The Abdus Salam
International Centre
for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

2442-1

Preparatory School to the Winter College on Optics

28 January - 1 February, 2013

Brief review of Fourier transforms

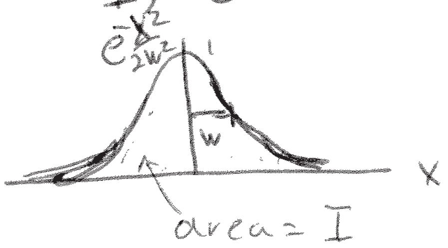
M. Bertolotti
Universita' di Roma "La Sapienza"
Roma
Italy

M. Alonso
University of Rochester
U.S.A.

Brief Review of Fourier Transforms

Before defining the Fourier transform, let us go through two derivations that will be very useful.

1) Gaussian integral



$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx$$

To solve, consider instead

$$I^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2w^2}} dy = \iint_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2w^2}} dx\,dy$$

let $r = \sqrt{x^2+y^2}$, $\tan\phi = y/x$, $dx\,dy = r\,dr\,d\phi$

$$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2w^2}} r\,d\phi\,dr = 2\pi \int_0^{\infty} e^{-\frac{r^2}{2w^2}} r\,dr$$

$u = \frac{r^2}{2}$, $du = r\,dr$

$$= 2\pi \int_0^{\infty} e^{-\frac{u}{w^2}} du = -2\pi w^2 e^{-\frac{u}{w^2}}\Big|_0^{\infty} = -2\pi w^2 (0 - 1)$$

assuming $Re[w^2] > 0$

$$I^2 = 2\pi w^2.$$

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}} dx = \sqrt{2\pi}\, w$$

2) Dirac's delta as the integral of an exponential

consider $I(x) = \int_{-\infty}^{\infty} e^{iKxp}\,dp$, K = real constant

To force the integral to converge, insert "unity" as

$$1 = \lim_{q\to\infty} e^{-\frac{p^2}{2q^2}}$$

$$I(x) = \lim_{q\to\infty} \int_{-\infty}^{\infty} 1\, e^{iKxp}\,dp = \lim_{q\to\infty} \int_{-\infty}^{\infty} e^{-\frac{p^2}{2q^2} + iKxp}\,dp$$

$$= \lim_{q\to\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2q^2}\left[p^2 - 2p\,iKxq^2\right]}\,dp = \lim_{q\to\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2q^2}\left[p^2 - 2p\,iKxq^2 + (iKxq^2)^2\right]}\,dp\, e^{\frac{[iKxq^2]^2}{2q^2}}$$

these two balance each other

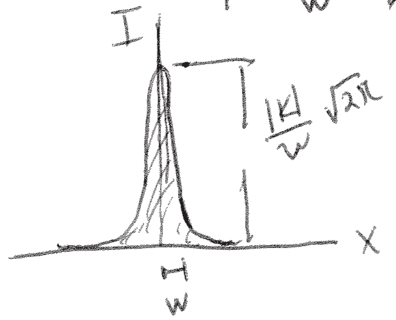
$$= \lim_{q\to\infty} \int_{-\infty}^{\infty} e^{-\frac{(p - iKxq^2)^2}{2q^2}}\,dp\, e^{-\frac{K^2q^2x^2}{2}}$$

$\tau = p - iKxq^2$, $d\tau = dp$

$$= \lim_{q\to\infty} \underbrace{\int_{-\infty - iKxq^2}^{\infty - iKxq^2} e^{-\frac{\tau^2}{2q^2}}\,d\tau}\; e^{-\frac{K^2q^2x^2}{2}} = \lim_{q\to\infty} \sqrt{2\pi}\, \frac{e^{-\frac{K^2q^2x^2}{2}}}{q}$$

can use result from previous page with "$\tau \to x$", "$q \to w$"!
The small imaginary parts of the limits of the integral have no effect because the integrand $\to 0$

Let $K^2q^2 = \frac{1}{w^2}$, then $I(x) = \lim_{w\to 0} \sqrt{2\pi}\, e^{-\frac{x^2}{2w^2}} \frac{w}{|K|}$



Area $= \sqrt{2\pi}\,\frac{w}{|K|} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2w^2}}\,dx = \frac{2\pi}{|K|}$

so $I(x) = \int_{-\infty}^{\infty} e^{iKxp}\,dp = \frac{2\pi}{|K|}\,\delta(x)$

We now define the Fourier transform as

$$\tilde{f}(p) = \hat{\mathcal{F}}_{x\to p} f(x) = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iKxp}\, dx$$

where K is a real constant chosen for convenience or convention. Common choices are:

$$K = \begin{cases} \pm 2\pi \\ k = \omega/c = \frac{2\pi}{\lambda} \text{ (wavenumber)} \\ \pm 1 \\ \hbar^{-1} = \frac{2\pi}{h} \text{ (in quantum mechanics).} \end{cases}$$

