The Abdus Salam International Centre for Theoretical Physics

2442-4

Preparatory School to the Winter College on Optics

28 January - 1 February, 2013

Electromagnetic waves in Matter, Reflection and Transmission

I. Ashraf
Quaid-i-Azam University Islamabad
Pakistan

ELECTROMAGNETIC WAVES IN VACUUM

Imrana Ashraf
Quaid-i-Azam University Islamabad Pakistan

Preparatory School to Winter College on Optics: Trends in Laser Development and Multidisciplinary Applications to Science and Industry
28th January-1st February 2013

ELECTROMAGNETIC WAVES IN VACUUM

➢ THE WAVE EQUATION

- In regions of free space (i.e. the vacuum) - where no electric charges - no electric currents and no matter of any kind are present - Maxwell's equations (in differential form) are:

1) $\vec{\nabla} \cdot \vec{E}(\vec{r},t) = 0$

2) $\vec{\nabla} \cdot \vec{B}(\vec{r},t) = 0$

3) $\vec{\nabla} \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu_o \varepsilon_o \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t} = \dfrac{1}{c^2} \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t}$

$\left(c^2 = 1/\varepsilon_o \mu_o\right)$

Set of coupled first-order partial differential equations

ELECTROMAGNETIC WAVES IN VACUUM . . .

- We can de-couple Maxwell's equations -by applying the curl operator to equations 3) and 4):

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= \vec{\nabla}\left(\overbrace{\vec{\nabla} \bullet \vec{E}}^{=0}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right)$$

$$= -\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right)$$

$$= \boxed{\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \vec{\nabla} \times \left(\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right)$$

$$= \vec{\nabla}\left(\overbrace{\vec{\nabla} \bullet \vec{B}}^{=0}\right) - \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{E}\right)$$

$$= -\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$= \boxed{\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}}$$

ELECTROMAGNETIC WAVES IN VACUUM . . .

- These are three-dimensional de-coupled wave equations.
- Have exactly the same structure – both are linear, homogeneous, 2nd order differential equations.
- Remember that each of the above equations is explicitly dependent on space and time,

$$i.e.\ \vec{E} = \vec{E}(\vec{r},t) \text{ and } \vec{B} = \vec{B}(\vec{r},t):$$

$$\nabla^2 \vec{E}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = 0$$

$$\nabla^2 \vec{B}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} = 0$$

ELECTROMAGNETIC WAVES IN VACUUM . . .

- Thus, Maxwell's equations implies that empty space – the vacuum {which is not empty, at the microscopic scale} – supports the propagation of {macroscopic} electromagnetic waves - which propagate at the speed of light {in vacuum}:

$$c = 1/\sqrt{\varepsilon_o \mu_o} = 3 \times 10^8 \text{ m/s}$$

MONOCHROMATIC EM PLANE WAVES

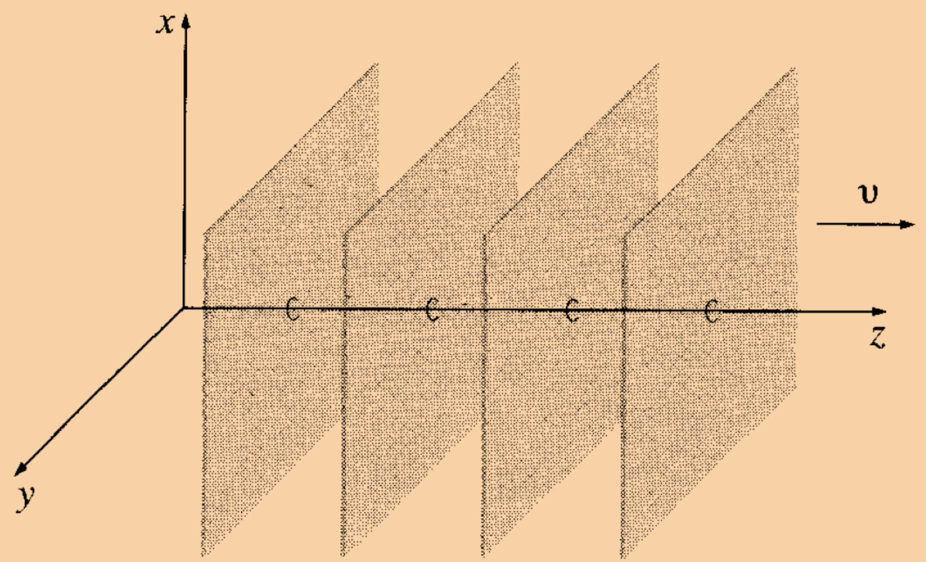
- EM plane waves propagating in free space with speed $c= f\lambda =\omega\ k$ - consisting of a single frequency f - wavelength $\lambda =c\ f$ - angular frequency $\omega = 2\pi\ f$ and wave-number $k = 2\pi\ /\lambda$ – called Monochromatic.
- Visible region of the EM spectrum *{~380 nm (violet) ≤ λ ≤ ~ 780 nm (red)}*- EM light waves of a given frequency / wavelength are perceived by the human eye as having a specific-single colour.

Single- frequency EM waves are called mono-chromatic.

MONOCHROMATIC EM PLANE WAVES

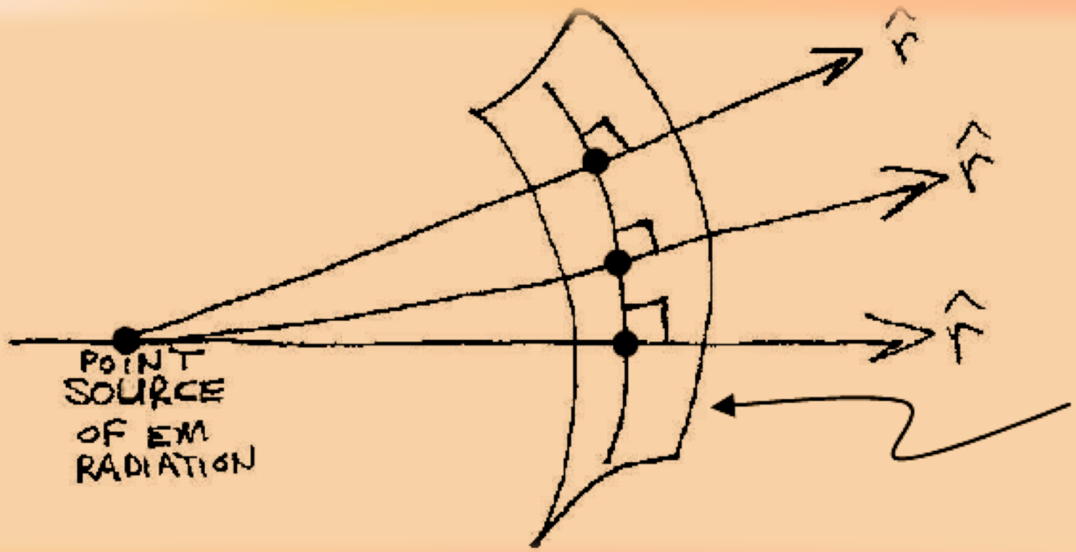
EM waves that propagate e.g. in the $+\hat{z}$ direction- with no explicit *x*- or *y*-dependence are known as plane waves.

For a given time= t the wave front(s) of the EM wave lie in a plane which is ⊥ to the $\hat{z}$ *-axis*,



MONOCHROMATIC EM PLANE WAVES

There also exist spherical EM waves – emitted from a point source – the wave-fronts associated with these EM waves are spherical - do not lie in a plane ⊥ to the direction of propagation of the EM wave



Portion of a spherical wavefront associated with a spherical wave

MONOCHROMATIC EM PLANE WAVES

If the point source is infinitely far away from observer- then a spherical wave → plane wave in this limit- the radius of curvature → ∞ and a spherical surface becomes planar as $R_C \to \infty$. Criterion for a plane wave: $\lambda \ll R_C$

Monochromatic plane waves associated with $\vec{E}$ and $\vec{B}$

$$\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_o e^{i(kz-\omega t)}$$

$$\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_o e^{i(kz-\omega t)}$$

MONOCHROMATIC EM PLANE WAVES

$$\tilde{\vec{E}}(z,t)=\tilde{\vec{E}}_o e^{\overbrace{i(kz-\omega t)}}$$

$$\tilde{\vec{B}}(z,t)=\tilde{\vec{B}}_o e^{\overbrace{i(kz-\omega t)}}$$

Propagating in $+\hat{z}$ direction

Propagating in $+\hat{z}$ direction

$n.b.$ complex vectors:

$e.g.$ $\tilde{\vec{E}}_o = E_o e^{i\delta}\hat{x}$

$n.b.$ complex vectors:

$e.g.$ $\tilde{\vec{B}}_o = B_o e^{i\delta}\hat{y}$

$n.b.$ The *real*, *physical* (instantaneous) fields are:

$$\left\{ \begin{array}{l} \vec{E}(\vec{r},t)\equiv \mathrm{Re}\left(\tilde{\vec{E}}(\vec{r},t)\right) \\ \vec{B}(\vec{r},t)\equiv \mathrm{Re}\left(\tilde{\vec{B}}(\vec{r},t)\right) \end{array} \right\}$$

Very important to keep in mind!!

MONOCHROMATIC EM PLANE WAVES

Maxwell's equations for free space impose additional constraints on $\tilde{\vec{E}}_o$ and $\tilde{\vec{B}}_o$

Since: $\vec{\nabla}\bullet\vec{E} = 0$ and: $\vec{\nabla}\bullet\vec{B} = 0$

$$= \mathrm{Re}\left(\vec{\nabla}\bullet\tilde{\vec{E}}\right) = 0 \qquad = \mathrm{Re}\left(\vec{\nabla}\bullet\tilde{\vec{B}}\right) = 0$$

These two relations can only be satisfied

$$\forall(\vec{r},t) \text{ if } \vec{\nabla}\bullet\tilde{\vec{E}} = 0 \ \forall(\vec{r},t) \text{ and } \vec{\nabla}\bullet\tilde{\vec{B}} = 0 \ \forall(\vec{r},t)$$

In Cartesian coordinates: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

Thus: $\left(\vec{\nabla}\bullet\tilde{\vec{E}}\right) = 0$ and $\left(\vec{\nabla}\bullet\tilde{\vec{B}}\right) = 0$ become:

MONOCHROMATIC EM PLANE WAVES

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\tilde{\vec{E}}_o e^{i(kz-\omega t)}\right)=0 \quad \text{and} \quad \left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\tilde{\vec{B}}_o e^{i(kz-\omega t)}\right)=0$$

Now suppose we do allow:

$$\tilde{\vec{E}}_o=\underbrace{\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{E}_o e^{i\delta}$$

$$\tilde{\vec{B}}_o=\underbrace{\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)}_{\text{polarization in } \hat{x}-\hat{y}-\hat{z}\ (3-D)}e^{i\delta}\equiv\vec{B}_o e^{i\delta}$$

Then

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(E_{ox}\hat{x}+E_{oy}\hat{y}+E_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

$$\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)\cdot\left(B_{ox}\hat{x}+B_{oy}\hat{y}+B_{oz}\hat{z}\right)e^{i\delta}e^{i(kz-\omega t)}=0$$

MONOCHROMATIC EM PLANE WAVES

E_{ox}, E_{oy}, E_{oz} = Amplitudes (constants) of the electric field components in x, y, z directions respectively.

B_{ox}, B_{oy}, B_{oz} =Amplitudes (constants) of the magnetic field components in *x, y, z* directions respectively.

$$\frac{\partial}{\partial x}\hat{x}\bullet E_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet E_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial x}\hat{x}\bullet B_{ox}\hat{x}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial y}\hat{y}\bullet B_{oy}\hat{y}e^{i(kz-\omega t)}e^{i\delta}=0$$

$$\frac{\partial}{\partial z}\left(e^{az}\right)=ae^{az}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\frac{\partial}{\partial z}\hat{z}\bullet E_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ikE_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \quad \text{true } \textit{iff} \ \boxed{E_{oz} \equiv 0} \ !!!$$

$$\frac{\partial}{\partial z}\hat{z}\bullet B_{oz}\hat{z}e^{i(kz-\omega t)}e^{i\delta} = ik\mathrm{E}_{oz}e^{i(kz-\omega t)}e^{i\delta} = 0 \quad \Leftarrow \quad \text{true } \textit{iff} \ \boxed{B_{oz} \equiv 0} \ !!!$$

- Maxwell's equations additionally impose the restriction that an electromagnetic plane wave cannot have any component of **E** or **B** || to (or anti- || to) the propagation direction (in this case here, the z -direction)
- Another way of stating this is that an EM wave cannot have any longitudinal components of **E** and **B** (i.e. components of **E** and **B** lying along the propagation direction).

MONOCHROMATIC EM PLANE WAVES . . .

- Thus, Maxwell's equations additionally tell us that an EM wave is a purely transverse wave (at least for propagation in free space) – the components of **E** and **B** must be ⊥ to propagation direction.
- The plane of polarization of an EM wave is defined (by convention) to be parallel to **E.**

MONOCHROMATIC EM PLANE WAVES . . .

