The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2442-3

Preparatory School to the Winter College on Optics

28 January - 1 February, 2013

Resonators and Gaussian optics

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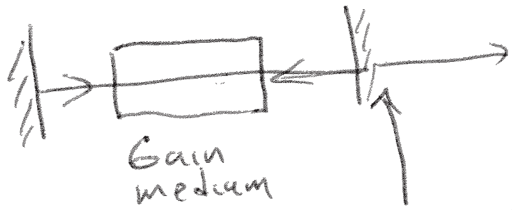
Resonator cavities

In a laser, light is forced to pass many times through a gain medium by placing reflecting surfaces at both sides

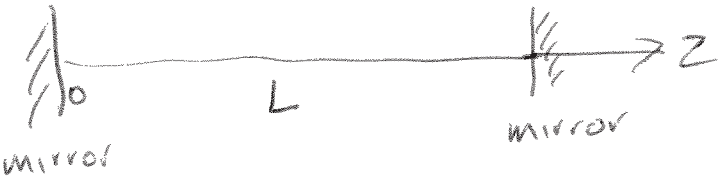


One of these surfaces has a small transmissivity which lets some light come out. This is the output of the laser.

In this lecture, we will study the effects of the cavity formed by the reflectors. We will assume that the reflectors are mirrors. (If they are not mirrors but some other type of reflector, some of the details of the results presented here will change slightly, but qualitatively the behavior will be similar.)

We will also ignore the presence of the gain medium for simplicity. (If this medium is well index-matched to the interior of the cavity, this is a good approximation.)

Let us start with a 1D treatment.



The field inside the cavity is composed of a forward-propagating component:

$$U^f \propto e^{ikz}$$

and a backward propagating component

$$U^b \propto e^{-ikz}$$

so the total field is

$$U = a e^{ikz} + b e^{-ikz}$$

$$= (a+b)\cos(kz) + i(a-b)\sin(kz)$$

Note that at the mirrors, $z=0$ & $z=L$, U must vanish:

$$U(0) = a+b = 0 \Rightarrow b = -a$$

$$U(L) = 2ia\sin(kL)$$

so the condition of vanishing at both mirrors is only satisfied if

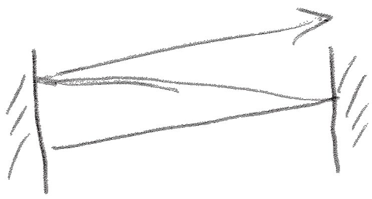
$$kL = m\pi$$
m: integer

Using $k = \frac{\omega}{c}$, the allowed frequencies are found to be

$\boxed{\omega_m = \frac{m\pi c}{L}}$ and the allowed wavelengths $\boxed{\lambda_m = \frac{2L}{m}}$

If the cavity is filled with a medium with index n, then we must replace L with Ln.

Now let us consider the effect of the transverse directions. It is easy to see that two parallel mirrors are no longer a stable cavity since rays that are not exactly normal to them would escape.



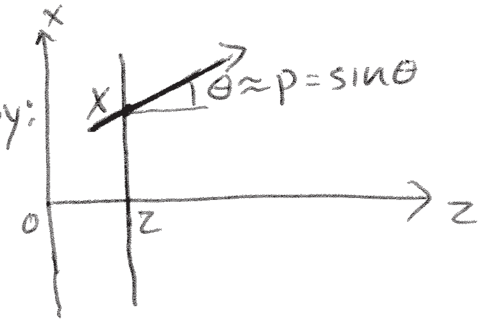
Even the rays that are exactly normal would escape under the smallest misalignment of the mirrors. It is therefore convenient to curve the mirrors:

But how much? And how does this affect the laser beam?

To study this, let us give a brief summary of paraxial ray optics.

Paraxial ray optics and ABCD systems

For simplicity let us consider the two-dimensional case $\vec{r}=(x,z)$, with rays traveling at small angles with respect to the z axis.