To find the inverse, consider

$$\int_{-\infty}^{\infty} \tilde{f}(p)\, e^{iKxp}\, dp = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')\, e^{-iKx'p}\, dx'\, e^{iKxp}\, dp$$

substitute

change order of $\int$.

$$= \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} f(x') \underbrace{\int_{-\infty}^{\infty} e^{iK(x-x')p}\, dp}_{\frac{2\pi}{|K|}\delta(x-x')}\, dx' = \sqrt{\frac{2\pi}{|K|}} \underbrace{\int_{-\infty}^{\infty} f(x')\,\delta(x'-x)\, dx'}_{f(x).}$$

therefore

$$f(x) = \hat{\mathcal{F}}^{-1}_{p\to x} \tilde{f}(p) = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p)\, e^{iKxp}\, dp$$

Inverse Fourier transform

$$\hat{f}(p) = \hat{\mathcal{F}}_{x\to p} f(x) = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iKxp}\, dx \qquad \text{F.T.}$$

$$f(x) = \hat{\mathcal{F}}^{-1}_{p\to x} \tilde{f}(p) = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p)\, e^{iKxp}\, dp \qquad \text{I.F.T.}$$

The Fourier transform satisfies many properties (see list at end of these notes)

Here, we will use these:

- $$\hat{\mathcal{F}}_{x\to p}\left[x^n f(x)\right] = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} x^n f(x)\, e^{-iKxp}\, dx = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} \frac{f(x)\, \frac{d^n}{dp^n} e^{-iKxp}}{(-iK)^n}\, dx$$

$$= \left(\frac{i}{K}\right)^n \frac{d^n}{dp^n} \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iKxp}\, dx = \underline{\left(\frac{i}{K}\right)^n \tilde{f}^{(n)}(p)}$$

Conversely

- $$\hat{\mathcal{F}}^{-1}_{p\to x}\left[p^n \tilde{f}(p)\right] = \left(\frac{-i}{K}\right)^n f^{(n)}(x), \quad \text{so}$$

$$\underline{\hat{\mathcal{F}}_{x\to p} f^{(n)}(x) = (iK)^n p^n \tilde{f}(p)}$$

- Scaling

$$\hat{\mathcal{F}}_{x\to p} f(\underbrace{a}_{\text{real, positive}} x) = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{-iKxp}\, dx = \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-iK\frac{x'}{a}p}\, \frac{dx'}{a}$$

$x' = ax,\ dx' = a\, dx$

$$= \underline{\frac{1}{a} \tilde{f}\left(\frac{p}{a}\right)}$$

- Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f^*(x) f(x)\, dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p)\, e^{iKxp}\, dp\right]^* f(x)\, dx = \int_{-\infty}^{\infty} \tilde{f}^*(p) \sqrt{\frac{|K|}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iKxp}\, dx\, dp$$

$$= \int_{-\infty}^{\infty} \tilde{f}^*(p)\, \tilde{f}(p)\, dp = \underline{\int_{-\infty}^{\infty} |\tilde{f}(p)|^2 dp}$$

- Uncertainty relation

First, derive Cauchy-Schwarz-Bunyakowski inequality:

$$\iint_{-\infty}^{\infty} |g(x)h(y)-g(y)h(x)|^2 dx\,dy \geqslant 0$$

"=" only if $h(x) = a\,g(x)$

a: constant

$$\iint_{-\infty}^{\infty} \left[g^*(x)h^*(y)-g^*(y)h^*(x)\right]\left[g(x)h(y)-g(y)h(x)\right]dx\,dy \geqslant 0$$

multiply:

$$\int_{-\infty}^{\infty}|g(x)|^2dx\int_{-\infty}^{\infty}|h(y)|^2dy - \int_{-\infty}^{\infty}g^*(x)h(x)dx\int_{-\infty}^{\infty}h^*(y)g(y)dy$$

$$-\int_{-\infty}^{\infty}g^*(y)h(y)dy\int_{-\infty}^{\infty}h^*(x)g(x)dx + \int_{-\infty}^{\infty}|g(y)|^2dy\int_{-\infty}^{\infty}|h(x)|^2dx \geqslant 0$$

x & y are dummy variables of integration, so we can change their names. Then, this reduces to

$$2\left[\int_{-\infty}^{\infty}|g(x)|^2dx\right]\left[\int_{-\infty}^{\infty}|h(x)|^2dx\right] - 2\left[\int_{-\infty}^{\infty}g^*(x)h(x)dx\right]\left[\int_{-\infty}^{\infty}h^*(x)g(x)dx\right] \geqslant 0$$

or

$$\left[\int_{-\infty}^{\infty}|g(x)|^2dx\right]\left[\int_{-\infty}^{\infty}|h(x)|^2dx\right] \geqslant \left|\int_{-\infty}^{\infty}g^*(x)h(x)dx\right|^2$$