Maxwell's equations impose another restriction on the allowed form of E and B for an EM wave:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = \mathrm{Re}\left(\vec{\nabla}\times\tilde{\vec{E}}\right) = \mathrm{Re}\left(-\frac{\partial\tilde{\vec{B}}}{\partial t}\right)$$

and/or:

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mathrm{Re}\left(\vec{\nabla}\times\tilde{\vec{B}}\right) = \mathrm{Re}\left(\frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}\right)$$

Can *only* be satisfied $\forall\ (\vec{r},t)$ ***iff***:

$$\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial\tilde{\vec{B}}}{\partial t}$$

and/or:

$$\vec{\nabla}\times\tilde{\vec{B}} = \frac{1}{c^2}\frac{\partial\tilde{\vec{E}}}{\partial t}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\nabla} \times \vec{\tilde{E}} = \left(\overbrace{\frac{\partial \tilde{E}_z}{\partial y}}^{=0} - \frac{\partial \tilde{E}_y}{\partial z} \right) \hat{x} + \left(\frac{\partial \tilde{E}_x}{\partial z} - \overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0} \right) \hat{y} + \left(\overbrace{\frac{\partial \tilde{E}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{E}_x}{\partial y}}^{=0} \right) \hat{z} = -\frac{\partial \tilde{B}_x}{\partial t} \hat{x} - \frac{\partial \tilde{B}_y}{\partial t} \hat{y} - \overbrace{\frac{\partial \tilde{B}_z}{\partial t}}^{=0} \hat{z}$$

$$\vec{\nabla} \times \tilde{B} = \left(\overbrace{\frac{\partial \tilde{B}_z}{\partial y}}^{=0} - \frac{\partial \tilde{B}_y}{\partial z} \right) \hat{x} + \left(\frac{\partial \tilde{B}_x}{\partial z} - \overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0} \right) \hat{y} + \left(\overbrace{\frac{\partial \tilde{B}_y}{\partial x}}^{=0} - \overbrace{\frac{\partial \tilde{B}_x}{\partial y}}^{=0} \right) \hat{z} = \frac{1}{c^2} \frac{\partial \tilde{E}_x}{\partial t} \hat{x} + \frac{1}{c^2} \frac{\partial \tilde{E}_y}{\partial t} \hat{y} + \frac{1}{c^2} \overbrace{\frac{\partial \tilde{E}_z}{\partial t}}^{=0} \hat{z}$$

$$\vec{\tilde{E}} = \tilde{E}_x \hat{x} + \tilde{E}_y \hat{y} + \overbrace{\tilde{E}_z}^{=0} \hat{z} = \left(E_{ox} \hat{x} + E_{oy} \hat{y} + \overbrace{E_{oz}}^{=0} \hat{z} \right) e^{i(kz - \omega t)} e^{i\delta}$$

$$\vec{\tilde{B}} = \tilde{B}_x \hat{x} + \tilde{B}_y \hat{y} + \overbrace{\tilde{B}_z}^{=0} \hat{z} = \left(B_{ox} \hat{x} + B_{oy} \hat{y} + \overbrace{B_{oz}}^{=0} \hat{z} \right) e^{i(kz - \omega t)} e^{i\delta}$$

MONOCHROMATIC EM PLANE WAVES . . .

$$\vec{\tilde{E}} = \tilde{E}_x \hat{x} + \tilde{E}_y \hat{y} = \left(E_{ox} \hat{x} + E_{oy} \hat{y} \right) e^{i(kz-\omega t)} e^{i\delta}$$

$$\vec{\tilde{B}} = \tilde{B}_x \hat{x} + \tilde{B}_y \hat{y} = \left(B_{ox} \hat{x} + B_{oy} \hat{y} \right) e^{i(kz-\omega t)} e^{i\delta}$$

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \tilde{E}_y}{\partial z} \hat{x} + \frac{\partial \tilde{E}_x}{\partial z} \hat{y} = -\frac{\partial \tilde{B}_x}{\partial t} \hat{x} - \frac{\partial \tilde{B}_y}{\partial t} \hat{y}$$

$$\vec{\nabla} \times \vec{\tilde{B}} = -\frac{\partial \tilde{B}_y}{\partial z} \hat{x} + \frac{\partial \tilde{B}_x}{\partial z} \hat{y} = \frac{1}{c^2} \frac{\partial \tilde{E}_x}{\partial t} \hat{x} + \frac{1}{c^2} \frac{\partial \tilde{E}_y}{\partial t} \hat{y}$$

Can only be satisfied / can only be true ***iff*** the $\hat{x}$ and $\hat{y}$ relations are <u>separately</u> / <u>independently</u> satisfied $\forall \ (\vec{r}, t)$!

MONOCHROMATIC EM PLANE WAVES . . .

$\vec{\nabla}\times\tilde{\vec{E}}:$

$$-\frac{\partial \tilde{E}_y}{\partial z}\hat{x} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} \Rightarrow \frac{\partial \tilde{E}_y}{\partial z} = \frac{\partial \tilde{B}_x}{\partial t} \Rightarrow ikE_{oy} = -i\omega B_{ox} \quad (1)$$

$$+\frac{\partial \tilde{E}_x}{\partial z}\hat{y} = -\frac{\partial \tilde{B}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{E}_x}{\partial z} = -\frac{\partial \tilde{B}_y}{\partial t} \Rightarrow ikE_{ox} = +i\omega B_{oy} \quad (2)$$

$\vec{\nabla}\times\tilde{B}:$

$$-\frac{\partial \tilde{B}_y}{\partial z}\hat{x} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t}\hat{x} \Rightarrow -\frac{\partial \tilde{B}_y}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_x}{\partial t} \Rightarrow -ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox} \quad (3)$$

$$+\frac{\partial \tilde{B}_x}{\partial z}\hat{y} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t}\hat{y} \Rightarrow \frac{\partial \tilde{B}_x}{\partial z} = \frac{1}{c^2}\frac{\partial \tilde{E}_y}{\partial t} \Rightarrow ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy} \quad (4)$$

From (1): $ik\tilde{E}_{oy} = -i\omega B_{ox} \Rightarrow E_{oy} = -\left(\frac{\omega}{k}\right)B_{ox}$ or: $B_{ox} = -\left(\frac{k}{\omega}\right)E_{oy}$

MONOCHROMATIC EM PLANE WAVES . . .

From (2): $ik\tilde{E}_{ox} = +i\omega B_{oy}$ $\Rightarrow$ $E_{ox} = +\left(\frac{\omega}{k}\right)B_{oy}$ or: $B_{oy} = +\left(\frac{k}{\omega}\right)E_{ox}$

From (3): $-ikB_{oy} = -\frac{1}{c^2}i\omega E_{ox}$ $\Rightarrow$ $B_{oy} = +\frac{1}{c^2}\left(\frac{\omega}{k}\right)E_{ox}$

From (4): $ikB_{ox} = -\frac{1}{c^2}i\omega E_{oy}$ $\Rightarrow$ $B_{ox} = -\frac{1}{c^2}\left(\frac{\omega}{k}\right)E_{oy}$

$$c = f\lambda = (2\pi f)\left(\frac{\lambda}{2\pi}\right) = \left(\frac{\omega}{k}\right) \qquad \frac{1}{c} = \left(k/\omega\right) \qquad \left(k = 2\pi/\lambda\right)$$

MONOCHROMATIC EM PLANE WAVES . . .

$\vec{\nabla} \times \tilde{\vec{E}}$:

(1) $$B_{ox} = -\frac{1}{c} E_{oy}$$

(2) $$B_{oy} = +\frac{1}{c} E_{ox}$$

$\vec{\nabla} \times \tilde{B}$:

(3) $$B_{oy} = +\frac{1}{c} E_{ox}$$

(4) $$B_{ox} = -\frac{1}{c} E_{oy}$$

Maxwell's Equations also have some redundancy encrypted into them!

Actually we have only two independent relations:

$$B_{ox} = -\frac{1}{c} E_{oy} \quad \text{and} \quad B_{oy} = +\frac{1}{c} E_{ox}$$

But: $\hat{z} \times \hat{y} = -\hat{x}$ $\qquad$ $\hat{z} \times \hat{x} = +\hat{y}$

MONOCHROMATIC EM PLANE WAVES . . .

Very Useful Table:

$\hat{x}\times\hat{y}=\hat{z}$	$\hat{y}\times\hat{x}=-\hat{z}$
$\hat{y}\times\hat{z}=\hat{x}$	$\hat{z}\times\hat{y}=-\hat{x}$
$\hat{z}\times\hat{x}=\hat{y}$	$\hat{x}\times\hat{z}=-\hat{y}$

Two relations can be written compactly into one relation:

$$\vec{\tilde{B}}_o=\frac{1}{c}\left(\hat{z}\times\vec{\tilde{E}}_o\right)$$

Physically this relation states that E and B are:

- in phase with each other.
- mutually perpendicular to each other - $(\mathrm{E}\perp\mathbf{B})\perp \hat{z}$

MONOCHROMATIC EM PLANE WAVES . . .

The **E** and **B** fields associated with this monochromatic plane EM wave are purely transverse

The real amplitudes of E and B are related to each other by:

$$B_o = \frac{1}{c} E_o$$ with $$B_o = \sqrt{B_{ox}^2 + B_{oy}^2}$$ and $$E_o = \sqrt{E_{ox}^2 + E_{oy}^2}$$

Instantaneous Poynting's Vector for a linearly polarized *EM wave*

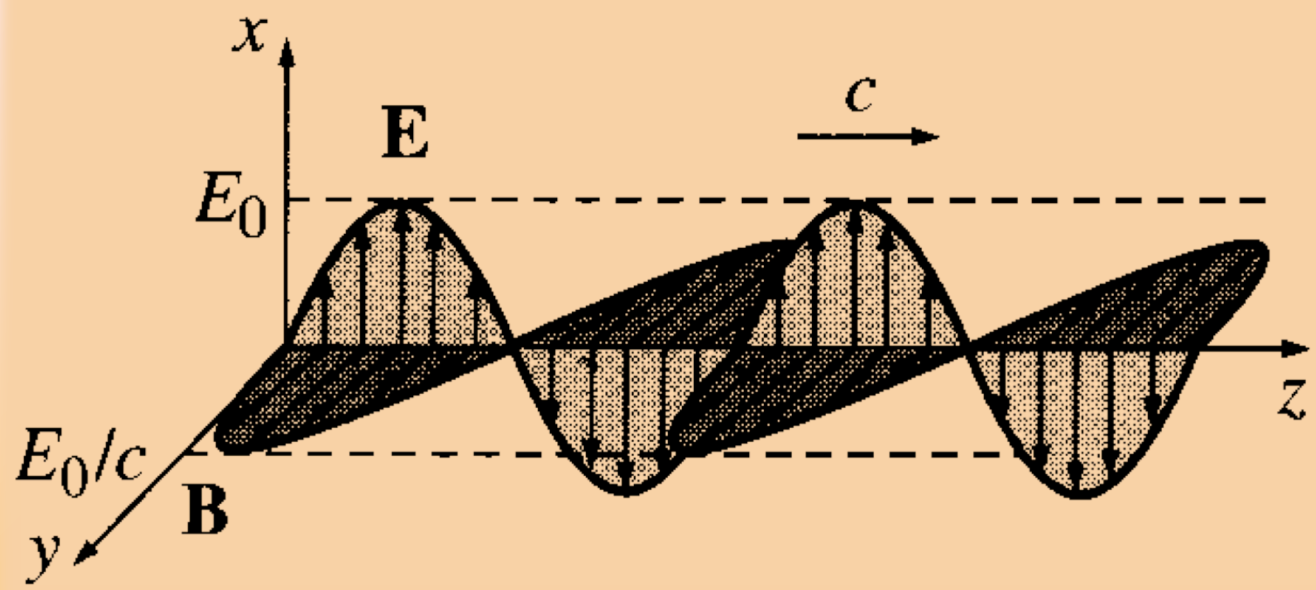
$$\vec{S}(z,t) = \frac{1}{\mu_o}\vec{E}(z,t)\times\vec{B}(z,t) = \frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\underbrace{(\hat{x}\times\hat{y})}_{=\hat{z}}$$

$$\vec{S}(z,t) = \frac{1}{\mu_o}E_oB_o\cos^2(kz-\omega t+\delta)\,\hat{z} \qquad \left(\frac{\mathrm{Watts}}{\mathrm{m}^2}\right)$$

⇒EM Power flows in the direction of propagation of the EM wave (here, the +zˆ direction)

Instantaneous Poynting's Vector for a linearly polarized *EM wave*



This is the paradigm for a monochromatic plane wave. The wave as a whole is said to be polarized in the x direction (by convention the direction of E to specify the polarization of an electromagnetic wave).

Instantaneous Energy & Linear Momentum & Angular Momentum in *EM Waves*

Instantaneous Energy Density Associated with an *EM Wave:*

$$u_{EM}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t)+\frac{1}{\mu_o}B^2(\vec{r},t)\right)=u_{elect}(\vec{r},t)+u_{mag}(\vec{r},t)$$

where $$u_{elect}(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

and $$u_{mag}(\vec{r},t)=\frac{1}{2\mu_o}B^2(\vec{r},t)=\frac{1}{2}\varepsilon_o E^2(\vec{r},t)$$

Instantaneous Energy & Linear Momentum & Angular Momentum in *EM Waves*

But $B^2 = \frac{1}{c^2} E^2$ - EM waves in vacuum, and $\frac{1}{c^2} = \varepsilon_o \mu_o$

$$u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{\varepsilon_o \mu_o}{\mu_o} E^2(\vec{r},t) \right) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \varepsilon_o E^2(\vec{r},t) \right)$$

$$u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2\left(\vec{k} \bullet \vec{r} - \omega t + \delta \right) \quad \left(\frac{\text{Joules}}{\text{m}^3} \right)$$

$u_{elect}(\vec{r},t) = u_{mag}(\vec{r},t)$ - EM waves propagating in the vacuum !!!!