At a given z, the ray is identified by:

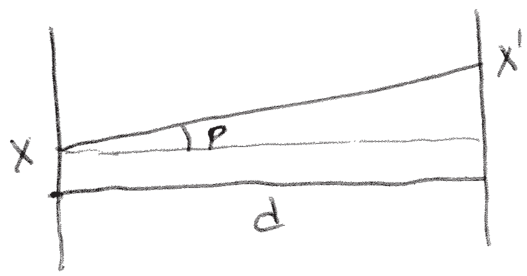
- its transverse position x
- its direction parameter
  $p = \sin\theta \approx \theta \approx \tan\theta$

[If the medium is not free space but a medium with index n, then we define $p = n\sin\theta$.]

We define the ray state vector as

$$\underline{V} = \begin{pmatrix} x \\ p \end{pmatrix}$$

How does $\underline{V}$ change upon propagation in free space



Note that the slope does not change, so

$$p' = p$$

but the position does change

$$x' = x + pd$$

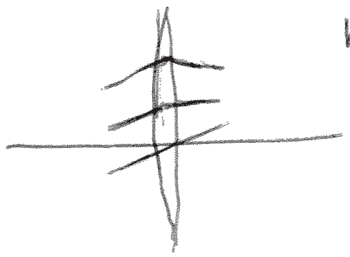
we can write this as

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \quad \text{or} \quad \underline{V}' = \mathbb{I}_d \underline{V}, \text{ where}$$

$$\boxed{\mathbb{I}_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}}$$

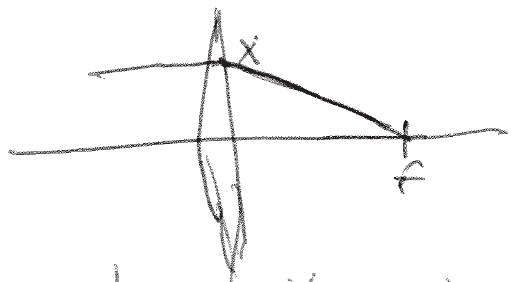
(If the medium has refractive index n, then we replace d with d/n).

How does $\underline{V}$ change when crossing a thin lens?



note that $x' = x$, but the direction changes proportionally to x.

In particular, if $p = 0$, $p' = \frac{-x}{f}$, where f is the focal distance:

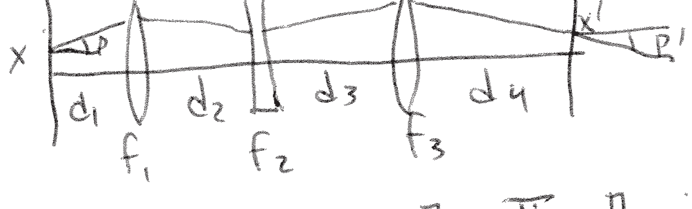


so $x' = x$, $p' = p - \frac{x}{f}$, which can be written as

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \text{ or } \underline{V}' = \mathbb{L}_f \underline{V}, \text{ where}$$

$$\mathbb{L}_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

If we have a system:



then $$\underline{V}' = \underbrace{\mathbb{T}_{d_4} \mathbb{L}_{f_3} \mathbb{T}_{d_3} \mathbb{L}_{f_2} \mathbb{T}_{d_2} \mathbb{L}_{f_1} \mathbb{T}_{d_1}}_{\mathbb{S} = \text{matrix for the whole system}} \underline{V} = \mathbb{S} \underline{V}$$

Note that $\mathrm{Det}[\mathbb{T}_d]=1$, $\mathrm{Det}[\mathbb{L}_f]=1$.