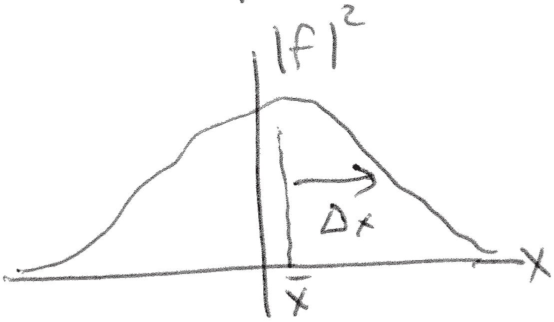
Cauchy-Schwarz-Bunyakowski inequality.

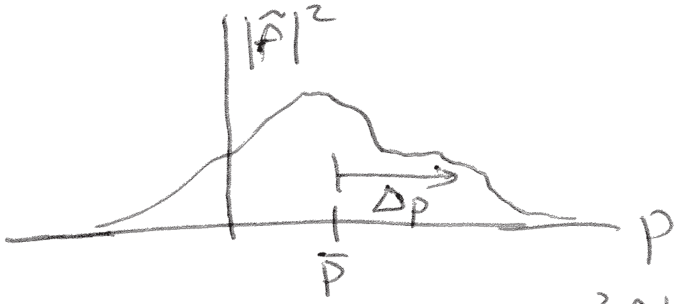
Now, let us define:

- squared norm of f: $\|f\|^2 = \int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|\tilde{f}(p)|^2dp$
- average or centroid: $\bar{x} = \dfrac{\int_{-\infty}^{\infty}x|f(x)|^2dx}{\|f\|^2}$
- standard deviation: $\Delta x = \left[\dfrac{\int_{-\infty}^{\infty}(x-\bar{x})^2|f(x)|^2dx}{\|f\|^2}\right]^{1/2}$

Similarly

$$\bar{p} = \frac{\int_{-\infty}^{\infty} p\,|\tilde{f}(p)|^2\,dp}{\|f\|^2}, \quad \Delta_p = \left[\frac{\int_{-\infty}^{\infty} (p-\bar{p})^2\,|\tilde{f}(p)|^2\,dp}{\|f\|^2}\right]^{1/2}$$





Δx & Δp estimate the widths of $|f(x)|^2$ & $|\tilde{f}(p)|^2$.

Let $g(x) = \dfrac{(x-\bar{x})\,f(x)}{\|f\|}$, so $\displaystyle\int_{-\infty}^{\infty} |g(x)|^2\,dx = \frac{\int_{-\infty}^{\infty}(x-\bar{x})^2|f(x)|^2\,dx}{\|f\|^2}$

$$= \Delta_x^2$$

Similarly, let

$$\tilde{h}(p) = \frac{(p-\bar{p})\,\tilde{f}(p)}{\|f\|}, \text{ so } \int_{-\infty}^{\infty} |h(x)|^2\,dx = \int_{-\infty}^{\infty} |\tilde{h}(p)|^2\,dp$$

$$= \Delta_p^2$$

Note:

$$h(x) = \hat{\mathfrak{f}}^{-1}_{p\to x}\,\tilde{h}(p) = \frac{1}{\|f\|}\left[\hat{\mathfrak{f}}^{-1}\left(p\tilde{f}(p)\right) - \bar{p}\,\hat{\mathfrak{f}}^{-1}\left(\tilde{f}(p)\right)\right]$$

$$= \frac{-iK^{-1}f'(x) - \bar{p}f(x)}{\|f\|}$$

$$\text{so } \int_{-\infty}^{\infty} g^*(x)\,h(x)\,dx = \frac{1}{\|f\|^2}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)\left[-iK^{-1}f'(x) - \bar{p}f(x)\right]dx \quad (i)$$

$$= \frac{-iK^{-1}}{\|f\|^2}\underbrace{\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)f'(x)\,dx}_{\text{integrate by parts } u=(x-\bar{x})f^*,\ dv=f'dx} - \underbrace{\frac{\bar{p}}{\|f\|^2}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)f(x)\,dx}_{\text{this term vanishes, but let's keep it anyway.}}$$

$$= \frac{-iK^{-1}}{\|f\|^2}\left[\underbrace{\cancel{(x-\bar{x})f^*(x)f(x)\Big|_{-\infty}^{\infty}}}_{\substack{\text{if we assume that } |f(x)|^2 \\ \text{goes to zero at } \pm\infty \text{ faster} \\ \text{than } \frac{1}{x}\text{, this goes to zero.}}} - \int_{-\infty}^{\infty}\underbrace{[(x-\bar{x})f^*(x)]'}_{\text{expand}} f(x)\,dx\right] - \frac{\bar{p}}{\|f\|^2}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)f(x)\,dx$$

