ell} = -\frac{d}{dt}\left(\oint_S \vec{B} \bullet d\vec{a} \right) = -\frac{d\Phi_m}{dt}$

At interface between two different media, taking a closed contour C of width l extending slightly into the material on either side of interface.

Maxwell's Equations and Boundary Conditions at Interfaces in Matter



Maxwell's Equations and Boundary Conditions at Interfaces in Matter

$$\vec{E}_2^{above}\cdot\vec{\ell} - \vec{E}_1^{below}\cdot\vec{\ell} = -\frac{d}{dt}\oint_S \vec{B}\cdot d\vec{a} = 0$$

(in limit area of contour loop → 0, magnetic flux enclosed → 0)

$$\boxed{\underset{above}{E_2^{\parallel}} - \underset{below}{E_1^{\parallel}} = 0}$$ (at interface) or: $$\boxed{\underset{above}{E_2^{\parallel}} = \underset{below}{E_1^{\parallel}}}$$ (at interface)

Since: $\vec{D} = \varepsilon_o\vec{E} + \vec{P}$ And: $\varepsilon_o\vec{E} = \vec{D} - \vec{P}$

Thus: $$\left(\vec{E}_2^{above}\cdot\vec{\ell} - \vec{E}_1^{below}\cdot\vec{\ell}\right) = \left(\vec{D}_2^{above}\cdot\vec{\ell} - \vec{D}_1^{below}\cdot\vec{\ell}\right) - \left(\vec{P}_2^{above}\cdot\vec{\ell} - \vec{P}_1^{below}\cdot\vec{\ell}\right) = 0$$

In limit area of contour loop → 0 magnetic flux enclosed → 0

$$\Rightarrow \boxed{\left(\underset{above}{\vec{D}_2^{\parallel}} - \underset{below}{\vec{D}_1^{\parallel}}\right) = \left(\underset{above}{\vec{P}_2^{\parallel}} - \underset{below}{\vec{P}_1^{\parallel}}\right)}$$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

4) Finally, for Ampere's Law: $\oint_C \vec{B} \bullet d\vec{\ell} = \mu_o \left(I_{TOT}^{encl} + I_D^{encl} \right)$

$$\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell} = \mu_o I_{TOT}^{encl} + \mu_o I_D^{encl}$$

$$I_D^{encl} = \int_S \vec{J}_D \bullet d\vec{a} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a}$$

$$I_{TOT}^{encl} = I_{free}^{encl} + I_{bound}^{encl} + I_{P_{bound}}^{encl}$$

$$I_{P_{bound}}^{encl} = \int_S \vec{J}_{P_{bound}} \bullet d\vec{a} = \int_S \frac{\partial \vec{P}}{\partial t} \bullet d\vec{a}$$

$$I_{bound}^{encl} = \int_S \vec{J}_m^{bound} \bullet d\vec{a} = \int_S \vec{\nabla} \times \vec{M} \bullet d\vec{a}$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Where I_{TOT}^{encl} = TOTAL current (free + bound + polarization) passing through enclosing Amperian loop contour C

No volume current density $\vec{J}_{TOT}, \vec{J}_{free}, \vec{J}^{m}_{bound}$ or $\vec{J}_P$ contributes to I_{TOT}^{encl} in the limit area of contour loop → 0, however a surface current $\vec{K}_{TOT}, \vec{K}_{free}, \vec{K}^{m}_{bound} = \vec{M} \times \hat{n}$ <u>can</u> contribute!

In the limit that the enclosing Amperian loop contour C shrinks to zero height above/below interface- the enclosed area of loop contour → 0,

Then: $$I_D^{encl} = \varepsilon_o \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \varepsilon_o \frac{d}{dt}\left[\int_S \vec{E} \cdot d\vec{a}\right] = \varepsilon \frac{d\Phi_E}{dt} \to 0$$

$$(\Phi_E \equiv \int_S \vec{E} \cdot d\vec{a} \; = \text{enclosed flux of electric field lines})$$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

Similarly:

$$I_{P_{bound}}^{encl} = \int_S \frac{\partial \vec{P}}{\partial t} \bullet d\vec{a} = \frac{d}{dt}\left[\int_S \vec{P} \bullet d\vec{a}\right] = \frac{d\Phi_P}{dt} \to 0$$

$$\left(\Phi_P \equiv \int_S \vec{P} \bullet d\vec{a} = \text{enclosed flux of electric polarization field lines}\right)$$