Since the Determinant of a product of matrices the product of the determinants:

$$\underline{\mathrm{Det}[\mathbb{S}]=1}$$

We call $\mathbb{S}$ the ABCD matrix for the system, since in general we write it as

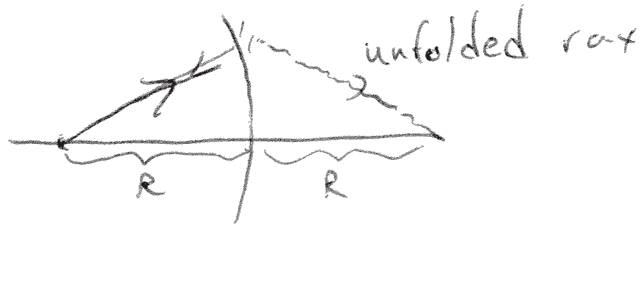
$$\mathbb{S}=\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with

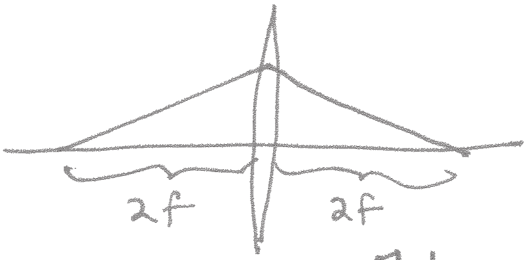
$$\underline{AD-BC=1}$$

For the study of cavities we are interested in curved mirrors. Let R be the radius of curvature of the mirror, defined as positive for convex mirrors and negative for convex mirrors. A curved mirror has a similar effect as a lens, if we "unfold" the system

In particular, if the ray comes from the center of curvature, then it returns on itself:



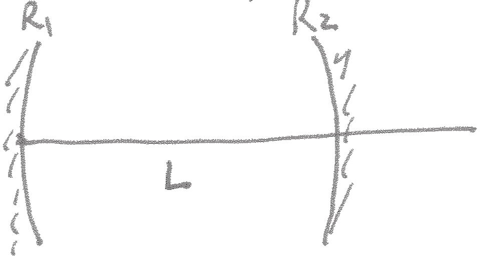
This unfolded picture looks like that of a lens for a ray coming from the axis at $2f$ away from the lens



so the focal distance of a mirror is $f = \frac{R}{2}$ and its matrix is:

$$\mathbb{M}_R = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$$

A round trip inside the cavity is then:



$$\mathbb{S} = \mathbb{M}_{R_1} \mathbb{T}_L \mathbb{M}_{R_2} \mathbb{T}_L$$

calculate

$$\mathbb{M}_R \mathbb{T}_L = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L \\ -\frac{2}{R} & 1 - \frac{2L}{R} \end{pmatrix}$$

so $$\mathbb{S} = \begin{pmatrix} 1 & L \\ -\frac{2}{R_1} & 1 - \frac{2L}{R_1} \end{pmatrix} \begin{pmatrix} 1 & L \\ -\frac{2}{R_2} & 1 - \frac{2L}{R_2} \end{pmatrix} = \begin{pmatrix} 1 - \frac{2L}{R_2}, & 2L - \frac{2L^2}{R_2} \\ -\frac{2}{R_1} - \frac{2}{R_2} + \frac{4L}{R_1 R_2}, & 1 - \frac{4L}{R_1} - \frac{2L}{R_2} + \frac{4L^2}{R_1 R_2} \end{pmatrix}$$

so $$A = 1 - \frac{2L}{R_2}$$

$$B = 2L\left(1 - \frac{L}{R_2}\right)$$

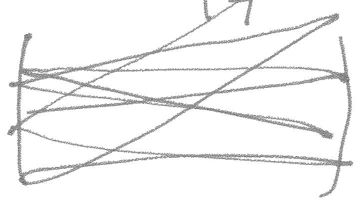
$$C = \frac{4L}{R_1 R_2} - \frac{2}{R_1} - \frac{2}{R_2}$$

$$D = 1 - \frac{4L}{R_1} - \frac{2L}{R_2} + \frac{4L^2}{R_1 R_2}.$$

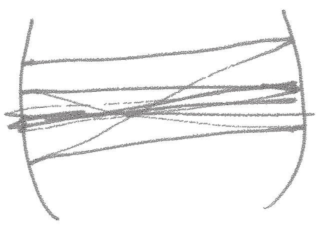
To verify the stability of the matrix, consider finding "eigenrays" such that