$$= \frac{+iK^{-1}}{\|f\|^2}\left[\underbrace{\int_{-\infty}^{\infty} f^*(x)f(x)\,dx}_{\|f\|^2} + \int_{-\infty}^{\infty}(x-\bar{x})f'^*(x)f(x)\,dx\right] - \frac{\bar{p}}{\|f\|^2}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)f(x)\,dx$$

regroup

$$= +iK^{-1} + \frac{1}{\|f\|^2}\left[\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)\left[-iK^{-1}f'(x) - \bar{p}f(x)\right]dx\right]^* \qquad (ii)$$

So relations (i) & (ii) can be written as:

$$\int_{-\infty}^{\infty} g^*(x)h(x)\,dx = \begin{cases} T & (i) \\ +iK^{-1} + T^* & (ii) \end{cases}$$

where $T = \frac{1}{\|f\|^2}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)\left(-iK^{-1}f'(x) - \bar{p}f(x)\right)dx$

averaging both options

$$\int_{-\infty}^{\infty} g^*(x)h(x)\,dx = \frac{(i)+(ii)}{2} = \underbrace{\mathrm{Re}(T)}_{\text{this term is called } \Delta xp} + \frac{iK^{-1}}{2}$$

so $\int_{-\infty}^{\infty}|g(x)|^2dx = \Delta_x^2$, $\int_{-\infty}^{\infty}|h(x)|^2dx = \Delta_p^2$, $\int_{-\infty}^{\infty}g^*(x)h(x)\,dx = \Delta xp + \frac{iK^{-1}}{2}$

Therefore, from Cauchy-Schwarz-Bunyakowski,

$$\Delta_x^2\Delta_p^2 \geq \left|\Delta xp + \frac{iK^{-1}}{2}\right|^2 = \Delta xp^2 + \frac{1}{4K^2}$$

Schrödinger inequality

Since $\Delta_{xp}^2 \geq 0$,

$$\Delta_x^2 \Delta_p^2 \geq \Delta_{xp}^2 + \frac{1}{4K^2} \geq \frac{1}{4K^2}$$

$$\boxed{\Delta_x \Delta_p \geq \frac{1}{2|K|}}$$ Uncertainty relation.

That is, for a given Δx, $\Delta p \geq \frac{|K|}{2\Delta x|K|}$ and vice-versa.

Cannot have very small Δx & Δp simultaneously!!!

Note: $\Delta x \Delta p = \frac{1}{2|K|}$ Only if:

- $h(x) = a g(x)$ & • $\Delta_{xp} = 0$

$h(x) = ag(x)$ means $-iK^{-1}f'(x) - \bar{p} f(x) = \underset{\text{constant}}{a}(x-\bar{x}) f(x)$

$$\frac{f'(x)}{f(x)} = i\bar{p}K + iKa(x-\bar{x})$$

Integrating:

$$\ln f(x) + \underset{\text{constant}}{c} = iK\bar{p}x + iKa\frac{(x-\bar{x})^2}{2}$$

$$f(x) = \underbrace{e^{-c}}_{f_0 = \text{constant}} e^{iK\bar{p}x} e^{iKa\frac{(x-\bar{x})^2}{2}}$$

$f'(x) = [iK\bar{p} + iKa(x-\bar{x})] f(x)$, so

$$T = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) [\bar{p} + a(x-\bar{x}) - \bar{p}] f(x) dx = a \frac{\overbrace{\int_{-\infty}^{\infty} (x-\bar{x})^2 |f(x)|^2 dx}^{\text{real}}}{\|f\|^2}$$

so $\Delta_{xp} = \mathrm{Re}(T) = 0$ only if a is imaginary, e.g. $a = \frac{i}{Kw^2}$

so $\Delta x \Delta p = \frac{1}{2|K|}$ for $\underbrace{f(x) = f_0 e^{iK\bar{p}x} e^{-\frac{(x-\bar{x})^2}{2w^2}}}_{\text{Gaussian}}$

Exercises

1) Using $K=1$, find FT of $e^{-\frac{x^2}{2}}$

2) Using the previous result, find the F.T. of $e^{\frac{-x^2}{2w^2}}$

3) Find Δx and Δp

4) Use your result from (1) to find the F.T. of: $x e^{-\frac{x^2}{2}}$, $x^2 e^{-\frac{x^2}{2}}$, $x^3 e^{-\frac{x^2}{2}}$