Instantaneous Poynting's Vector Associated with an *EM Wave*

$$\vec{S}(\vec{r},t)=\frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)=\frac{1}{\mu_o}\mathrm{Re}\left\{\tilde{\vec{E}}(z,t)\right\}\times\mathrm{Re}\left\{\tilde{\vec{B}}(z,t)\right\}\quad\left(\frac{\mathrm{Watts}}{\mathrm{m}^2}\right)$$

For a linearly polarized monochromatic plane EM wave propagating in the vacuum,

$$\vec{S}(\vec{r},t)=c\left(\frac{\varepsilon_o\mu_o}{\mu_o}\right)E_o^2\cos^2(kz-\omega t+\delta)\hat{z}=c\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)\hat{z}$$

But $u_{EM}(\vec{r},t)=\varepsilon_oE^2(\vec{r},t)=\varepsilon_oE_o^2\cos^2(kz-\omega t+\delta)$

$$\vec{S}(\vec{r},t)=cu_{EM}(\vec{r},t)\hat{z}$$

Instantaneous Poynting's Vector Associated with an *EM Wave*

The propagation velocity of energy $\vec{v}_{prop} = c\hat{z}$

Poynting's Vector = Energy Density * Propagation Velocity

$$\vec{S}(\vec{r},t) = u_{EM}(\vec{r},t)\vec{v}_{prop}$$

Instantaneous Linear Momentum Density Associated with an EM Wave:

$$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Instantaneous Linear Momentum Density Associated with an *EM Wave*

For linearly polarized monochromatic plane EM waves propagating in the vacuum:

$$\vec{\wp}_{EM} = \frac{1}{c^{\not{2}}} \not{c}\, \varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)\, \hat{z} = \frac{1}{c} \underbrace{\varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)}_{=u_{EM}} \hat{z}$$

But: $$u_{EM}(\vec{r},t) = \varepsilon_o E^2(\vec{r},t) = \varepsilon_o E_o^2 \cos^2(kz - \omega t + \delta)$$

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)\,\hat{z}} \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

Instantaneous Angular Momentum Density Associated with an *EM wave*

$$\vec{\ell}_{EM}(\vec{r},t) = \vec{r} \times \vec{\wp}_{EM}(\vec{r},t) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

But:
$$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)\hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

For an EM wave propagating in the +z^ direction:

$$\vec{\ell}_{EM}(\vec{r},t) = \frac{1}{c^2}\vec{r} \times \vec{S}(\vec{r},t) = \frac{1}{c} u_{EM}(\vec{r},t)(\vec{r} \times \hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

↑ Depends on the choice of origin

Instantaneous Power Associated with an *EM wave*

The instantaneous EM power flowing into/out of volume v with bounding surface S enclosing volume v (containing EM fields in the volume v) is:

$$P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = \int_v \frac{\partial u_{EM}(\vec{r},t)}{\partial t} d\tau = -\oint_S \vec{S}(\vec{r},t) \cdot d\vec{a}$$

The instantaneous EM power crossing (imaginary) surface is:

$$P_{EM}(t) = -\int_S \vec{S}(\vec{r},t) \cdot d\vec{a}_\perp$$

The instantaneous total *E*M energy contained in volume v

$$U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\, d\tau \quad \text{(Joules)}$$

Instantaneous Angular Momentum Density Associated with an *EM wave*

The instantaneous total EM linear momentum contained in the volume v is:

$$\boxed{\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau} \qquad \left(\frac{\text{kg-m}}{\text{sec}}\right)$$

The instantaneous total EM angular momentum contained in the volume v is:

$$\boxed{\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\,d\tau} \qquad \left(\frac{\text{kg-m}^2}{\text{sec}}\right)$$

Time-Averaged Quantities Associated with EM Waves

Usually we are not interested in knowing the instantaneous power P(t), energy / energy density, Poynting's vector, linear and angular momentum, *etc.*- because experimental measurements of these quantities are very often averages over many extremely fast cycles of oscillation. For example period of oscillation of light wave

$$\tau_{light} = 1/f_{light} \simeq \frac{1}{10^{15}\,\text{cps}} = 10^{-15}\ \text{sec/cycle} = 1\,\text{femto-sec}\ )$$

We need time averaged expressions for each of these quantities - in order to compare directly with experimental data- for monochromatic plane EM light waves:

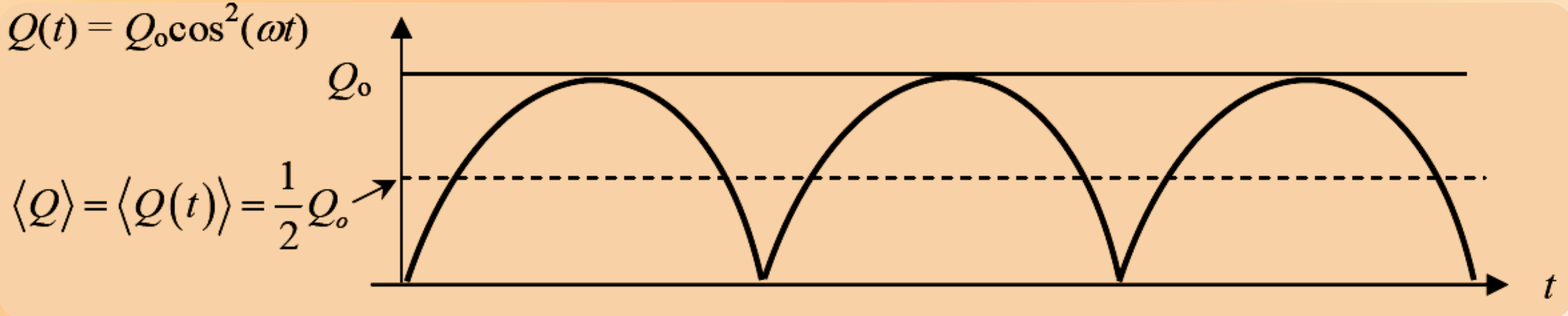
Time-Averaged Quantities Associated with EM Waves

If we have *a "generic"* instantaneous physical quantity of the form:

$$Q(t) = Q_o \cos^2(\omega t)$$

The time-average of $Q(t)$ is defined as:

$$\langle Q(t) \rangle \equiv \langle Q \rangle = \frac{1}{\tau} \int_{t=0}^{t=\tau} Q(t)\,dt = \frac{Q_o}{\tau} \int_{t=0}^{t=\tau} \cos^2(\omega t)\,dt$$



Time-Averaged Quantities Associated with EM Waves

The time average of the $\cos^2(\omega t)$ function:

$$\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{\tau}\left[\frac{t}{2}+\frac{\sin 2\omega t}{4\omega}\right]_{t=0}^{t=\tau} = \frac{1}{2\tau}\left[(\tau-0)+\left(\frac{\sin 2\omega\tau}{2\omega}-0\right)\right] = \frac{1}{2\tau}\left[\tau+\frac{\sin 2\omega\tau}{2\omega}\right]$$

$$\omega\tau = 2\pi f\tau \qquad f = 1/\tau \qquad \omega\tau = 2\pi(\tau/\tau) = 2\pi \qquad \sin(\omega\tau) = \sin(2\pi) = 0$$

$$\frac{1}{\tau}\int_0^{\tau}\cos^2(\omega t)\,dt = \frac{1}{2\not\tau}\left[\not\tau\right] = \frac{1}{2} \qquad \langle Q(t)\rangle = \langle Q\rangle = \frac{1}{2}Q_o$$

Thus, the time-averaged quantities associated with an EM wave propagating in free space are:

Time-Averaged Quantities Associated with EM Waves

EM Energy Density: $u_{EM}(\vec{r},t) \Rightarrow \langle u_{EM}(\vec{r},t) \rangle$

Total EM Energy: $U_{EM}(t) \Rightarrow \langle U_{EM}(t) \rangle$

Poynting's Vector: $\vec{S}(\vec{r},t) \Rightarrow \langle \vec{S}_{EM}(\vec{r},t) \rangle$

EM Power: $P_{EM}(t) \Rightarrow \langle P_{EM}(t) \rangle$

Time-Averaged Quantities Associated with EM Waves

Linear Momentum Density: $\vec{\wp}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\wp}_{EM}(\vec{r},t) \rangle$

Linear Momentum: $\vec{p}_{EM}(t) \Rightarrow \langle \vec{p}_{EM}(t) \rangle$

Angular Momentum Density: $\vec{\ell}_{EM}(\vec{r},t) \Rightarrow \langle \vec{\ell}_{EM}(\vec{r},t) \rangle$

Angular Momentum: $\vec{\mathcal{L}}_{EM}(t) \Rightarrow \langle \vec{\mathcal{L}}_{EM}(t) \rangle$

Time-Averaged Quantities Associated with EM Waves

For a monochromatic EM plane wave propagating in free space / vacuum in ˆz direction:

$$\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon_o E_o^2 \quad \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

$$\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{2}c\varepsilon_o E_o^2\hat{z} = c\langle u_{EM}(\vec{r},t)\rangle\hat{z} \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

$$\langle \vec{\wp}_{EM}(\vec{r},t)\rangle = \frac{1}{2c}\varepsilon_o E_o^2\hat{z} = \frac{1}{c^2}\langle \vec{S}(\vec{r},t)\rangle = \frac{1}{c}\langle u_{EM}(\vec{r},t)\rangle\hat{z} \quad \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

$$\langle \ell_{EM}(\vec{r},t)\rangle = \left(\vec{r}\times\langle \vec{\wp}_{EM}(\vec{r},t)\rangle\right) = \frac{1}{c^2}\left(\vec{r}\times\langle \vec{S}(\vec{r},t)\rangle\right) = \frac{1}{c}\langle u_{EM}(\vec{r},t)\rangle(\hat{r}\times\hat{z}) \quad \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Time-Averaged Quantities Associated with EM Waves

Intensity of an *EM* wave:

$$I(\vec{r}) \equiv \langle S(\vec{r},t) \rangle = \langle |\vec{S}(\vec{r},t)| \rangle = c \langle u_{EM}(\vec{r},t) \rangle = \frac{1}{2} c \varepsilon_o E_o^2 \quad \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

The intensity of an EM wave is also known as the irradiance of the EM wave – it is the radiant power incident per unit area upon a surface.

ELECTROMAGNETIC WAVES IN MATTER

Imrana Ashraf
Quaid-i-Azam University Islamabad Pakistan

Preparatory School to Winter College on Optics: Trends in Laser Development and Multidisciplinary Applications to Science and Industry
28th January-1st February 2013

Electromagnetic Wave Propagation in Linear Media

Consider EM wave propagation inside matter - in regions where there are NO free charges and/or free currents (the medium is an insulator/non-conductor).

For this situation, Maxwell's equations become:

1) $$\vec{\nabla} \bullet \vec{D}(\vec{r},t) = 0$$

2) $$\vec{\nabla} \bullet \vec{B}(\vec{r},t) = 0$$

3) $$\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}$$

4) $$\vec{\nabla} \times \vec{H}(\vec{r},t) = \frac{\partial \vec{D}(\vec{r},t)}{\partial t}$$

Electromagnetic Wave Propagation in Linear Media

The medium is assumed to be linear, homogeneous and isotropic- thus the following relations are valid in this medium:

$$\vec{D}(\vec{r},t) = \varepsilon \vec{E}(\vec{r},t)$$

and

$$\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)$$

- ε = electric permittivity of the medium.
- $\varepsilon = \varepsilon_o(1 + \chi_e)$, χ_e = electric susceptibility of the medium.
- μ = magnetic permeability of the medium.
- $\mu = \mu_o(1 + \chi_m)$, χ_m = magnetic susceptibility of the medium.
- ε_o = electric permittivity of free space = 8.85×10^{-12} Farads/m.
- μ_o = magnetic permeability of free space = $4\pi \times 10^{-7}$ Henrys/m.