$$\mathbb{S}\,\underline{v} = \lambda\,\underline{v}$$

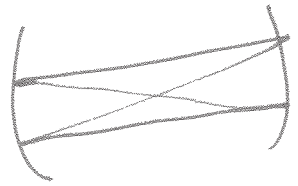
I.e. if the ray initially was $\begin{pmatrix} x \\ p \end{pmatrix}$ then, after a round trip, is $\begin{pmatrix} \lambda x \\ \lambda p \end{pmatrix}$, and after N round trips it is $\begin{pmatrix} \lambda^N x \\ \lambda^N p \end{pmatrix}$. If $\lambda > 1$ then it is clear that the cavity is unstable because x & p keep growing:



If $\lambda < 1$, then the ray "damps down", but because ray optics is reversible, this also means that the cavity is unstable



If $\lambda = 1$, then the ray is periodic



If λ is complex this means that there are no eigenrays for a single bounce. However, λ^m can be real so there are eigenrays for m bounces. If $\lambda^m = 1$, the cavity is stable. In summary, the cavity is stable if

$$|\lambda| = 1.$$

Let us find λ:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} = \lambda \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

or

$$\begin{pmatrix} A-\lambda & B \\ C & D-\lambda \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} = 0$$

For this to be true, we need

$$\operatorname{Det}\begin{pmatrix} A-\lambda & B \\ C & D-\lambda \end{pmatrix} = (A-\lambda)(D-\lambda) - BC = 0$$

$$\lambda^2 - \lambda(A+D) + \underbrace{AD - BC}_{1} = 0 \quad \therefore \quad \lambda_{\frac{1}{2}} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1}$$

$\lambda_{\frac{1}{2}}$ are real and $\neq 1$ if the squareroot is real & $\neq 0$. That is

The cavity is unstable if $\left(\frac{A+D}{2}\right)^2 > 1$

$|\lambda_{\frac{1}{2}}| = 1$ if the squareroot is 0 or imaginary, that is, the cavity is stable if

$$\left(\frac{A+D}{2}\right)^2 \leq 1$$

Therefore, the condition for stability is:

$$-1 \leq 1 - \frac{2L}{R_1} - \frac{2L}{R_2} + \frac{2L^2}{R_1 R_2} \leq 1$$

or

$$0 \leq 2\left(1 - \frac{L}{R_1} - \frac{L}{R_2} + \frac{L^2}{R_1 R_2}\right) \leq 2$$

which can be written as

$$\boxed{0 \leq g_1 g_2 \leq 1}$$

where $g_1 = 1 - \frac{L}{R_1}$, $g_2 = 1 - \frac{L}{R_2}$

For a symmetric cavity, $R_1 = R_2 = R$, this means that the difference between R & L must be smaller than R, i.e. $\frac{L}{2} \leq R \leq \infty$, where $R = \infty$ means a flat mirror.

The "=" are at the boundary between stability and instability so it is better to avoid them.

Therefore

$$0 < g_1 g_2 < 1.$$

Wave-optical treatment.

In the wave regime, it turns out that paraxial propagation can also be treated according to ABCD matrices. Let us assume that the system has rotational symmetry, so the same matrix is applied to x & y. The "Collins formula" states

$$U(x,y,z) = e^{ik\bar{z}}\,\hat{G}_{\mathcal{S}}\,U(x,y,0)$$

$$= e^{ik\bar{z}}\,\frac{k}{2\pi i B}\iint U(x',y',0)\,e^{\frac{ik}{2B}\left[A(x'^2+y'^2)+D(x^2+y^2)-2(xx'+yy')\right]}\,dx'dy'$$

where $\bar{z}$ is the optical path length of the ray along the axis from 0 to z.

Note that, for free space, $\mathcal{S} = T_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $\bar{z} = z$, the Collins formula reduces to the Fresnel propagation formula. For thin lenses and mirrors, $B=0$ and the formula seems to be ill-defined. However, one can take the limit $B \to 0$, leading to the correct result.