If $\hat{n}$ *is unit normal/perpendicular to interface*, note that $\left(\hat{n} \times \vec{\ell}\right)$ is normal/perpendicular to plane of the Amperian loop contour.

$$I_{TOT}^{encl} = \vec{K}_{TOT} \bullet \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{TOT} \times \hat{n}\right) \bullet \vec{\ell}$$

$$I_{free}^{encl} = \vec{K}_{free} \bullet \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{free} \times \hat{n}\right) \bullet \vec{\ell}$$

$$I_{bound}^{encl} = \vec{K}_{bound} \bullet \left(\hat{n} \times \vec{\ell}\right) = \left(\vec{K}_{bound}^{m} \times \hat{n}\right) \bullet \vec{\ell}$$

Using: $\vec{A} \bullet \left(\vec{B} \times \vec{C}\right) = \vec{B} \bullet \left(\vec{C} \times \vec{A}\right)$

$= \vec{C} \bullet \left(\vec{A} \times \vec{B}\right)$

$\left\{= \left(\vec{A} \times \vec{B}\right) \bullet \vec{C}\right\}$

$I_{TOT} = I_{free} + I_{bound}$ $\qquad$ $\vec{K}_{TOT} = \vec{K}_{free} + \vec{K}_{bound}$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

In the limit that the enclosing Amperian loop contour C (of width l) shrinks to zero height above/below interface, causing area of enclosed loop contour → 0, then:

$$\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell} = \mu_o I_{TOT}^{encl} + \underbrace{\cancel{\mu_o I_D^{encl}}}_{=0} = \mu_o I_{TOT}^{encl} = \left(\vec{K}_{TOT} \times \hat{n}\right) \bullet \vec{\ell}$$

$$\boxed{\underset{above}{B_2^{\parallel}} - \underset{below}{B_1^{\parallel}} = \mu_o \vec{K}_{TOT} \times \hat{n} = \mu_o \left(\vec{K}_{free} + \vec{K}_{bound}^{m}\right) \times \hat{n}} \quad \text{(at interface)}$$

Since: $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$ and: $\frac{1}{\mu_o}\vec{B} = \vec{H} + \vec{M}$ then:

$$\boxed{\frac{1}{\mu_o}\left(\vec{B}_2^{above} \bullet \vec{\ell} - \vec{B}_1^{below} \bullet \vec{\ell}\right) = \left(\vec{H}_2^{above} \bullet \vec{\ell} - \vec{H}_1^{below} \bullet \vec{\ell}\right) + \left(\vec{M}_2^{above} \bullet \vec{\ell} - \vec{M}_1^{below} \bullet \vec{\ell}\right) = \left[\left(\vec{K}_{free} \times \hat{n}\right) + \left(\vec{K}_{bound} \times \hat{n}\right)\right]}$$

(at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

We also see that: $\boxed{\underset{above}{H_2^{\parallel}} - \underset{below}{H_1^{\parallel}} = \vec{K}_{free} \times \hat{n}}$ (at interface)

and: $\boxed{\underset{above}{M_2^{\parallel}} - \underset{below}{M_1^{\parallel}} = \vec{K}^{m}_{bound} \times \hat{n}}$ (at interface)

∥- components of ***B*** are discontinuous at interface by $\mu_o \vec{K}_{TOT} \times \hat{n}$

∥- components of ***H*** are discontinuous at interface by $\vec{K}_{free} \times \hat{n}$

∥- components of **M** are discontinuous at interface by $\vec{K}^{m}_{bound} \times \hat{n}$

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

If $\vec{B} = \vec{\nabla} \times \vec{A}$

where ***A*** is the magnetic vector potential, then:

$$\left(\frac{1}{\mu_0}\right)\left[\underset{above}{B_2^{\parallel}} - \underset{below}{B_1^{\parallel}}\right] = \vec{K}_{TOT} \times \hat{n}$$ (at interface) is equivalent to:

$$\left(\frac{1}{\mu_o}\right)\left(\frac{\partial \vec{A}_2^{above}}{\partial n} - \frac{\partial \vec{A}_1^{below}}{\partial n}\right)\Bigg|_{interface} = -\vec{K}_{TOT}$$ (at interface)