5) Based on your results find the values of the constants $\alpha, \beta, \gamma, \delta, a, b, c, d, h$, so that

$$\hat{f}_{x\to p}\, e^{-\frac{x^2}{2}} = \alpha e^{-\frac{p^2}{2}}$$

$$\hat{f}_{x\to p}\left[x e^{-\frac{x^2}{2}}\right] = \beta p e^{-\frac{p^2}{2}}$$

$$\hat{f}_{x\to p}\left(\left[x^2+ax+b\right]e^{-\frac{x^2}{2}}\right) = \gamma\left[p^2+ap+b\right]e^{-\frac{p^2}{2}}$$

$$\hat{f}_{x\to p}\left(\left[x^3+cx^2+dx+h\right]e^{-\frac{x^2}{2}}\right) = \delta\left[p^3+cp^2+dp+h\right]e^{-\frac{p^2}{2}}$$

2D Fourier transform

$$\tilde{f}(p_x, p_y) = \hat{\mathcal{F}}_{\substack{x \to p_x \\ y \to p_y}} f(x,y) = \frac{|K|}{2\pi} \iint_{-\infty}^{\infty} f(x,y)\, e^{-iK(xp_x + yp_y)}\, dx\,dy$$

$$f(x,y) = \hat{\mathcal{F}}^{-1}_{\substack{p_x \to x \\ p_y \to y}} \tilde{f}(p_x, p_y) = \frac{|K|}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(p_x, p_y)\, e^{iK(xp_x + yp_y)}\, dp_x\,dp_y$$

Properties of Fourier transforms

$$\tilde{f}(p) = \hat{\mathcal{F}}_{x\to p} f(x) = \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\eta x p} dx$$

$$f(x) = \hat{\mathcal{F}}^{-1}_{p\to x} \tilde{f}(p) = \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(p)\, e^{i\eta x p} dp$$

- Shift

$$\hat{\mathcal{F}}_{x\to p}\left[f(x-a)\right] = \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(\underbrace{x-a}_{x'})\, e^{-i\eta x p} dx$$

$x = x' + a$
$dx = dx'$

$$= \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-i\eta x' p} dx'\, e^{-i\eta a p}$$

$$= \tilde{f}(p)\, e^{-i\eta a p}$$

- Phase

$$\hat{\mathcal{F}}_{x\to p}\left[f(x)\, e^{i\eta a x}\right] = \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} \left[f(x)\, e^{i\eta a x}\right] e^{-i\eta x p} dx$$

$$= \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\eta x (p-a)} dx = \tilde{f}(p-a)$$

- Scale

$$\hat{\mathcal{F}}_{x\to p}\left[f(\underset{\text{real}}{a} x)\right] = \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(\underbrace{ax}_{x'})\, e^{-i\eta x p} dx$$

$x = x'/a$
$dx = dx'/a$

$$= \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty\,\mathrm{sgn}(a)}^{\infty\,\mathrm{sgn}(a)} f(x')\, e^{-i\eta \frac{x'}{a} p} \frac{dx'}{a} = \frac{\mathrm{sgn}(a)}{a} \hat{f}\left(\frac{p}{a}\right)$$

$$= \frac{1}{|a|} \hat{f}\left(\frac{p}{a}\right)$$

• Product

$$\hat{\mathfrak{F}}_{x\to p}\left[f(x)g(x)\right]=\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\underbrace{f(x)}_{\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(p')e^{i\eta x p'}dp'}g(x)e^{-i\eta x p}dx$$

$$=\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(p')e^{i\eta x p'}dp'\,g(x)e^{-i\eta x p}dx$$

$$=\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(p')\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-i\eta x(p-p')}dx\,dp'$$

$$=\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(p')\tilde{g}(p-p')\,dp'=\underline{\sqrt{\frac{|\eta|}{2\pi}}\,\tilde{f}*\tilde{g}(p)}$$

• Convolution

$$\hat{\mathfrak{F}}_{x\to p}\left[f*g(x)\right]=\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x')g(x-x')dx'\,e^{-i\eta x p}dx$$

$$=\int_{-\infty}^{\infty}\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}g(\underbrace{x-x'}_{x''})\,e^{-i\eta(x-x')p}dx\;f(x')e^{-i\eta x'p}dx'$$

$x''=x-x'$, $dx''=dx$

$$=\int_{-\infty}^{\infty}\underbrace{\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}g(x'')e^{-ix''p}dx''}_{\tilde{g}(p)}\,f(x')e^{-i\eta x'p}dx'$$

$$=\tilde{g}(p)\int_{-\infty}^{\infty}f(x')e^{-i\eta x'p}dx'=\underline{\sqrt{\frac{2\pi}{|\eta|}}\,\tilde{f}(p)\,\tilde{g}(p)}$$

• Derivative

$$\hat{\mathcal{F}}_{x\to p}\left[f'(x)\right] = \sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{-i\eta x p} dx$$