If $\mathcal{S}$ describes the cavity, $\bar{z} = 2L$, and the condition for wave stability is

$$U(x,y,0) = e^{ik2L}\,\hat{G}_{\mathcal{S}}\,U(x,y,0)$$

Let us try a Gaussian beam with waist at $z=z_0$, Rayleigh range z_R:

$$U^G(x,y,z) = \frac{u_0}{1+i\frac{z-z_0}{z_R}}\, e^{ikz}\, e^{\frac{ik(x^2+y^2)}{2(z-z_0-iz_R)}}$$

at $z=0$,

$$U^G(x,y,0) = \bar{u}_0\, e^{\frac{-ik(x^2+y^2)}{z_0+iz_R}} = \bar{u}_0\, e^{\frac{-k(x^2+y^2)}{2(z_R-iz_0)}}$$

with $\bar{u}_0 = \dfrac{u_0}{1-i\frac{z_0}{z_R}}$

$$\hat{\mathcal{C}}_{\$}\, U^G(x,y,0) = \frac{k\bar{u}_0}{2\pi i B}\iint_{-\infty}^{\infty} e^{-\frac{k(x'^2+y'^2)}{2(z_R-iz_0)}+\frac{ik}{2B}\left[A(x'^2+y'^2)-2(xx'+yy')\right]}\,dx'dy' \cdot e^{\frac{ik}{2B}D(x^2+y^2)}$$

$$= \frac{k\bar{u}_0}{2\pi i B}\int_{-\infty}^{\infty} e^{-\frac{\cancel{k}}{2}\underbrace{\left(\frac{1}{z_R-iz_0}-\frac{iA}{B}\right)}_{a}x'^2-\frac{ikxx'}{B}}\,dx'\, e^{\frac{ik}{2B}Dx^2}$$

$$\cdot\int_{-\infty}^{\infty} e^{-\frac{k}{2}\overbrace{\left(\frac{1}{z_R-iz_0}-\frac{iA}{B}\right)}^{a}y'^2-\frac{ikyy'}{B}}\,dy'\, e^{\frac{ikDy^2}{2B}}$$

Evaluate

$$\int_{-\infty}^{\infty} e^{-\frac{kax'^2}{2}-\frac{ikxx'}{B}}\,dx'\, e^{\frac{ikDx^2}{2B}} = \int_{-\infty}^{\infty} e^{-\frac{ka}{2}\left(x'^2+2\frac{ix}{aB}x'+\left(\frac{ix}{aB}\right)^2\right)}\,dx'$$

$$\cdot\, e^{\frac{-kx^2}{2aB^2}}\, e^{\frac{ikDx^2}{2B}} = \sqrt{\frac{2\pi}{ka}}\, e^{\frac{-kx^2}{2aB^2}(1-iDBa)}$$

$$= \sqrt{\frac{2\pi}{k\left(\frac{1}{z_R-iz_0}-\frac{iA}{B}\right)}}\; e^{-\frac{kx^2}{2aB^2}\left(1-\frac{iDB}{z_R-iz_0}-DA\right)}$$

$1-DA=-BC$

$$\hat{\mathcal{C}}_\$ U^G(x,y,0) = \frac{k\bar{U}_0}{2\pi i B}\,\frac{2\pi}{k\left(\frac{1}{z_R - iz_0} - \frac{iA}{B}\right)}\, e^{+\frac{k}{2aB}\left(\frac{iD}{z_R-iz_0}+C\right)(x^2+y^2)}$$

$$= \frac{\bar{U}_0}{i\left(\frac{B}{z_R-iz_0} - iA\right)}\, e^{\frac{k}{2}\frac{iD + C(z_R-iz_0)}{B - iA(z_R-iz_0)}(x^2+y^2)}$$