Maxwell's Equations and Boundary Conditions at Interfaces in Matter

For linear magnetic media:

$$\vec{B} = \mu \vec{H} \quad \text{or:} \quad \vec{H} = \frac{1}{\mu}\vec{B}$$

$$\boxed{\left[\underset{above}{H_2^{\parallel}} - \underset{below}{H_1^{\parallel}} \right] = \vec{K}_{free} \times \hat{n}}$$

(at interface) is equivalent to:

$$\boxed{\left(\frac{1}{\underset{above}{\mu_2}}\right) \frac{\partial \vec{A}_2^{above}}{\partial n}\bigg|_{\text{interface}} - \left(\frac{1}{\underset{below}{\mu_1}}\right) \frac{\partial \vec{A}_1^{below}}{\partial n}\bigg|_{\text{interface}} = -\vec{K}_{free}}$$

(at interface)

Potential Formulation of Electrodynamics 1

- In electrostatic

$$\vec{\nabla} \times \vec{E} = 0$$

$$\Rightarrow \vec{E} = -\vec{\nabla} V$$

In electrodynamics

$$\vec{\nabla} \times \vec{E} \neq 0$$

But

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Putting this in Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right)$$

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = 0$$

$$\Rightarrow \left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = -\nabla V$$

$$\Rightarrow \vec{E} = -\vec{\nabla} V - \frac{\partial}{\partial t}\vec{A}$$

Potential Formulation of Electrodynamics 2

$$\vec{\nabla}.\vec{B} = \vec{\nabla}.\left(\vec{\nabla} \times \vec{A}\right) = 0$$

and from

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla} V - \frac{\partial}{\partial t} \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{E} = 0 - \frac{\partial}{\partial t} \vec{B}$$

Explain Maxwell's ii and iii equations

Potential Formulation of Electrodynamics 3

Now consider Maxwell's i and iv equations

As

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_o}$$

Gauss's Law

$$\vec{\nabla}\cdot\left(\vec{\nabla}V+\frac{\partial\vec{A}}{\partial t}\right)=-\frac{\rho}{\varepsilon_o}$$

$$\nabla^2 V+\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}=-\frac{\rho}{\varepsilon_o}$$

This replaces Poisson's Equation in electrodynamics

Potential Formulation of Electrodynamics 4

Now consider

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Putting values of E and B we get

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \mu_o \vec{J} - \mu_o \varepsilon_o \vec{\nabla} \left(\frac{\partial V}{\partial t} \right) - \mu_o \varepsilon_o \left(\frac{\partial^2 \vec{A}}{\partial t^2} \right)$$

Potential Formulation of Electrodynamics 5

Using vector identity

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot A\right)-\nabla^2 A$$

Re-arranging

$$\left(\nabla^2\vec{A}-\mu_o\varepsilon_o\left(\frac{\partial^2\vec{A}}{\partial t^2}\right)\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_o\varepsilon_o\left(\frac{\partial V}{\partial t}\right)\right)=-\mu_o\vec{J}$$

These equation carry all information in Maxwell's equations

Potential Formulation of Electrodynamics 6

$$\nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\frac{\rho}{\varepsilon_o}$$

$$\left(\nabla^2\vec{A}-\mu_o\varepsilon_o\left(\frac{\partial^2\vec{A}}{\partial t^2}\right)\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_o\varepsilon_o\left(\frac{\partial V}{\partial t}\right)\right)=-\mu_o\vec{J}$$

Four Maxwell's equations reduced to two equations using potential formulation. Potentials V and A are not uniquely defined by above equations.

Gauge Transformations

- Two sets of potentials, (V, A) and (V',A')- corresponds to same electric and Magnetic fields.
- Write;

A'=A+α and V'=V+β- as A's give same B

→ curl of α=0, which implies α= grad. of λ. As the two potentials also give same E, then from

$$\vec{E} = -\vec{\nabla} V' - \frac{\partial}{\partial t} A'$$

$$\vec{E} = -\vec{\nabla} V - \vec{\nabla} \beta - \frac{\partial}{\partial t} A - \frac{\partial}{\partial t} \alpha$$