$u = e^{-i\eta x p},\ dv = f' dx$

$du = -i\eta p\, e^{-i\eta x p} dx,\ v = f$

$$= \sqrt{\frac{|\eta|}{2\pi}}\left[\cancel{f(x) e^{-i\eta x p}}\Big|_{-\infty}^{\infty} \to 0 - \int_{-\infty}^{\infty} -i\eta p\, e^{-i\eta x p} f(x)\, dx\right]$$

assume $\lim_{x\to\pm\infty} f(x) = 0$

$$= \underline{i\eta p\, \tilde{f}(p)}$$

Using this repeatedly:

$$\hat{\mathcal{F}}_{x\to p}\left[f^{(n)}(x)\right] = \underline{(i\eta p)^n \tilde{f}(p)}$$

• Powers of x

$$\hat{\mathcal{F}}_{x\to p}\left[x^n f(x)\right] = \sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty} f(x)\, \underbrace{x^n e^{-i\eta x p}}_{\left(\frac{1}{-i\eta}\right)^n \frac{d^n}{dp^n} e^{-i\eta x p}} dx$$

$$= \left(\frac{i}{\eta}\right)^n \frac{d^n}{dp^n} \sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\eta x p} dx$$

$$= \underline{\left(\frac{i}{\eta}\right)^n \tilde{f}^{(n)}(p)}$$

- Parseval's theorem

$$\int_{-\infty}^{\infty} f^*(x)\, g(x)\, dx = \int_{-\infty}^{\infty}\left[\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\hat{f}(p)\, e^{i\eta x p}\, dp\right]^* g(x)\, dx$$

$$= \sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty}\tilde{f}^*(p)\int_{-\infty}^{\infty} g(x)\, e^{-i\eta x p}\, dx\, dp$$

$$= \underline{\int_{-\infty}^{\infty}\tilde{f}^*(p)\,\tilde{g}(p)\, dp}$$

in particular $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(p)|^2 dp$

- Real functions

if $f(x) = f^*(x)$

$$\tilde{f}^*(p) = \left[\sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\eta x p}\, dx\right]^*$$

$$= \sqrt{\frac{|\eta|}{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\eta x p}\, dx = \underline{\hat{f}(-p)}$$

so

$\mathrm{Re}\{\tilde{f}(p)\}$ is even

$\mathrm{Im}\{\hat{f}(p)\}$ is odd

• Asymptotic behavior for discontinuous functions.

If $f(x)$ is discontinuous in its m^{th} derivative at x_0:

$$\hat{f}(p) = \sqrt{\frac{|\eta|}{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\eta x p}\, dx = \sqrt{\frac{|\eta|}{2\pi}} \left[\int_{-\infty}^{x_0} f(x) e^{-i\eta x p} dx + \int_{x_0}^{\infty} f(x) e^{-i\eta x p} dx \right]$$

For large p, can use the trick from edge effects in stationary phase.

$$\int_{-\infty}^{x_0} f(x) \underbrace{e^{-i\eta x p}}_{\left(\frac{i}{\eta p}\right) \frac{d}{dx} e^{-i\eta x p}} dx = \frac{i}{\eta p} f(x)\, e^{-i\eta x p} \Big|_{-\infty}^{x_0} - \frac{i}{\eta p} \int_{-\infty}^{x_0} f'(x) e^{-i\eta x p} dx$$

$$= \cdots = \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{\eta p}\right)^{n+1} f^{(n)}(x_0^-)\, e^{-i\eta x_0 p}$$

similarly

$$\int_{x_0}^{\infty} f(x) e^{-i\eta x p} dx = -\sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{\eta p}\right)^{n+1} f^{(n)}(x_0^+) e^{-i\eta x_0 p}$$

so

$$\tilde{f}(p) = \sum_{n=0}^{\infty} \sqrt{\frac{|\eta|}{2\pi}} (-1)^{n+1} \left(\frac{i}{\eta p}\right)^{n+1} \Big[\underbrace{f^{(n)}(x_0^+) - f^{(n)}(x_0^-)}_{\text{these cancel for } n<m} \Big] e^{-i\eta x_0 p}$$

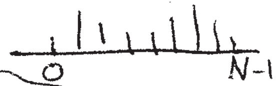
$$\approx \sqrt{\frac{|\eta|}{2\pi}} \frac{e^{-i\eta x_0 p}}{(i\eta p)^{m+1}} \left[f^{(m)}(x_0^+) - f^{(m)}(x_0^-) \right] \propto \frac{e^{-i\eta x_0 p}}{p^{m+1}},\ p \to \pm\infty$$

Similarly, if $\tilde{f}(p)$ is discontinuous in m^{th} derivative,

$$f(x) \approx e^{i\eta x p_0} \sqrt{\frac{|\eta|}{2\pi}} \left(\frac{i}{\eta x}\right)^{m+1} \left[\hat{f}^{(m)}(p_0^+) - \hat{f}^{(m)}(p_0^-) \right] \propto \frac{1}{x^{m+1}},\ x \to \pm\infty$$