Electromagnetic Wave Propagation in Linear Media

Maxwell's equations inside the linear, homogeneous and isotropic non-conducting medium become:

1) $\vec{\nabla}\bullet\vec{E}(\vec{r},t)=0$

2) $\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0$

3) $\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}$

4) $\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}$

In a linear /homogeneous/isotropic medium, the speed of propagation of EM waves is:

$$v'_{prop}=\frac{1}{\sqrt{\varepsilon\mu}}$$

Electromagnetic Wave Propagation in Linear Media

The ***E*** and ***B*** fields in the medium obey the following wave equation:

$$\nabla^2 \vec{E}(\vec{r},t) = \varepsilon\mu \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

$$\nabla^2 \vec{B}(\vec{r},t) = \varepsilon\mu \frac{\partial \vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2}$$

Electromagnetic Wave Propagation in Linear Media

For linear / homogeneous / isotropic media:

$$\varepsilon = K_e \varepsilon_o = (1+\chi_e)\varepsilon_o \qquad K_e = \frac{\varepsilon}{\varepsilon_o} = (1+\chi_e) = \text{relative electric permittivity}$$

$$\mu = K_m \mu_0 = (1+\chi_m)\mu_o \qquad K_m = \frac{\mu}{\mu_o} = (1+\chi_m) = \text{relative magnetic permeability}$$

$$v'_{prop} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{K_e \varepsilon_o K_m \mu_o}} = \frac{1}{\sqrt{K_e K_m}} \frac{1}{\sqrt{\varepsilon_o \mu_o}} = \frac{1}{\sqrt{K_e K_m}} c$$

If $K_e K_m \geq 1$ thus $\frac{1}{\sqrt{K_e K_m}} \leq 1 \Rightarrow v'_{prop} = \frac{1}{\sqrt{K_e K_m}} c \leq c$

Electromagnetic Wave Propagation in Linear Media

Note also that since $K_e = \dfrac{\varepsilon}{\varepsilon_o}$ and $K_m = \dfrac{\mu}{\mu_o}$ are dimensionless

quantities, then so is $\dfrac{1}{\sqrt{K_e K_m}}$

Define the index of refraction { *a dimensionless quantity*} of the linear / homogeneous / isotropic medium as:

$$n \equiv \sqrt{K_e K_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o\mu_o}}$$

Electromagnetic Wave Propagation in Linear Media

Thus, for linear / homogeneous / isotropic media:

$$v'_{prop} = c/n \;\; (\leq c)$$ because $$n \geq 1$$

Now for many (but not all) linear/homogeneous/isotropic materials:

$$\mu = \mu_o (1 + \chi_m) \simeq \mu_o$$

(*True for many paramagnetic and diamagnetic-type materials*)

$$|\chi_m| \sim \vartheta(10^{-8}) \sim 0$$

Thus $$K_m = \frac{\mu}{\mu_o} = (1 + \chi_m) \simeq 1$$ $\Rightarrow$ $$n \simeq \sqrt{K_e}$$ and $$v'_{prop} = \frac{c}{n} \simeq \frac{c}{\sqrt{K_e}}.$$

Electromagnetic Wave Propagation in Linear Media

The instantaneous EM energy density associated with a linear/homogeneous/isotropic material

$$u_{EM}(\vec{r},t)=\frac{1}{2}\left(\varepsilon E^2(\vec{r},t)+\frac{1}{\mu}B^2(\vec{r},t)\right)=\frac{1}{2}\left(\vec{E}(\vec{r},t)\cdot\vec{D}(\vec{r},t)+\vec{B}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right)\quad\left(\frac{\text{Joules}}{\text{m}^3}\right)$$

with $\vec{D}(\vec{r},t)=\varepsilon\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)=\frac{1}{\mu}\vec{B}(\vec{r},t)$

Electromagnetic Wave Propagation in Linear Media

The instantaneous Poynting's vector associated with a linear/homogeneous/isotropic material

$$\vec{S}(\vec{r},t) = \frac{1}{\mu}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \left(\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right) \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The intensity of an EM wave propagating in a linear/homogeneous /isotropic medium is:

$$I(\vec{r}) \equiv \left\langle\left|\vec{S}(\vec{r},t)\right|\right\rangle = v'_{prop}\left\langle u_{EM}(\vec{r},t)\right\rangle = \frac{1}{2}v'_{prop}\varepsilon E_o^2(\vec{r}) = \frac{1}{2}\left(\frac{c}{n}\right)\varepsilon E_o^2(\vec{r}) = \left(\frac{c}{n}\right)\varepsilon E_{o_{rms}}^2(\vec{r}) \quad \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Where

$$E_{o_{rms}} \equiv \frac{1}{\sqrt{2}}E_o$$

Electromagnetic Wave Propagation in Linear Media

The instantaneous linear momentum density associated with an EM wave propagating in a linear/homogeneous/isotropic medium is:

$$\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\mu\vec{S}(\vec{r},t) = \frac{1}{v'^{2}_{prop}}\vec{S}(\vec{r},t) = \varepsilon\not{\mu}\frac{1}{\not{\mu}}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \varepsilon\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) \left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

The instantaneous angular momentum density associated with an EM wave propagating in a linear/homogeneous/isotropic medium is:

$$\vec{\ell}_{EM}(\vec{r},t) = \vec{r}\times\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\,\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) \left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Electromagnetic Wave Propagation in Linear Media

Total instantaneous EM energy: $U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\,d\tau$ (Joules)

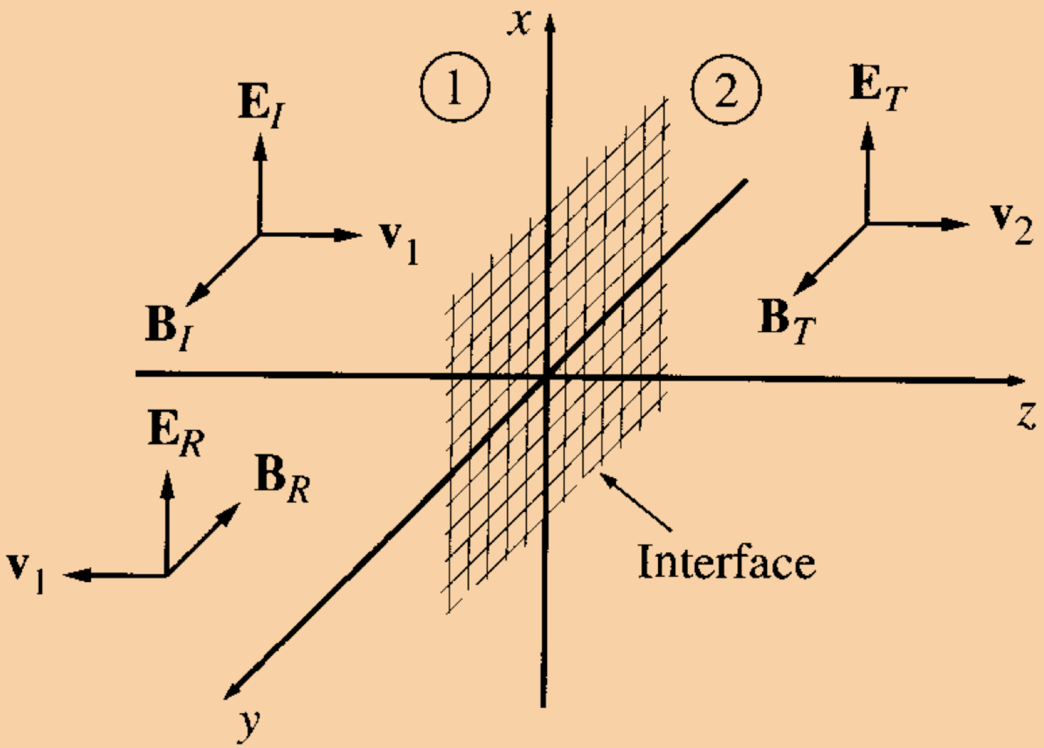
Total instantaneous linear momentum: $\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau$ $\left(\frac{\text{kg-m}}{\text{sec}}\right)$

Instantaneous *EM* Power: $P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = -\oint_S \vec{S}(\vec{r},t)\cdot d\vec{a}$ (Watts)

Total instantaneous angular momentum: $\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\,d\tau$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Suppose the x-y plane forms the boundary between two linear media. A plane wave of frequency ω- travelling in the z- direction and polarized in the x- direction- approaches the interface from the left



Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Incident EM plane wave (in medium 1):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with polarization $\boxed{\hat{n}_{inc} = +\hat{x}}$

$$\boxed{\vec{\tilde{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x}} \quad \text{with:} \quad \boxed{k_{inc} = \left|\vec{k}_{inc}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$$

$$\boxed{\vec{\tilde{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \vec{\tilde{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y}} \quad \underline{\text{since}}: \quad \boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z} \times \hat{x} = +\hat{y}}$$

Reflected EM plane wave (in medium 1):

Propagates in the $-\hat{z}$-direction (*i.e.* $\hat{k}_{refl} = -\hat{k}_1 = -\hat{z}$), with polarization $\boxed{\hat{n}_{refl} = +\hat{x}}$

$$\boxed{\vec{\tilde{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{x}} \quad \text{with:} \quad \boxed{k_{refl} = \left|\vec{k}_{refl}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$$

$$\boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \vec{\tilde{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)} \hat{y}} \quad \underline{\text{since}}: \quad \boxed{\hat{k}_{refl} \times \hat{n}_{refl} = -\hat{z} \times \hat{x} = -\hat{y}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Transmitted EM plane wave (in medium 2):

Propagates in the $+\hat{z}$ -direction (*i.e.* $\hat{k}_{trans} = +\hat{k}_2 = +\hat{z}$), with polarization $\hat{n}_{trans} = +\hat{x}$

$$\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{x}$$ with: $$k_{trans} = \left|\vec{k}_{trans}\right| = k_2 = \left|\vec{k}_2\right| = 2\pi/\lambda_2 = \omega/v_2$$

$$\tilde{\vec{B}}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{y}$$ since: $$\hat{k}_{trans} \times \hat{n}_{trans} = +\hat{z} \times \hat{x} = +\hat{y}$$

Note that {*here, in this situation*} *the* E -field / polarization vectors are all oriented in the same direction, i.e.

$$\hat{n}_{inc} = \hat{n}_{refl} = \hat{n}_{trans} = +\hat{x}$$ or equivalently:

$$\vec{E}_{inc}(\vec{r},t) \parallel \vec{E}_{refl}(\vec{r},t) \parallel \vec{E}_{trans}(\vec{r},t)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At the interface between the two linear / homogeneous / isotropic media -at z = 0 {in the x-y plane} the boundary conditions 1) – 4) must be satisfied for the total E and B -fields immediately present on either side of the interface:

BC 1) Normal $\vec{D}$ continuous: $\boxed{\varepsilon_1 E^{\perp}_{1_{Tot}} = \varepsilon_2 E^{\perp}_{2_{Tot}}}$

(*n.b.* $\perp$ refers to the *x*-*y* boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous: $\boxed{E^{\parallel}_{1_{Tot}} = E^{\parallel}_{2_{Tot}}}$

(*n.b.* $\parallel$ refers to the *x*-*y* boundary, *i.e.* in the *x*-*y* plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 3) Normal $\vec{B}$ continuous: $$B^{\perp}_{1_{Tot}} = B^{\perp}_{2_{Tot}}$$

(⊥ to x-y boundary, i.e. in the +z^ direction)

BC 4) Tangential $\vec{H}$ continuous: $$\frac{1}{\mu_1} B^{\parallel}_{1_{Tot}} = \frac{1}{\mu_2} B^{\parallel}_{2_{Tot}}$$

(∥ to x-y boundary, i.e. in x-y plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For plane EM waves at normal incidence on the boundary at z = 0 lying in the x-y plane- no components of *E or B* (incident, reflected or transmitted waves) - allowed to be along the ±zˆ propagation direction(s) - the E and *B*-field are transverse fields {*constraints imposed by Maxwell's equations*}.

BC 1) and BC 3) impose no restrictions on such *EM* waves since:

$$\{ E^{\perp}_{1_{Tot}} = \mathrm{E}^{z}_{1_{Tot}} = 0 ;\ E^{\perp}_{2_{Tot}} = E^{z}_{2_{Tot}} = 0 \} \text{ and } \{ B^{\perp}_{1_{Tot}} = B^{z}_{1_{Tot}} = 0 ;\ B^{\perp}_{2_{Tot}} = B^{z}_{2_{Tot}} = 0 \}$$

⇒ The only restrictions on plane EM waves propagating with normal incidence on the boundary at z = 0 are imposed by BC 2) and BC 4).

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At z = 0 in medium 1) (i.e. z ≤ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{1_{Tot}}(z=0,t) = \vec{\tilde{E}}_{inc}(z=0,t) + \vec{\tilde{E}}_{refl}(z=0,t)} \text{ and}$$

$$\boxed{\frac{1}{\mu_1}\vec{\tilde{B}}^{\parallel}_{1_{Tot}}(z=0,t) = \frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t) + \frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t)}$$

While at z = 0 in medium 2) (i.e. z ≥ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{2_{Tot}}(z=0,t) = \vec{\tilde{E}}_{trans}(z=0,t)} \text{ and}$$

$$\boxed{\frac{1}{\mu_2}\vec{\tilde{B}}^{\parallel}_{2_{Tot}}(z=0,t) = \frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential *E* is continuous @ z = 0) requires that:

$$\left.\vec{\tilde{E}}^{\parallel}_{1_{Tot}}\right|_{z=0} = \left.\vec{\tilde{E}}^{\parallel}_{2_{Tot}}\right|_{z=0}$$

or:

$$\vec{\tilde{E}}_{inc}(z=0,t) + \vec{\tilde{E}}_{refl}(z=0,t) = \vec{\tilde{E}}_{trans}(z=0,t)$$

BC 4) (Tangential *H* is continuous @ z = 0) requires that:

$$\frac{1}{\mu_1}\left.\vec{\tilde{B}}^{\parallel}_{1_{Tot}}\right|_{z=0} = \frac{1}{\mu_2}\left.\vec{\tilde{B}}^{\parallel}_{2_{Tot}}\right|_{z=0}$$

or:

$$\frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t) + \frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t) = \frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Using explicit expressions for the complex **E** and **B** fields

$$\vec{\tilde{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z-\omega t)}\hat{x}$$

$$\vec{\tilde{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \vec{\tilde{E}}_{inc}(z,t) = \frac{1}{v_1}\vec{\tilde{E}}_{o_{inc}} e^{i(k_1 z-\omega t)}\hat{y}$$

$$\vec{\tilde{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z-\omega t)}\hat{x}$$

$$\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \vec{\tilde{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z-\omega t)}\hat{y}$$

$$\vec{\tilde{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z-\omega t)}\hat{x}$$

$$\tilde{B}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \vec{\tilde{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z-\omega t)}\hat{y}$$

into the above boundary condition relations- equations become

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\tilde{E}_{0_{inc}} e^{-i\omega t} + \tilde{E}_{0_{refl}} e^{-i\omega t} = \tilde{E}_{0_{trans}} e^{-i\omega t}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\frac{1}{\mu_1 v_1} \tilde{E}_{0_{inc}} e^{-i\omega t} - \frac{1}{\mu_1 v_1} \tilde{E}_{0_{refl}} e^{-i\omega t} = \frac{1}{\mu_2 v_2} \tilde{E}_{0_{trans}} e^{-i\omega t}$

Cancelling the common $e^{-i\omega t}$ factors on the LHS & RHS of above equations - we have at z = 0 { everywhere in the x-y plane- must be independent of any time t}:

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$

Assuming that {μ_1 and μ_2} and {v_1 and v_2} are known / given for the two media, we have <u>two</u> equations {from BC 2) and BC 4)} and *<u>three</u>* unknowns {$\tilde{E}_{o_{inc}}$, $\tilde{E}_{o_{refl}}$, $\tilde{E}_{o_{trans}}$ }

→ Solve above equations simultaneously for

{ $\tilde{E}_{o_{refl}}$ and $\tilde{E}_{o_{trans}}$ } in terms of / scaled to $\tilde{E}_{o_{inc}}$.