Gauge Transformations 1

$$\Rightarrow \nabla \beta + \frac{\partial}{\partial t}\alpha = 0$$

Putting value of α we get

$$\nabla\left(\beta + \frac{\partial}{\partial t}\lambda\right) = 0$$

The term in paratheses is independent of position

$$\Rightarrow \beta = -\frac{\partial}{\partial t}\lambda$$

Using this we get

$$A' = A + \nabla\lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

Such changes in V and A are called Gauge Transformations

Coulomb's and Lorentz Gauges

Coulomb Gauge $\nabla \bullet A = 0$

Using this we get $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$

It is Poisson's equation, setting V=0, we get

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r} d\tau$$

Scalar potential is easy to calculate in Coulomb's gauge but vector potential is difficult to calculate

Coulomb's Gauge

The differential equations for V and A in Coulombs gauge reads

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o}$$

$$\left(\nabla^2 \vec{A} - \mu_o \varepsilon_o \left(\frac{\partial^2 \vec{A}}{\partial t^2}\right)\right) = -\mu_o \vec{J} + \mu_o \varepsilon_o \nabla \left(\frac{\partial V}{\partial t}\right)$$

Lorentz Gauge

The Lorentz gauge:

$$\nabla \cdot \vec{A} = -\mu_o \varepsilon_o \left(\frac{\partial V}{\partial t} \right)$$

This is design to eliminate the middle term in eqn. for A

$$\left(\nabla^2 \vec{A} - \mu_o \varepsilon_o \left(\frac{\partial^2 \vec{A}}{\partial t^2} \right) \right) = -\mu_o \vec{J}$$

and equation for V will become

$$\nabla^2 V - \mu_o \varepsilon_o \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$$

Lorentz Gauge

The Lorentz gauge treats V and A on equal footing. The same differential operator

$$\nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} = \Box^2$$

called the d'Alembertian

$$\Box^2 A = -\mu_0 j$$

and

$$\Box^2 V = -\frac{1}{\varepsilon_0}\rho$$

The Electromagnetic Waves

Electromagnetic Wave Equation

The electromagnetic wave equation is a second-order partial differential equation that describes the propagation of electromagnetic waves through a medium or in a vacuum.

To obtain the electromagnetic wave equation in a vacuum we begin with the modern 'Heaviside' form of Maxwell's equations.

From Maxwell's Equations to the Electromagnetic Waves 1

The Wave Equation

Maxwell's equation in free space – no charge or no current are given as

$$\vec{\nabla} \bullet \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

From Maxwell's Equations to the Electromagnetic Waves 2

Take curl of

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} \times [-\frac{\partial \vec{B}}{\partial t}]$$

Change the order of differentiation on the R.H.S

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

From Maxwell's Equations to the Electromagnetic Waves 3

As

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Substituting for $\vec{\nabla} \times \vec{B}$ **we have**

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}]$$

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

- As μ_o and ε_o are constant in time

From Maxwell's Equations to the Electromagnetic Waves 4

Using the vector identity

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E}$$

gives,

$$\vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E} = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

In free space

$$\vec{\nabla} \bullet \vec{E} = 0$$

And we are left with the wave equation

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

From Maxwell’s Equations to the Electromagnetic Waves 5

Similarly the wave equation for magnetic field

$$\nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

where,

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$$

Electromagnetic Wave Equation in Vacuum

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

The solutions to the wave equations, when there is no source charge present can be plane waves - obtained by method of separation of variables