Discrete Fourier transform (DFT)

Instead of $f(x)$ we have f_n, $n = 0, \ldots, N-1$

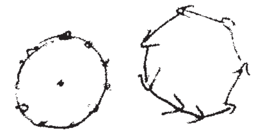


$$\boxed{F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n \, e^{-i2\pi \frac{mn}{N}}}$$ Discrete Fourier transform

Inverse: try:

$$f_{n'} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m e^{i2\pi \frac{mn'}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f_n \underbrace{\sum_{m=0}^{N-1} e^{i2\pi \frac{(n'-n)m}{N}}}_{N\delta_{n'-n}}$$



so:

$$\boxed{f_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m \, e^{i2\pi \frac{mn}{N}}}$$ Inverse Discrete Fourier transform

Approximating FT with DFT

(Notice that the sums can be shifted,) if we define

$$f_{n-N} = f_n$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N}{2} \rfloor} f_n \, e^{-i2\pi \frac{mn}{N}}$$

Let f_n be a sampling of $f(x)$:

$$f_n = f(n\Delta x)$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N}{2} \rfloor} f(n\Delta x) \, e^{-i2\pi \frac{mn}{N}}$$

For very large N, and small Δx,
can approximate the sum as an integral

$$F_m \approx \frac{1}{\sqrt{N}} \int_{-X_1}^{X_2} f(x)\, e^{\frac{-i2\pi m x}{N\Delta x}} \frac{dx}{\Delta x}$$

where $n\Delta x \rightarrow x$

$X_1 = \lfloor \frac{N-1}{2} \rfloor \Delta x$, $X_2 = \lfloor \frac{N}{2} \rfloor \Delta x$

Assume $\underset{\text{big}}{N} \underset{\text{small}}{\Delta x} = \text{big} >>$ width of $f(x)$.

note $X_1 \approx X_2 \approx \frac{N\Delta x}{2} = \text{big}$.

Then

$$F_m \approx \frac{1}{\sqrt{N}\,\Delta x} \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x \left(\frac{m}{N\Delta x}\right)} dx$$

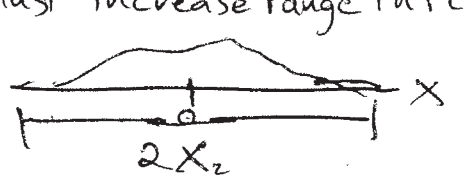
$$= \boxed{\frac{\tilde{f}\left(\frac{m}{N\Delta x}\right)}{\sqrt{N}\,\Delta x}}$$

So the sampling distance in ν is $\frac{1}{N\Delta x} \approx \frac{1}{2X_2}$

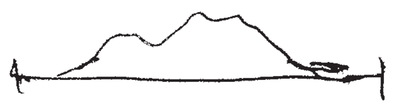
where $2X_2$ is the width over which we're sampling $f(x)$.

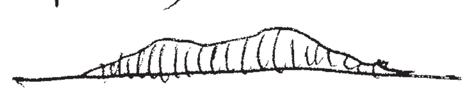
Therefore:

- To increase resolution in $\tilde{f}(\nu) \longrightarrow$ must increase range in $f(x)$



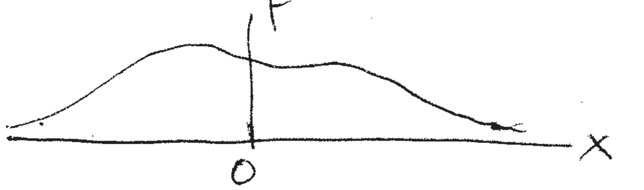
- To increase range in $\tilde{f}(\nu)$ and avoid aliasing $\longrightarrow$ must decrease sampling spacing in $f(x)$





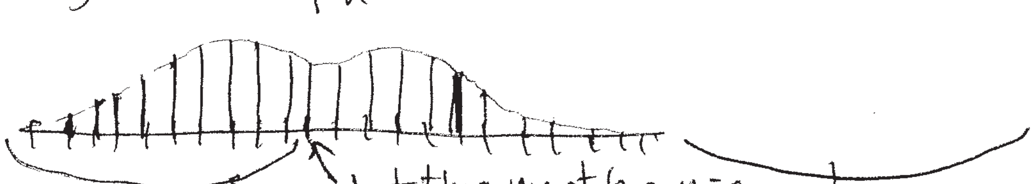
Shifting the functions.