So $e^{ik2L}\hat{\mathcal{C}}_\$ U^G(x,y,0) = U^G(x,y,0)$ if

$$\bar{U}_0\, e^{\frac{-k(x^2+y^2)}{2(z_R-iz_0)}} = \frac{e^{ik2L}\bar{U}_0}{A + \frac{iB}{z_R-iz_0}}\, e^{\frac{ik}{2}\frac{D-iC(z_R-iz_0)}{B-iA(z_R-iz_0)}(x^2+y^2)}$$

i.e. if

$$e^{ik2L} = A + \frac{iB}{z_R - iz_0} \qquad (i)$$

$$\frac{-1}{(z_R-iz_0)} = i\,\frac{D-iC(z_R-iz_0)}{B-iA(z_R-iz_0)} \qquad (ii)$$

These equations have the solution

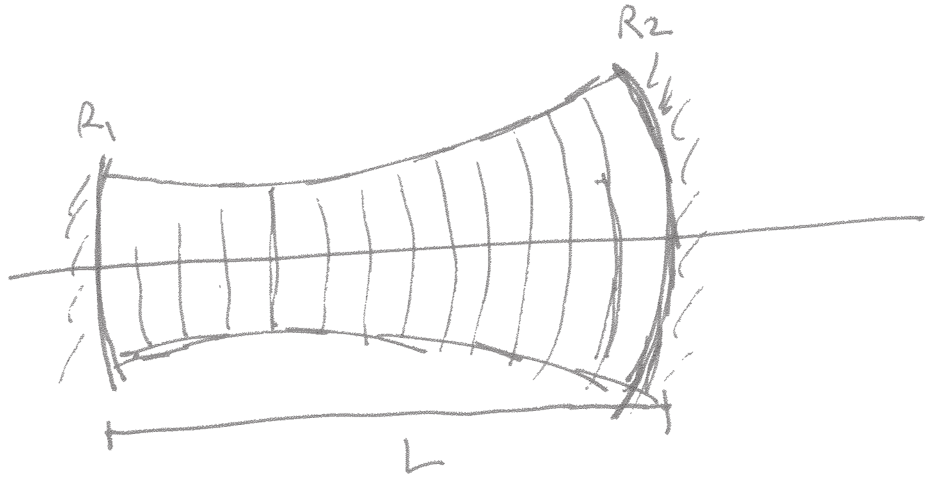
$$z_0 = \frac{D-A}{2C} = \frac{-\frac{4L}{R_1}\left(1-\frac{L}{R_2}\right)}{\left(\frac{2L}{R_1R_2} - \frac{1}{R_1} - \frac{1}{R_2}\right)} = \frac{L(L-R_2)}{2L-R_1-R_2}$$

$$z_R = \frac{1}{C}\sqrt{1-\left(\frac{A+D}{2}\right)^2} = \frac{\sqrt{1-\left(1-\frac{2L}{R_1}-\frac{2L}{R_2}+\frac{2L^2}{R_1R_2}\right)^2}}{\frac{4L}{R_1R_2}-\frac{2}{R_1}-\frac{2}{R_2}}$$

$$2kL = 2\pi \overset{\text{integer}}{M} - \arccos\left(\frac{A+D}{2}\right) \Rightarrow \omega_M = \pi\frac{M}{L}c - \frac{c}{2L}\arccos\left[1-\frac{2L}{R_1}-\frac{2L}{R_2}+\frac{2L^2}{R_1R_2}\right]$$

so there is a small correction to the 1D formula

These formulas are consistent with matching the mirrors to wavefronts of the Gaussian beam



Since the wavefronts of Hermite-Gaussian are the same as for Gaussians, spherical pairs of mirrors that support Gaussian beams <u>also</u> support Hermite-Gaussian beams. For any m,n, the relation between z_0 & z_R and L, R_1 & R_2 are the same as for Gaussian beams. However, the allowed frequencies change slightly:

$$2kL = 2\pi M - (1+m+n)\arccos\left(\frac{A+D}{2}\right)$$

$$\omega_M = \frac{\pi M c}{L} - \frac{c}{2L}(1+m+n)\arccos\left(1-\frac{2L}{R_1}-\frac{2L}{R_2}+\frac{2L^2}{R_1R_2}\right)$$