Solution of Electromagnetic Wave

- Plane electromagnetic waves can be expressed as

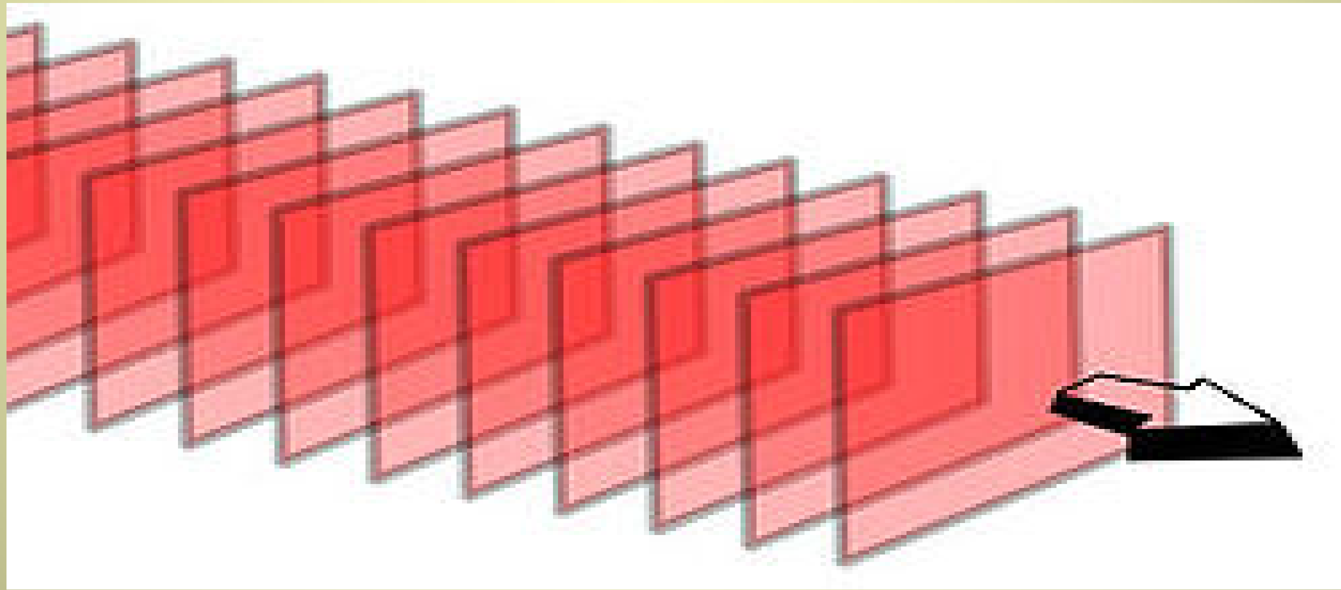
$$\vec{E} = E_o e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)} \hat{n}$$

$$\vec{B} = \frac{1}{c} E_o e^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)} \left(\hat{k} \times \hat{n}\right) = \frac{1}{c}\left(\hat{k} \times \vec{E}\right)$$

Where $\hat{n}$ is the polarization vector and. $\hat{k}$ is a propagation vector.

Electromagnetic Plane waves

- **Plane wave -** a constant-frequency wave whose wave-fronts (surfaces of constant phase) are infinite parallel planes of constant amplitude normal to the direction of propagation



Real Electromagnetic Plane waves

The real electric and magnetic fields in the form of a mono-chromatic plane wave with propagation vector $\hat{k}$ and polarization $\hat{n}$

$$\vec{E}\left(\vec{r},t\right) = E_o \cos(\vec{k}\cdot\vec{r} - \omega t)\hat{n}$$

$$\vec{B}\left(\vec{r},t\right) = \frac{1}{c} E_o \cos(\vec{k}\cdot\vec{r} - \omega t)\left(\vec{k}\times\hat{n}\right)$$

Homogenous Wave Equations Inside Matter

The homogeneous form of the equation - written in terms of either the electric field **E** or the magnetic field **B** - takes the form:

Vacuum

$$\frac{1}{\mu_o \varepsilon_o} \nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{1}{\mu_o \varepsilon_o} \nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}$$

Matter

$$\frac{1}{\mu \varepsilon} \nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{1}{\mu \varepsilon} \nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}$$

Homogenous Wave Equations Inside Matter 1

Permittivity: $\varepsilon=\varepsilon_r\varepsilon_o$ (ε_r is dielectric constant)
Permeability: $\mu=\mu_r\mu_o$ (μ_r is relative permeability ≈1

$$v=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mu_r\mu_0\varepsilon_r\varepsilon_0}}=\frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{1}{\sqrt{\mu_r\varepsilon_r}}$$

=c

= n

$$v=\frac{c}{n}$$

n=Refractive Index

Energy and Momentum of Electromagnetic Waves

The energy per unit volume stored in electromagnetic field is

$$U = \frac{1}{2}\left(\varepsilon_o E^2 + \frac{1}{\mu_o} B^2\right)$$

In the case of monochromatic plane wave

$$B^2 = \frac{1}{c^2} E^2 = \mu_o \varepsilon_o E^2$$

$$\Rightarrow U = \varepsilon_o E^2 = \varepsilon_o E_o^{\ 2} \cos^2(kx - \omega t)$$

Energy and Momentum of Electromagnetic Waves 1

As the wave propagates, it carries this energy along with it. The energy flux density (energy per unit area per unit time) transported by the field is given by the poynting vector

$$\vec{S} = \frac{1}{\mu_o}\left(\vec{E} \times \vec{B}\right)$$

For monochromatic plane waves

$$\vec{S} = c\varepsilon_o E_o{}^2 \cos^2\left(kx - \omega t\right)\hat{i} = cU\hat{i}$$

Polarization of Electromagnetic Waves

- Polarization of electromagnetic waves is very complex- consider the optical (light) part of EM waves.