First (for convenience) let us define: $\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 4) (Tangential ***H*** continuous @ z = 0) relation becomes:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}$$

BC 2) (Tangential **E** continuous @ *z = 0):*

$$\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}$$

BC 4) (Tangential ***H*** continuous @ *z = 0):*

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\, \tilde{E}_{o_{trans}}$$

with

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Add and Subtract BC 2) and BC 4) relations:

$$2\tilde{E}_{o_{inc}} = (1+\beta)\,\tilde{E}_{o_{trans}} \quad \Rightarrow \quad \tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}} \quad (2+4)$$

$$2\tilde{E}_{o_{refl}} = (1-\beta)\,\tilde{E}_{o_{trans}} \quad \Rightarrow \quad \tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{2}\right)\tilde{E}_{o_{trans}} \quad (2-4)$$

Insert the result of eqn. (2+4) into eqn. (2−4):

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{\not{2}}\right)\left(\frac{\not{2}}{1+\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}} \quad \text{and} \quad \tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}$$

<u>Now</u>: $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2}$ and: $v_1 = \dfrac{c}{n_1}$, $v_2 = \dfrac{c}{n_2}$ where: $n_1 = \sqrt{\dfrac{\varepsilon_1\mu_1}{\varepsilon_o\mu_o}}$ and $n_2 = \sqrt{\dfrac{\varepsilon_2\mu_2}{\varepsilon_o\mu_o}}$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 (c/n_1)}{\mu_2 (c/n_2)} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1\sqrt{\varepsilon_2\mu_2/\varepsilon_o\mu_o}}{\mu_2\sqrt{\varepsilon_1\mu_1/\varepsilon_o\mu_o}} = \frac{\mu_1}{\mu_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} = \sqrt{\left(\frac{\varepsilon_2}{\mu_2}\right)\Big/\left(\frac{\varepsilon_1}{\mu_1}\right)} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now if the two media are both paramagnetic and/or diamagnetic, such that $\left|\chi_{m_{1,2}}\right| \ll 1$

$$i.e.\ \mu_1 = \mu_o\left(1+\chi_{m_1}\right) \approx \mu_o \text{ and: } \mu_2 = \mu_o\left(1+\chi_{m_2}\right) \approx \mu_o$$

Very common for many (but not all) non-conducting linear/ homogeneous/isotropic media

Then $\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2} \simeq \left(\dfrac{v_1}{v_2}\right) = \left(\dfrac{n_2}{n_1}\right)$ for $\mu_1 \approx \mu_2 \approx \mu_o$ or $\left|\chi_{m_{1,2}}\right| \ll 1$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Then

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{1-(v_1/v_2)}{1+(v_1/v_2)}\right)\tilde{E}_{o_{inc}} = \left(\frac{v_2-v_1}{v_2+v_1}\right)\tilde{E}_{o_{inc}}$$

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{2}{1+(v_1/v_2)}\right)\tilde{E}_{o_{nc}} = \left(\frac{2v_2}{v_2+v_1}\right)\tilde{E}_{o_{inc}}$$

We can alternatively express these relations in terms of the indices of refraction n_1 & n_2:

$$\tilde{E}_{o_{refl}} = \left(\frac{n_1-n_2}{n_1+n_2}\right)\tilde{E}_{o_{inc}} \quad \text{and} \quad \tilde{E}_{o_{trans}} = \left(\frac{2n_1}{n_1+n_2}\right)\tilde{E}_{o_{inc}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now since:

$$\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}$$

$$\tilde{E}_{o_{refl}} = E_{o_{refl}} e^{i\delta}$$

$$\tilde{E}_{o_{trans}} = E_{o_{trans}} e^{i\delta}$$

δ = phase angle (in radians) defined at the zero of time - $t = 0$

Then for the purely real amplitudes $(E_{o_{inc}}, E_{o_{refl}}, E_{o_{trans}})$

these relations become:

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

for $\mu_1 \simeq \mu_2 \simeq \mu_o$

$$E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}$$

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}$$

for $\mu_1 \simeq \mu_2 \simeq \mu_o$

Monochromatic plane *EM wave at normal* incidence on a boundary between two linear / homogeneous / isotropic media

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, $\mu_1 \simeq \mu_2 \simeq \mu_o$ note the following points:

If $v_2 > v_1$ $(i.e.\ n_2 < n_1)$ $\{e.g.$ medium 1) = glass $\Rightarrow$ medium 2) = air$\}$:

$$E_{o_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}} \Rightarrow$$

$E_{o_{refl}}$ is precisely in-phase with $E_{o_{inc}}$ because $(v_2 - v_1) > 0$.

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

If $v_2 < v_1$ $(i.e.\ n_2 > n_1)$ $\{e.g.$ medium 1) = air $\Rightarrow$ medium 2) = glass$\}$:

$$E_{o_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}} \Rightarrow$$

$E_{o_{refl}}$ is 180° out-of-phase with $E_{o_{inc}}$ because $(v_2 - v_1) < 0$.

i.e.

$$E_{o_{refl}} = -\left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{o_{inc}} = -\left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{o_{inc}} \Rightarrow$$

The minus sign indicates a 180° phase shift occurs upon reflection for $v_2 < v_1$ (i.e. $n_2 > n_1$) !!!

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$E_{o_{trans}}$ is always in-phase with $E_{o_{inc}}$ for all possible v_1 & v_2 $(n_1$ & $n_2)$ because:

$$E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1+v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1+n_2}\right) E_{o_{inc}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

What fraction of the incident *EM* wave energy is reflected?

What fraction of the incident *EM* wave energy is transmitted?

In a given linear/homogeneous/isotropic medium with

$$v = \sqrt{\frac{\varepsilon_o \mu_o}{\varepsilon \mu}}\, c = c/n$$

The time-averaged energy density in the EM wave is:

$$\langle u_{EM}(\vec{r},t) \rangle = \frac{1}{2} \varepsilon E_o^2(\vec{r}) = \varepsilon E_{o_{rms}}^2(\vec{r}) \left(\frac{\text{Joules}}{\text{m}^3} \right)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The time-averaged Poynting's vector is:

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{\mu} \left\langle \vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \right\rangle \quad \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

The intensity of the EM wave is:

$$I(\vec{r}) \equiv \left\langle \left| \vec{S}(\vec{r},t) \right| \right\rangle = v \left\langle u_{EM}(\vec{r},t) \right\rangle = v \left(\frac{1}{2} \varepsilon E_o^2(\vec{r}) \right) = \frac{1}{2} \varepsilon v E_o^2(\vec{r}) = \varepsilon v E_{o_{rms}}^2(\vec{r}) \quad \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

Note that the three Poynting's vectors associated with this problem are such that

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$$\vec{S}_{inc} \parallel (+\hat{z}), \quad \vec{S}_{refl} \parallel (-\hat{z}) \text{ and } \vec{S}_{trans} \parallel (+\hat{z})$$

For a monochromatic plane *EM wave at normal incidence on a boundary between two linear* /homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

$$E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}$$

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Take the ratios $\left(E_{o_{refl}}/E_{o_{inc}}\right)$ and $\left(E_{o_{trans}}/E_{o_{inc}}\right)$ - then square them:

$$\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

and

$$\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \approx \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Define the reflection coefficient as:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \frac{\left\langle\left|\vec{S}_{refl}(\vec{r},t)\right|\right\rangle}{\left\langle\left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle} = \frac{v_1\left\langle u_{EM}^{refl}(\vec{r},t)\right\rangle}{v_1\left\langle u_{EM}^{inc}(\vec{r},t)\right\rangle} = \frac{\left\langle u_{EM}^{refl}(\vec{r},t)\right\rangle}{\left\langle u_{EM}^{inc}(\vec{r},t)\right\rangle} = \frac{\frac{1}{2}\varepsilon_1 v_1 E_{o_{refl}}^2(\vec{r})}{\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})} = \frac{E_{o_{refl}}^2(\vec{r})}{E_{o_{inc}}^2(\vec{r})}$$

Define the transmission coefficient as:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \frac{\left\langle\left|\vec{S}_{trans}(\vec{r},t)\right|\right\rangle}{\left\langle\left|\vec{S}_{inc}(\vec{r},t)\right|\right\rangle} = \frac{v_2\left\langle u_{EM}^{trans}(\vec{r},t)\right\rangle}{v_1\left\langle u_{EM}^{inc}(\vec{r},t)\right\rangle} = \frac{\left(\frac{1}{2}\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})\right)}{\left(\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})\right)} = \frac{\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})}{\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a linearly-polarized monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection coefficient:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})} \right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o\,inc}(\vec{r})} \right)^2$$

Transmission coefficient:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})} \right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \right) \left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})} \right)^2$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

But:

$$\left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \quad \&$$

$$\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \simeq \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Thus Reflection and Transmission coefficient:

$$R(\vec{r}) \equiv \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$T(\vec{r}) \equiv \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 \simeq \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2v_2}{v_2 + v_1}\right)^2 = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now: $$\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\dfrac{\varepsilon_2 \mu_2 v_2}{\mu_2}}{\dfrac{\varepsilon_1 \mu_1 v_1}{\mu_1}}$$

but: $$v_2^2 = \frac{1}{\varepsilon_2 \mu_2} \Rightarrow \varepsilon_2 \mu_2 = \frac{1}{v_2^2}$$
$$v_1^2 = \frac{1}{\varepsilon_1 \mu_1} \Rightarrow \varepsilon_1 \mu_1 = \frac{1}{v_1^2}$$

$$\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\left(\dfrac{1}{v_2^{\,2}} \cdot v_2\right) \Big/ \mu_2}{\left(\dfrac{1}{v_1^{\,2}} \cdot v_1\right) \Big/ \mu_2} = \frac{1 \big/ \mu_2 v_2}{1 \big/ \mu_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} \equiv \beta$$!!! *i.e.* $$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

for $\mu_1 \approx \mu_2 \approx \mu_o$

$$T(\vec{r}) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2+v_1)^2} = \frac{4n_1 n_2}{(n_1+n_2)^2}$$

Thus:

$$R(\vec{r})+T(\vec{r}) = \frac{(1-\beta)^2}{(1+\beta)^2} + \frac{4\beta}{(1+\beta)^2} = \frac{(1-\beta)^2+4\beta}{(1+\beta)^2} = \frac{1-2\beta+\beta^2+4\beta}{(1+\beta)^2} = \frac{1+2\beta+\beta^2}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1$$

$$R(\vec{r})+T(\vec{r}) = 1$$

⇒EM energy is conserved at the interface/boundary between two L/H/I media

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a linearly-polarized monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection coefficient:

$\mu_1 \simeq \mu_2 \simeq \mu_o$

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})}\right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{(1-\beta)^2}{(1+\beta)^2} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

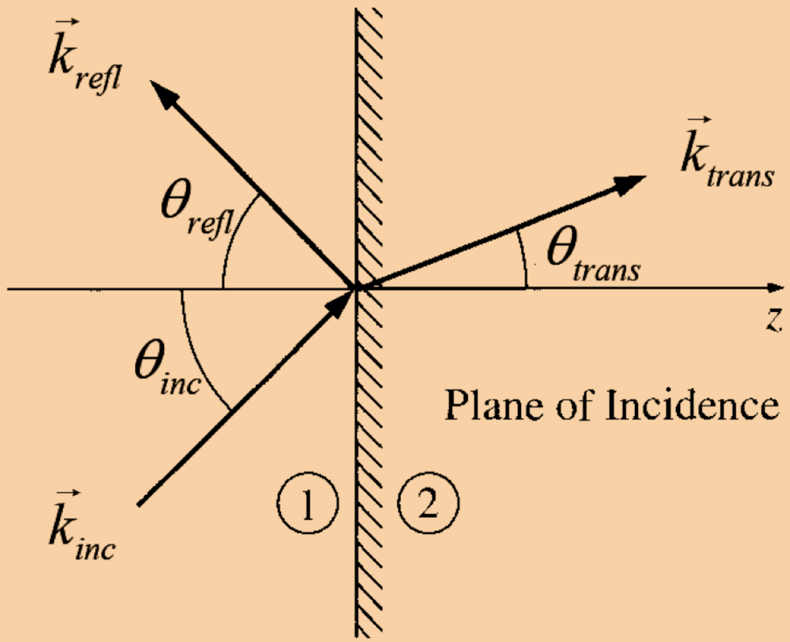
Transmission coefficient:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})}\right) = \beta\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2v_1}{(v_2+v_1)^2} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