Notice that, if we sample:



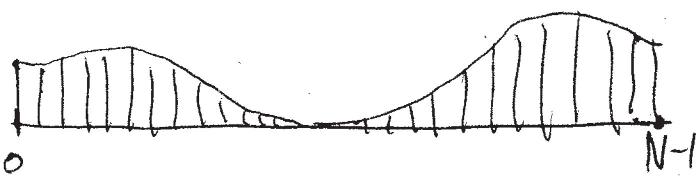
we get

f_n



but this must be $n=0$
so we must shift
this to here

so we get



→ this is f_n

Similarly, once we get F_m, it will look like

To reconstruct $\tilde{f}(\nu)$ we must cut the second half and place it before the first. We also need to multiply by $\sqrt{N}\Delta x$.

Fast Fourier transform (FFT)

Notice that the, for each m, the DFT involves the sum of N terms. Since m runs from 0 to $N-1$, then N^2 must be performed. The time of computation can therefore be expected to be proportional to N^2.

The FFT is an algorithm for performing the DFT whose time of computation is proportional to $N \log N$. While it can work for any N, its simplest form can be understood if $N = 2^M$ (so that $M = \log_2 N$):

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-\frac{i2\pi nm}{N}} = \frac{1}{\underset{\text{write as } 2\frac{N}{2}}{\sqrt{N}}} \left[\underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{-\frac{i2\pi(2n')m}{N}}}_{\text{terms with even } n} + \underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{-\frac{i2\pi(2n'+1)m}{N}}}_{\text{terms with odd } n} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{-\frac{i2\pi n'm}{(N/2)}}} + e^{-\frac{i2\pi m}{N}} \underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{-\frac{i2\pi n'm}{(N/2)}}} \right]$$

Each of these two sums is itself a DFT of size $\frac{N}{2}$.

They can be joined:

$$F_m = \frac{1}{\sqrt{N}} \sum_{n'=0}^{\frac{N}{2}-1} \left(f_{2n'} + e^{-\frac{i2\pi m}{N}} f_{(2n'+1)} \right) e^{-\frac{i2\pi n'm}{(N/2)}}$$

The same separation can be done M times.

2D DF

$$\boxed{F_{m_1,m_2} = \frac{1}{N} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2}\, e^{-i2\pi\frac{(m_1 n_1 + m_2 n_2)}{N}}}$$

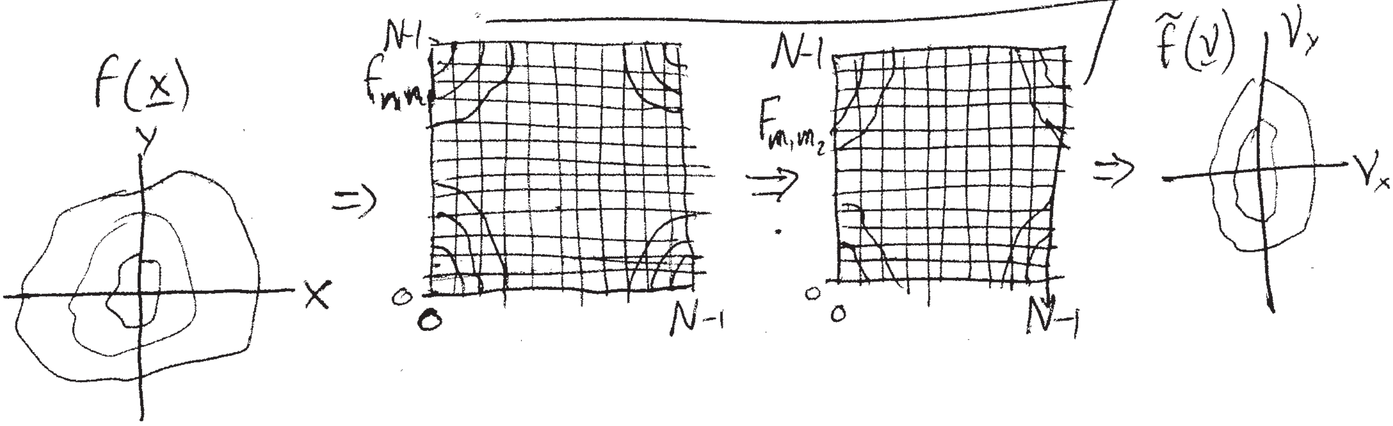
$$\boxed{f_{n_1,n_2} = \frac{1}{N} \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} F_{m_1,m_2}\, e^{i2\pi\frac{(m_1 n_1 + m_2 n_2)}{N}}}$$

Using 2D DFT to approximate 2D FT:

if $f_{n_1,n_2} = f(n_1\Delta x, n_2\Delta x)$,

and $N\Delta x$ is bigger than width of f, then:

$$F_{m_1,m_2} \approx \frac{1}{N\Delta x^2}\, \tilde{f}\left(\frac{m_1}{N\Delta x}, \frac{m_2}{N\Delta x}\right)$$



Fast Fourier transform: time $\propto N^2 \log N$