- Historically, the orientation of a polarized electromagnetic wave has been defined in the optical regime by the orientation of the electric field vector.

- Natural light is generally un-polarized- all planes of propagation being equally probable.

- Light is a transverse electromagnetic wave.

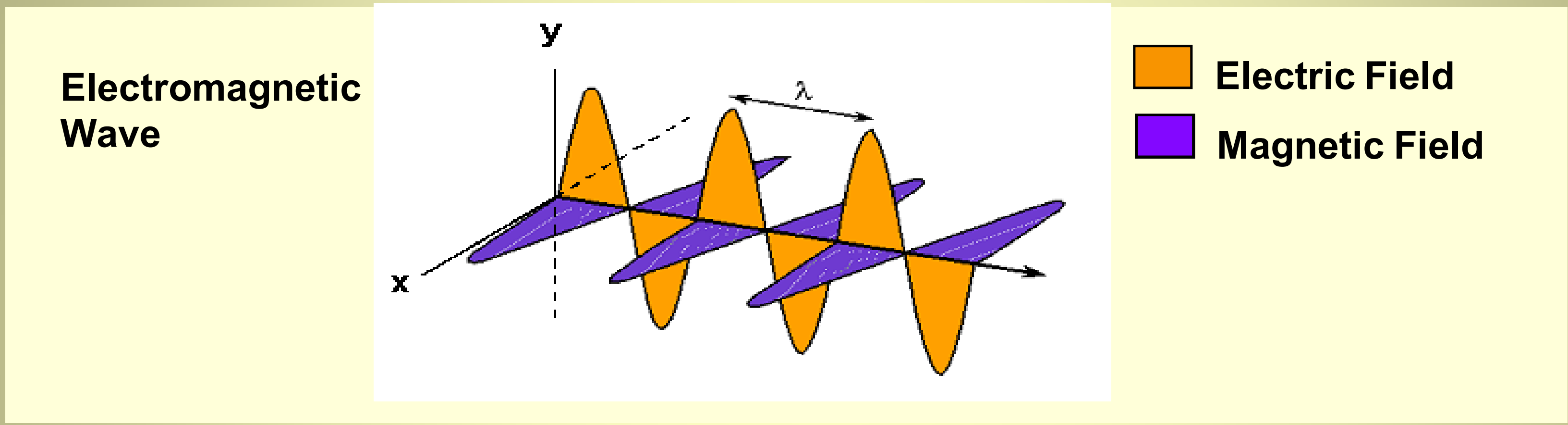
Linear Polarization

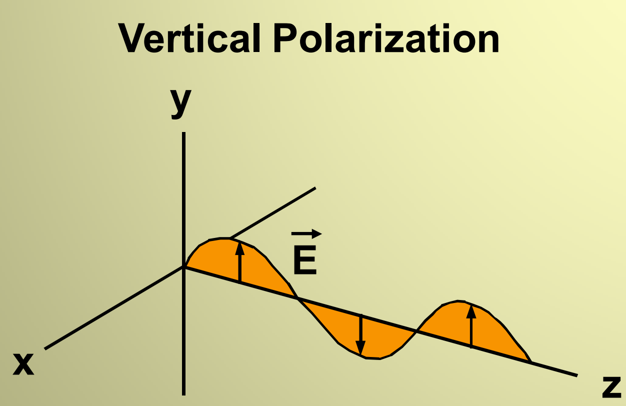
- In electrodynamics, linear polarization or plane polarization of electromagnetic radiation is a confinement of the electric field vector to a given plane along the direction of propagation.

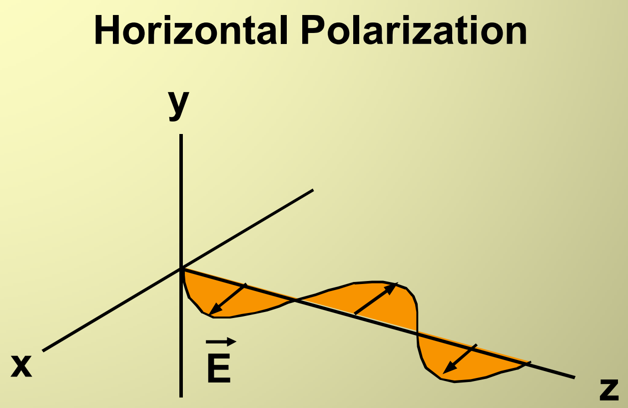
- The plane containing the electric field is called the plane of polarization.