$\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

A monochromatic plane EM wave incident at an oblique angle θ_{inc} on a boundary between two linear/homogeneous/isotropic media, defined with respect to the normal to the interface- as shown in the figure below:



Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

The incident EM wave is:

$$\vec{\tilde{E}}_{inc}(\vec{r},t) = \vec{\tilde{E}}_{o_{inc}} e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \vec{\tilde{B}}_{inc}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{inc}\times\vec{\tilde{E}}_{inc}(\vec{r},t)$$

The reflected EM wave is:

$$\vec{\tilde{E}}_{refl}(\vec{r},t) = \vec{\tilde{E}}_{o_{refl}} e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \vec{\tilde{B}}_{refl}(\vec{r},t) = \frac{1}{v_1}\hat{k}_{refl}\times\vec{\tilde{E}}_{refl}(\vec{r},t)$$

The transmitted EM wave is:

$$\vec{\tilde{E}}_{trans}(\vec{r},t) = \vec{\tilde{E}}_{o_{trans}} e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \vec{\tilde{B}}_{trans}(\vec{r},t) = \frac{1}{v_2}\hat{k}_{trans}\times\vec{\tilde{E}}_{trans}(\vec{r},t)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

All three EM waves have the same frequency- $f = \omega/2\pi$

$$\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2$$

$$k_{inc} = k_{refl} = k_1 = \left(\frac{v_2}{v_1}\right) k_{trans} = \left(\frac{v_2}{v_1}\right) k_2 = \left(\frac{n_1}{n_2}\right) k_{trans} = \left(\frac{n_1}{n_2}\right) k_2$$

$$v_i = c/n_i \quad i = 1,2$$

The total EM fields in medium 1

$$\tilde{\vec{E}}_{Tot_1}(\vec{r},t) = \tilde{\vec{E}}_{inc}(\vec{r},t) + \tilde{\vec{E}}_{refl}(\vec{r},t) \quad \text{and} \quad \tilde{\vec{B}}_{Tot_1}(\vec{r},t) = \tilde{\vec{B}}_{inc}(\vec{r},t) + \tilde{\vec{B}}_{refl}(\vec{r},t)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Must match to the total EM fields in medium 2:

$$\vec{\tilde{E}}_{Tot_2}(\vec{r},t) = \vec{\tilde{E}}_{trans}(\vec{r},t) \quad \text{and} \quad \vec{\tilde{B}}_{Tot_2}(\vec{r},t) = \vec{\tilde{B}}_{trans}(\vec{r},t)$$

Using the boundary conditions BC1) → BC4) at z = 0.

At $z = 0$- four boundary conditions are of the form:

$$(\sim)\, e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)} + (\sim)\, e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)} = (\sim)\, e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)}$$

They must hold for all (x,y) on the interface at z = 0 - and also must hold for all times, t. The above relation is already satisfied for arbitrary time, t - the factor $e^{-i\omega t}$ is common to all terms.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The following relation must hold for all (x,y) on interface at at z = 0:

$$(\sim)\, e^{i(\vec{k}_{inc}\cdot\vec{r})} + (\sim)\, e^{i(\vec{k}_{refl}\cdot\vec{r})} = (\sim)\, e^{i(\vec{k}_{trans}\cdot\vec{r})}$$

When *z* = *0* - at interface we must have:

$$\vec{k}_{inc} \bullet \vec{r} = \vec{k}_{refl} \bullet \vec{r} = \vec{k}_{trans} \bullet \vec{r}$$

$$k_{inc_x} x + k_{inc_y} y = k_{refl_x} x + k_{refl_y} y = k_{trans_x} x + k_{trans_y} y \qquad @\ z = 0$$

The above relation can only hold for arbitrary (x, y, z = 0) **iff (= if and only if):**

Reflection & Transmission of Monochromatic Plane *EM Waves* *at Oblique Incidence*

The above relation can only hold for arbitrary (x, y, z = 0) **iff (= if and only if):**

$$k_{inc_x} x = k_{refl_x} x = k_{trans_x} x \quad \Rightarrow \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

$$k_{inc_y} y = k_{refl_y} y = k_{trans_y} y \quad \Rightarrow \quad k_{inc_y} = k_{refl_y} = k_{trans_y}$$

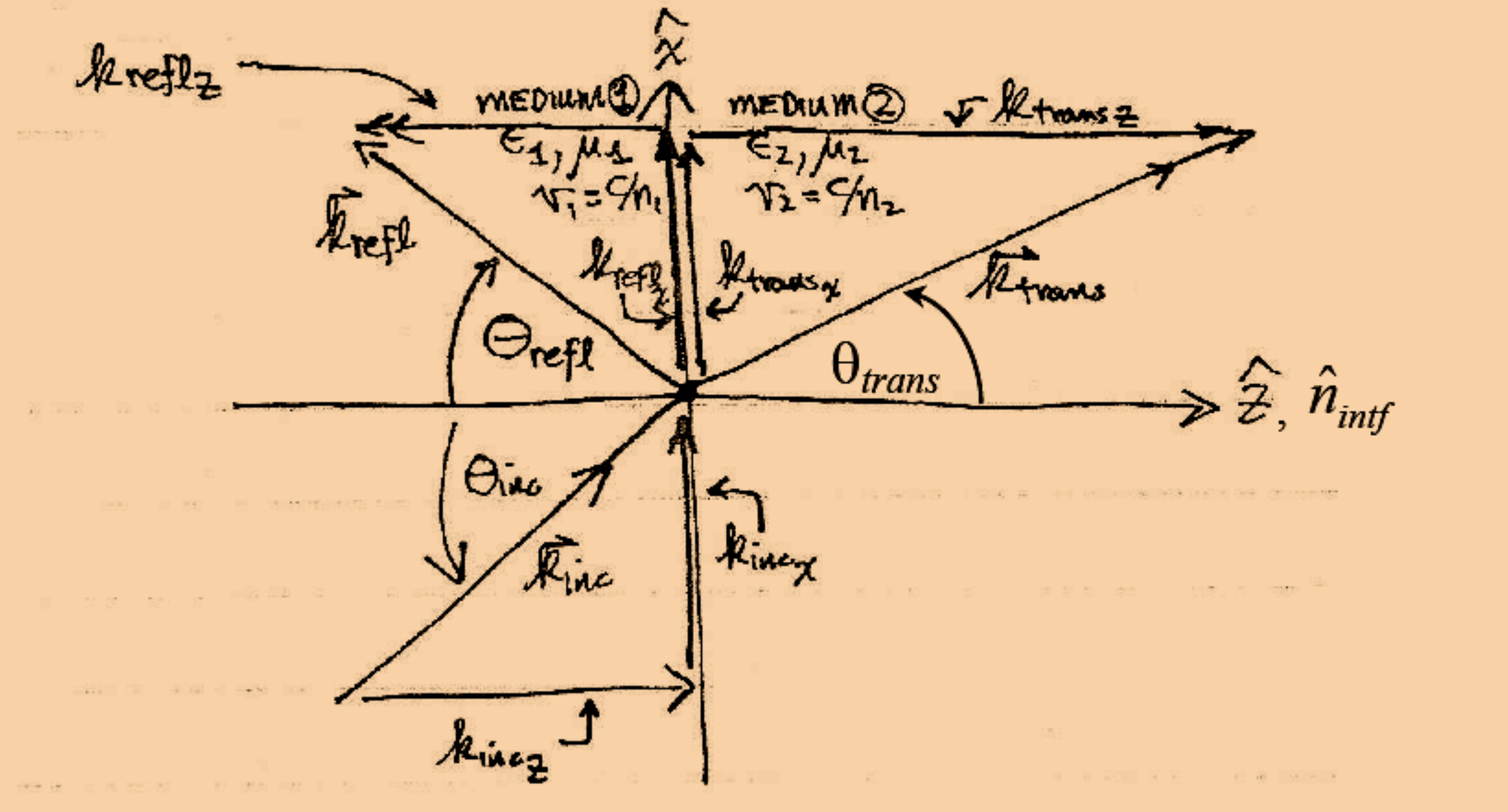
Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The problem has rotational symmetry about the z –axis- then without any loss of generality we can choose k to lie entirely within the x-z plane, as shown in the figure

$$k_{inc_y} = k_{refl_y} = k_{trans_y} = 0 \quad \text{and thus:} \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

The transverse components of $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ are all equal and point in the +x^ direction.

Reflection & Transmission of Monochromatic Plane *EM Waves* at Oblique Incidence



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The First Law of Geometrical Optics:

The incident, reflected, and transmitted wave vectors form a plane (called the plane of incidence), which also includes the normal to the surface (here, the z axis).

The Second Law of Geometrical Optics (Law of Reflection):

From the figure, we see that:

$$\boxed{k_{inc_x} = k_{inc}\sin\theta_{inc}} = \boxed{k_{refl_x} = k_{refl}\sin\theta_{refl}} = \boxed{k_{trans_x} = k_{trans}\sin\theta_{trans}}$$

$$\boxed{k_{inc} = k_{refl} = k_1} \Rightarrow \boxed{\sin\theta_{inc} = \sin\theta_{refl}}$$

Angle of Incidence = Angle of Reflection $\boxed{\theta_{inc} = \theta_{refl}}$ Law of Reflection!

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Third Law of Geometrical Optics (Law of Refraction – Snell's Law):

For the transmitted angle, θ_{trans} we see that:

$$k_{inc} \sin\theta_{inc} = k_{trans} \sin\theta_{trans}$$

In medium 1): $k_{inc} = k_1 = \omega/v_1 = n_1\omega/c = n_1 k_o$

where $k_o =$ vacuum wave number $= 2\pi/\lambda_o$

and $\lambda_o =$ vacuum wave length

In medium 2): $k_{trans} = k_2 = \omega/v_2 = n_2\omega/c = n_2 k_o$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$k_{inc} \sin\theta_{inc} = k_{trans} \sin\theta_{trans} \Rightarrow k_1 \sin\theta_{inc} = k_2 \sin\theta_{trans}$$

$$k_{inc} = k_1 = n_1 k_o \text{ and } k_{trans} = k_2 = n_2 k_o$$

$$k_1 \sin\theta_{inc} = k_2 \sin\theta_{trans} \Rightarrow n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}$$

Law of Refraction
(Snell's Law)

Which can also be written as: $$\frac{\sin\theta_{trans}}{\sin\theta_{inc}} = \frac{n_1}{n_2}$$

Since θ_{trans} refers to medium 2) and θ_{inc} refers to medium 1)

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

(incident) (transmitted)

or: $$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Because of the three laws of geometrical optics, we see that:

$$\vec{k}_{inc} \bullet \vec{r}\Big|_{z=0} = \vec{k}_{refl} \bullet \vec{r}\Big|_{z=0} = \vec{k}_{trans} \bullet \vec{r}\Big|_{z=0}$$

everywhere on the interface at *z = 0* *{in the x-y plane}*

Thus we see that: $$e^{i\left(\vec{k}_{inc} \bullet \vec{r} - \omega t\right)}\Big|_{z=0} = e^{i\left(\vec{k}_{refl} \bullet \vec{r} - \omega t\right)}\Big|_{z=0} = e^{i\left(\vec{k}_{trans} \bullet \vec{r} - \omega t\right)}\Big|_{z=0}$$

everywhere on the interface at z = 0 {in the x-y plane}, valid also for arbitrary/any/all time(s) t, since ω is the same in either medium (1 or 2).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The BC 1) → BC 4) for a monochromatic plane *EM* wave incident on an interface at an oblique angle between two linear/homogeneous/isotropic media become:

BC 1): Normal (*z-)* component of ***D*** continuous at $z = 0$ (no free surface charges):

$$\varepsilon_1\left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}}\right) = \varepsilon_2 \tilde{E}_{o_{trans_z}} \qquad \left\{\text{using } \vec{D} = \varepsilon\vec{E}\right\}$$

BC 2): Tangential (*x-, y-)* components of ***E*** continuous at $z = 0$:

$$\left(\tilde{E}_{o_{inc_{x,y}}} + \tilde{E}_{o_{refl_{x,y}}}\right) = \tilde{E}_{o_{trans_{x,y}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3): Normal (z-) component of ***B*** continuous at *z = 0:*

$$\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}$$

BC 4): Tangential (x-, y-) components of ***H*** continuous at *z = 0* (no free surface currents):

$$\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_{x,y}}} + \tilde{B}_{o_{refl_{x,y}}}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_{x,y}}}$$

Note that in each of the above, we also have the relation

$$\vec{\tilde{B}}_o = \frac{1}{v}\hat{k} \times \vec{\tilde{E}}_o$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For a monochromatic plane EM wave incident on a boundary between two L / H/ I media at an oblique angle of incidence, there are three possible polarization cases to consider:

Case I): $\vec{E}_{inc} \perp$ plane of incidence

$\{\vec{B}_{inc} \parallel$ plane of incidence$\}$

Transverse Electric (TE) Polarization

Case II): $\vec{E}_{inc} \parallel$ plane of incidence

$\{\vec{B}_{inc} \perp$ plane of incidence$\}$

Transverse Magnetic (TM) Polarization

Case III): <u>The most general case</u>: $\vec{E}_{inc}$ is neither $\perp$ nor $\parallel$ to the <u>plane of incidence</u>.

$\{\Rightarrow \vec{B}_{inc}$ is neither $\parallel$ nor $\perp$ to the <u>plane of incidence</u>$\}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Case I): Electric Field Vectors Perpendicular to the Plane of Incidence: Transverse Electric (*TE*) *Polarization*

- A monochromatic plane EM wave is incident on a boundary at z = 0 -in the x-y plane between two L/H/I media - at an oblique angle of incidence.
- The polarization of the incident EM wave is transverse (⊥) to the plane of incidence {containing the three wave-vectors and the unit normal to the boundary $\hat{n} = +\hat{z}$ }).
- The three B-field vectors are related to their respective E -field vectors by the right hand rule - all three B-field vectors lie in the x-z plane {the plane of incidence},

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the {complex} *E* and *B* fields on the boundary at z = 0 are:

BC 1) Normal (*z-*) component of D continuous at z = 0 (no free surface charges)

$$\varepsilon_1\left(\underbrace{\tilde{E}_{o_{inc_z}}}_{=0} + \underbrace{\tilde{E}_{o_{refl_z}}}_{=0}\right) = \varepsilon_2 \underbrace{\tilde{E}_{o_{trans_z}}}_{=0} \Rightarrow \boxed{0+0=0}$$

BC 2) Tangential (*x-*, *y-*) components of E continuous at *z = 0:*

$$\left(\tilde{E}_{o_{inc_y}} + \tilde{E}_{o_{refl_y}}\right) = \tilde{E}_{o_{trans_y}} \Rightarrow \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (*z-)* component of B continuous at z = 0:

$$\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}$$

$$\hat{k}_{inc} = \hat{k}_{inc_x} + \hat{k}_{inc_z} = \sin\theta_{inc}\hat{x} + \cos\theta_{inc}\hat{z}$$

$$\hat{k}_{refl} = \hat{k}_{refl_x} + \hat{k}_{refl_z} = \sin\theta_{refl}\hat{x} - \cos\theta_{refl}\hat{z}$$

$$\hat{k}_{trans} = \hat{k}_{trans_x} + \hat{k}_{trans_z} = \sin\theta_{trans}\hat{x} + \cos\theta_{trans}\hat{z}$$

$$\left(\tilde{B}_{o_{inc_z}}\hat{z} + \tilde{B}_{o_{refl_z}}\hat{z}\right) = \tilde{B}_{o_{trans_z}}\hat{z} = \frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\right)\hat{z} = \frac{1}{v_2}\tilde{E}_{o_{trans}}\sin\theta_{trans}\hat{z}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 4) Tangential (*x*-, *y*-) components of H continuous at z = 0 (no free surface currents):

$$\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_x}}\hat{x}+\tilde{B}_{o_{refl_x}}\hat{x}\right)=\frac{1}{\mu_2}\tilde{B}_{o_{trans_x}}\hat{x}$$

$$=\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}(-\cos\theta_{inc})+\tilde{E}_{o_{refl}}\cos\theta_{refl}\right)\hat{x}=\frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}(-\cos\theta_{trans})\hat{x}$$

$$\tilde{E}_{o_{inc}}+\tilde{E}_{o_{refl}}=\tilde{E}_{o_{trans}} \quad \text{(from BC 2))}$$

Using the Law of Reflection on the BC 3) result:

$$\tilde{E}_{o_{inc}}+\tilde{E}_{o_{refl}}=\left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using Snell's Law / Law of Refraction:

$$\boxed{n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}} \Rightarrow \boxed{\frac{n_1}{c}\sin\theta_{inc} = \frac{n_2}{c}\sin\theta_{trans}} \Rightarrow \boxed{\frac{1}{v_1}\sin\theta_{inc} = \frac{1}{v_2}\sin\theta_{trans}}$$

or: $\boxed{v_2 \sin\theta_{inc} = v_1 \sin\theta_{trans}}$ or: $\boxed{\left(\frac{v_1}{v_2}\cdot\frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right) = 1}$

From BC 1) → BC 4) actually have only two independent relations for the case of transverse electric (TE) polarization:

1) $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

2) $\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\cdot\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Now we define:

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

$$\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)$$

Then eqn. 2) becomes:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \alpha\beta\,\tilde{E}_{o_{trans}}$$

Adding and subtracting Eqn's 1 &2 to get:

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}} \quad \text{eqn. (1+2)}$$

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\tilde{E}_{o_{trans}} \quad \text{eqn. (2–1)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Plug eqn. (2+1) into eqn. (2−1) to obtain:

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}$$

$$\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) \quad \text{and} \quad \frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\frac{2}{1+\alpha\beta}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations for $\vec{E} \parallel$ to Interface

$= \vec{E} \perp$ Plane of Incidence = Transverse Electric (TE) Polarization

$$E_{o_{refl}}^{TE} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right) E_{o_{inc}}^{TE} \quad \text{and} \quad E_{o_{trans}}^{TE} = \left(\frac{2}{1+\alpha\beta}\right) E_{o_{inc}}^{TE}$$

with

$$\alpha \equiv \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) \quad \text{and} \quad \beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For *TE polarization:*

Incident Intensity

$$I_{inc}^{TE} = \left|\left\langle \vec{S}_{inc}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{inc}}^{TE}\right)^2\right)\left|\hat{k}_{inc}\bullet\hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{inc}}^{TE}\right)^2\right)\cos\theta_{inc} = \frac{1}{2}\varepsilon_1 v_1\left(E_{o_{inc}}^{TE}\right)^2\cos\theta_{inc}$$

Reflection Intensity

$$I_{refl}^{TE} = \left|\left\langle \vec{S}_{refl}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{refl}}^{TE}\right)^2\right)\cos\theta_{refl} = \frac{1}{2}\varepsilon_1 v_1\left(E_{o_{refl}}^{TE}\right)^2\cos\theta_{inc}$$

Transmission Intensity

$$I_{trans}^{TE} = \left|\left\langle \vec{S}_{trans}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_2\varepsilon_2\left(E_{o_{trans}}^{TE}\right)^2\right)\cos\theta_{trans} = \frac{1}{2}\varepsilon_2 v_2\left(E_{o_{trans}}^{TE}\right)^2\cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients for transverse electric (*TE*) *polarization*

$$R_{TE} \equiv \frac{I_{refl}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{refl}}^{TE}\right)^2 \cos \overbrace{\theta_{inc}}^{=\theta_{refl}}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2$$

$$T_{TE} \equiv \frac{I_{trans}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_2 v_2 \left(E_{o_{trans}}^{TE}\right)^2 \cos\theta_{trans}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The reflection and transmission coefficients for transverse electric (*TE*) polarization

$$R_{TE} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}} \right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta} \right)^2$$

$$T_{TE} = \alpha\beta \left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}} \right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

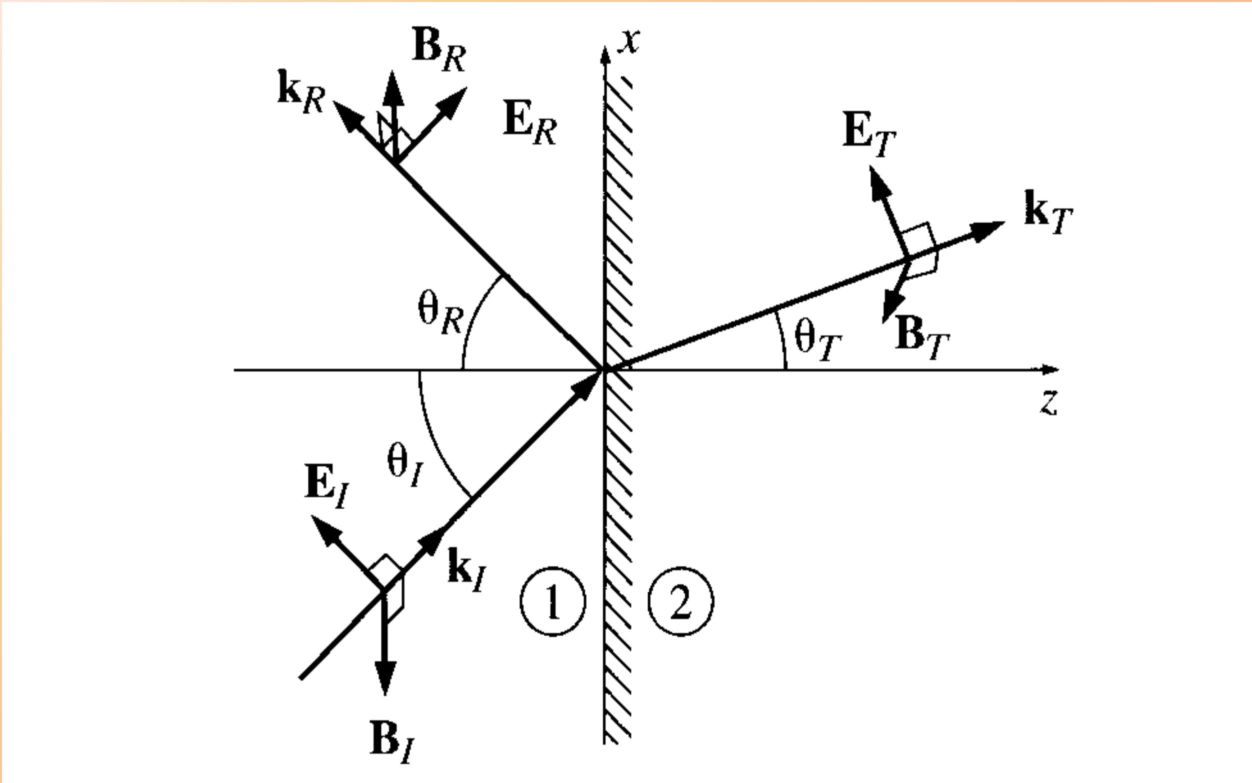
Case II): Electric Field Vectors Parallel to the Plane of Incidence: Transverse Magnetic (*TM) Polarization*

•A monochromatic plane EM wave is incident on a boundary at z = 0 in the x-y plane between two L / H/ I media at an oblique angle of incidence.

•The polarization of the incident EM wave is now parallel to the plane of incidence {containing the three wavevectors and the unit normal to the boundary $\hat{n} = +\hat{z}$ }).

• The three B -field vectors are related to E -field vectors by the right hand rule –then all three B-field vectors are ⊥ to the plane of incidence {hence the origin of the name transverse magnetic polarization}.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the {complex}E and B-fields on the boundary at z = 0 are:

BC 1) Normal (z-) component of D continuous at z = 0 (no free surface charges)

$$\varepsilon_1\left(\tilde{E}_{o_{inc_z}} + \tilde{E}_{o_{refl_z}}\right) = \varepsilon_2 \tilde{E}_{o_{trans_z}}$$

$$\varepsilon_1\left(-\tilde{E}_{o_{inc}} \sin\theta_{inc} + \tilde{E}_{o_{refl}} \sin\theta_{refl}\right) = \varepsilon_2\left(-\tilde{E}_{o_{trans}} \sin\theta_{trans}\right)$$

BC 2) Tangential (x-, y-) components of E continuous at *z = 0:*

$$\left(\tilde{E}_{o_{inc_x}} + \tilde{E}_{o_{refl_x}}\right) = \tilde{E}_{o_{trans_x}}$$

$$\left(\tilde{E}_{o_{inc}} \cos\theta_{inc} + \tilde{E}_{o_{refl}} \cos\theta_{refl}\right) = \tilde{E}_{o_{trans}} \cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (*z-*) *component of B* **continuous at** ***z = 0:***

$$\left(\cancelto{=0}{\tilde{B}_{o_{inc_z}}} + \cancelto{=0}{\tilde{B}_{o_{refl_z}}} \right) = \cancelto{=0}{\tilde{B}_{o_{trans_z}}} \quad \Rightarrow \quad \boxed{0 + 0 = 0}$$

BC 4) Tangential (x-, y-) components of H continuous at z = 0 (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right)} \quad \Rightarrow \quad \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 1) at $z = 0$:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\varepsilon_2}{\varepsilon_1} \frac{n_1}{n_2} \right) \tilde{E}_{o_{trans}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \right) \tilde{E}_{o_{trans}} = \beta \, \tilde{E}_{o_{trans}}$$

From BC 4) at $z = 0$:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\mu_1 v_1}{\mu_2 v_2} \right) \tilde{E}_{o_{trans}} = \beta \tilde{E}_{o_{trans}}$$

where:

$$\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2} \right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 2) at *z* = *0:*

$$\left(\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}}\right) = \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \alpha\tilde{E}_{o_{trans}}$$

where:

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

Thus for the case of transverse magnetic (TM) polarization:

$$\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\tilde{E}_{o_{trans}} \quad \text{and} \quad \tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \alpha\tilde{E}_{o_{trans}}$$

Solving these two above equations simultaneously, we obtain:

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}$$

$$\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{2}\right)\tilde{E}_{o_{trans}}$$

$$\tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}$$

The Fresnel Equations for $\vec{B} \parallel$ to Interface

$= \vec{B} \perp$ Plane of Incidence = Transverse Magnetic (TM) Polarization

$$\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right) \quad \text{and} \quad \left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{2}{\alpha+\beta}\right)$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflected & transmitted intensities at oblique incidence for the *TM case*

$$I_{inc}^{TM} = v_1 \left| \left\langle \vec{S}_{inc}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{inc}}^{TM} \right)^2 \right) \cos\theta_{inc} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{inc}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{refl}^{TM} = v_1 \left| \left\langle \vec{S}_{refl}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{refl}}^{TM} \right)^2 \right) \cos\theta_{refl} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{refl}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{trans}^{TM} = v_2 \left| \left\langle \vec{S}_{trans}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_2 \varepsilon_2 \left(E_{o_{trans}}^{TM} \right)^2 \right) \cos\theta_{trans} = \frac{1}{2} \varepsilon_2 v_2 \left(E_{o_{trans}}^{TM} \right)^2 \cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients

$$R_{TM} = \left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}} \right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2$$

$$T_{TM} = \alpha\beta \left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}} \right)^2 = \frac{4\alpha\beta}{(\alpha + \beta)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{2}{(1+\alpha\beta)}$$

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{2}{(\alpha+\beta)}$$

$$v_1 = c/n_1 = 1/\sqrt{\varepsilon_1\mu_1}$$

$$v_2 = c/n_2 = 1/\sqrt{\varepsilon_2\mu_2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission Coefficients *R* & *T*

$$R + T = 1$$

TE Polarization

$$R_{TE} \equiv \frac{I_{refl}^{TE}}{I_{inc}^{TE}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T_{TE} \equiv \left(\frac{I_{trans}^{TE}}{I_{inc}^{TE}}\right) = \alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$$

TM Polarization

$$R_{TM} \equiv \frac{I_{refl}^{TM}}{I_{inc}^{TM}} = \left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T_{TM} \equiv \left(\frac{I_{trans}^{TM}}{I_{inc}^{TM}}\right) = \alpha\beta\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$v_1 = c/n_1 = 1/\sqrt{\varepsilon_1\mu_1}$$

$$v_2 = c/n_2 = 1/\sqrt{\varepsilon_2\mu_2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Alternate versions of the Fresnel Relations

Fresnel Equations

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} - \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} + \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) = \frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc} + \left(\frac{n_2}{\mu_2}\right)\cos\theta_{trans}}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} - \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} + \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) = \frac{2\left(\frac{n_1}{\mu_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{\mu_2}\right)\cos\theta_{inc} + \left(\frac{n_1}{\mu_1}\right)\cos\theta_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Ignoring the magnetic properties of the two media

$|\chi_m| \ll 1$ then $\mu_1 \simeq \mu_2 \simeq \mu_o$ the Fresnel Relations become:

TE Polarization

$$\left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{n_1 \cos\theta_{inc} - n_2 \cos\theta_{trans}}{n_1 \cos\theta_{inc} + n_2 \cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right) \simeq \frac{2n_1 \cos\theta_{inc}}{n_1 \cos\theta_{inc} + n_2 \cos\theta_{trans}}$$

TM Polarization

$$\left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{-n_2 \cos\theta_{inc} + n_1 \cos\theta_{trans}}{n_2 \cos\theta_{inc} + n_1 \cos\theta_{trans}}$$

$$\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right) \simeq \frac{2n_1 \cos\theta_{inc}}{n_2 \cos\theta_{inc} + n_1 \cos\theta_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using Snell's Law and various trigonometric identities

TE Polarization

$$\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) \simeq -\frac{\sin(\theta_{inc}-\theta_{trans})}{\sin(\theta_{inc}+\theta_{trans})}$$

$$\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin(\theta_{inc}+\theta_{trans})}$$

TM Polarization

$$\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) \simeq -\frac{\tan(\theta_{inc}-\theta_{trans})}{\tan(\theta_{inc}+\theta_{trans})}$$

$$\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{trans}}}\right) \simeq \frac{2\cos\theta_{inc}\bullet\sin\theta_{trans}}{\sin(\theta_{inc}+\theta_{trans})\cos(\theta_{inc}-\theta_{trans})}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Use Snell's Law $n_{inc} \sin\theta_{inc} = n_{trans} \sin\theta_{trans}$ to eliminate θ_{trans} :

TE Polarization

$$\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{\cos\theta_{inc} - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

$$\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right) \simeq \frac{2\cos\theta_{inc}}{\cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

TM Polarization

$$\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) \simeq \frac{-\left(\frac{n_2}{n_1}\right)^2 \cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}{\left(\frac{n_2}{n_1}\right)^2 \cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

$$\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) \simeq \frac{2\left(\frac{n_2}{n_1}\right)\cos\theta_{inc}}{\left(\frac{n_2}{n_1}\right)^2 \cos\theta_{inc} + \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2\theta_{inc}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves* at Oblique Incidence

- Now explore the physics associated with the Fresnel Equations -the reflection and transmission coefficients.
- Comparing results for TE vs. TM polarization for the cases of external reflection (n1 < n2) and internal reflection n1 > n2)

Comment 1):

- When $(E_{refl} / E_{inc}) < 0$ - E_{orefl} *is* 180^o out-of-phase with E_{oinc} since the numerators of the original Fresnel Equations for TE & TM polarization are $(1-\alpha\beta)$ and $(\alpha-\beta)$ respectively.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Comment 2):

- For TM Polarization (only)- there exists an angle of incidence where $(E_{refl} / E_{inc}) = 0$ - no reflected wave occurs at this angle for TM polarization!
- This angle is known as Brewster's angle θ_B (also known as the polarizing angle θ_P - because an incident wave which is a linear combination of TE and TM polarizations will have a reflected wave which is 100% pure-TE polarized for an incidence angle $\theta_{inc} = \theta_B = \theta_P$!!).
- Brewster's angle θ_B exists for both external $(n_1 < n_2)$ & internal reflection $(n_1 > n_2)$ for TM polarization (only).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B/ *the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

From the numerator of $\left(E_{0_{refl}}^{TM} / E_{0_{inc}}^{TM}\right) = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)$ -the originally-derived expression for TM polarization- when this ratio = 0 at Brewster's angle θ_B = polarizing angle θ_P - this occurs when $(\alpha - \beta) = 0$, i.e. when $\alpha = \beta$.

$\cos\theta_{trans} = \sqrt{1 - \sin^2\theta_{trans}}$ and Snell's Law: $\sin\theta_{trans} = \left(\frac{n_1}{n_2}\right)\sin\theta_{inc}$

$$\alpha = \frac{\sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc}}}{\cos\theta_{inc}} \simeq \left(\frac{n_2}{n_1}\right) = \beta$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B / *the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

$$1-\frac{1}{\beta^2}\sin^2\theta_{inc} = \beta^2\cos^2\theta_{inc} = \beta^2\left(1-\sin^2\theta_{inc}\right) \leftarrow \text{Solve for } \sin^2\theta_{inc}$$

$$1-\beta^2 = \left(\frac{1}{\beta^2}-\beta^2\right)\sin^2\theta_{inc} \Rightarrow \sin^2\theta_{inc} = \frac{1-\beta^2}{1/\beta^2-\beta^2} = \frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^4\right)}$$

$$1-\beta^4 = \left(1-\beta^2\right)\left(1+\beta^2\right)$$

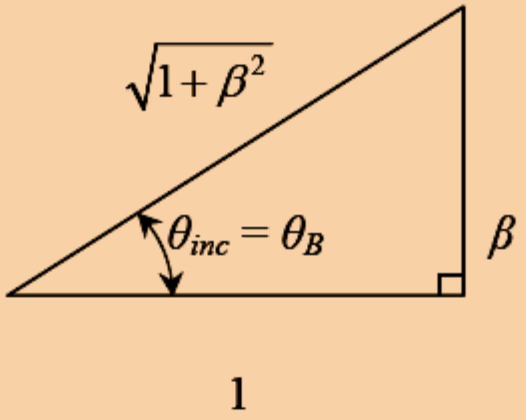
$$\sin^2\theta_{inc} = \frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^2\right)\left(1+\beta^2\right)} = \frac{\beta^2}{1+\beta^2} \Rightarrow \sin\theta_{inc} = \frac{\beta}{\sqrt{1+\beta^2}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Geometrically: $\boxed{\sin\theta_{inc} = \dfrac{\beta}{\sqrt{1+\beta^2}}} = \boxed{\dfrac{\text{opp. side}}{\text{hypotenuse}}}$

$\boxed{\cos\theta_{inc} = \dfrac{1}{\sqrt{1+\beta^2}}} = \boxed{\dfrac{\text{adjacent}}{\text{hypotenuse}}}$

$\boxed{\tan\theta_{inc} = \beta} = \boxed{\dfrac{\text{opp. side}}{\text{adjacent}}} \simeq \left(\dfrac{n_2}{n_1}\right)$



Thus, at an angle of incidence $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ = Brewster's angle / the polarizing angle for a *TM* polarized incident wave, where <u>*no*</u> <u>*reflected*</u> wave exists, we have:

$$\boxed{\tan\theta_B^{inc} \equiv \tan\theta_P^{inc} \simeq \left(\frac{n_2}{n_1}\right)} \quad \text{for} \quad \mu_1 \simeq \mu_2 \simeq \mu_o$$

From Snell's Law: $n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}$ we also see that: $\tan\theta_B^{inc} = \dfrac{\sin\theta_B^{inc}}{\cos\theta_B^{inc}} \simeq \dfrac{n_2}{n_1}$

or: $n_1 \sin\theta_B^{inc} \simeq n_2 \cos\theta_B^{inc}$ for $\mu_1 \simeq \mu_2 \simeq \mu_o$.

Thus, from Snell's Law we see that: $\cos\theta_B^{inc} = \sin\theta_{trans}$ when $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

So what's so interesting about this???

Well: $\cos\theta_B^{inc} = \sin\left(\frac{\pi}{2} - \theta_B^{inc}\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta_B^{inc} - \overbrace{\cos\left(\frac{\pi}{2}\right)}^{=0}\sin\theta_B^{inc} = \sin\theta_{trans}$ *i.e.* $\sin\left(\frac{\pi}{2} - \theta_B^{inc}\right) = \sin\theta_{trans}$

$\therefore$ When $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ for an incident *TM*-polarized *EM* wave, we see that $\theta_{trans} = \pi/2 - \theta_B^{inc}$

Thus: $\theta_B^{inc} + \theta_{trans} = \pi/2$, *i.e.* $\theta_B^{inc} \equiv \theta_P^{inc}$ and θ_{trans} are *complimentary* angles !!!

Comment 3):

For internal reflection ($n_1 > n_2$) there exists a critical angle of incidence past which no transmitted beam exists for either TE or TM polarization. The critical angle does not depend on polarization – it is actually dictated / defined by Snell's Law:

$n_1 \sin\theta_{critical}^{inc} = n_2 \sin\theta_{trans}^{max} = n_2 \sin\left(\frac{\pi}{2}\right) = n_2$ or: $\sin\theta_{critical}^{inc} = \left(\frac{n_2}{n_1}\right)$ or: $\theta_{critical}^{inc} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For $\theta_{inc} \geq \theta_{critical}^{inc}$, no transmitted beam exists → incident beam is totally internally reflected.

For $\theta_{inc} > \theta_{critical}^{inc}$, the transmitted wave is actually exponentially damped – becomes a so-called:

Evanescent Wave:

$$\vec{\tilde{E}}_{trans}(\vec{r},t) = \vec{\tilde{E}}_{o_{trans}} \underbrace{e^{-\alpha z}}_{\text{Exp. damping in } z} \underbrace{e^{i\left(k_2 x \sin\theta_{inc}\left(\frac{n_1}{n_2}\right) - \omega t\right)}}_{\text{Oscillatory along interface in } x\text{-direction}}$$

$$\alpha = k_2 \sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc} - 1}$$

Exp. damping in z Oscillatory along interface in x-direction

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's angle for *TE polarization:*

$$\theta^{B}_{\substack{inc\\TE}} = \sin^{-1}\sqrt{\frac{\left(\dfrac{\varepsilon_2}{\varepsilon_1}\right)-\left(\dfrac{\mu_2}{\mu_1}\right)}{\left(\dfrac{\mu_1}{\mu_2}\right)-\left(\dfrac{\mu_2}{\mu_1}\right)}} = \sin^{-1}\sqrt{A}$$

$$\sin\theta^{B}_{inc} = \sqrt{\frac{\left(\dfrac{\varepsilon_2}{\varepsilon_1}\right)-\left(\dfrac{\mu_2}{\mu_1}\right)}{\left(\dfrac{\mu_1}{\mu_2}\right)-\left(\dfrac{\mu_2}{\mu_1}\right)}} \equiv \sqrt{A}$$

i.e.

$$A \equiv \left[\frac{\left(\dfrac{\varepsilon_2}{\varepsilon_1}\right)-\left(\dfrac{\mu_2}{\mu_1}\right)}{\left(\dfrac{\mu_1}{\mu_2}\right)-\left(\dfrac{\mu_2}{\mu_1}\right)}\right]$$

References

1. CLASSICAL ELECTRODYNAMICS
   By J. D. Jackson (WILEY)

2. INTRODUCTION TO ELECTRODYNAMICS
   By David. J. Griffiths (PRENTICE HALL)

THANK YOU