Linear Polarization

- Linear polarization can be horizontal or vertical



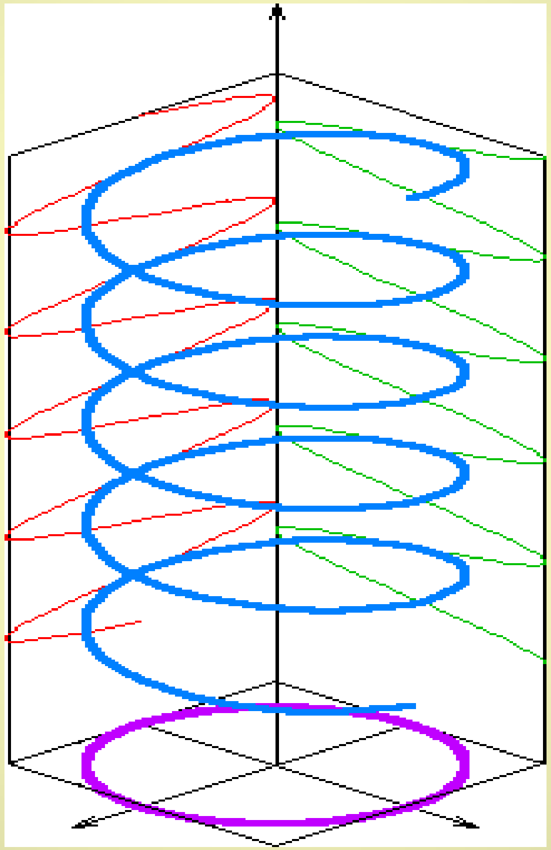




Circular Polarization

- A polarization in which the tip of the electric field vector- at a fixed point in space - describes a circle as time progresses.
- The electric vector - at one point in time - describes a helix along the direction of wave propagation.
- The magnitude of the electric field vector is constant as it rotates.
- Circular polarization is a limiting case of the more general condition of elliptical polarization.

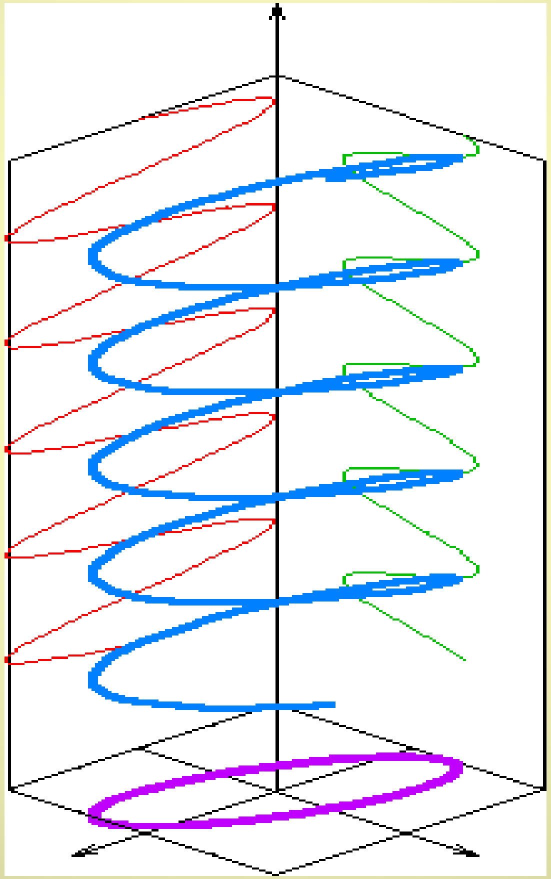
Circular Polarization



Elliptical Polarization

- **Elliptical polarization -** is the polarization of electromagnetic radiation such that the tip of the electric field vector describes an ellipse in any fixed plane intersecting - and normal to - the direction of propagation.
- An elliptically polarized wave may be resolved into two linearly polarized waves in phase quadrature- with their polarization planes at right angles to each other.

Elliptical Polarization



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THANK YOU