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Paraxial and far field approximations

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Paraxial propagation

As discussed in the EM lectures, a free-space monochromatic electric field can be written as $\vec{\mathcal{E}}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})\, e^{-i\omega t}\right]$, where $\vec{E}$ is a complex function that satisfies

$$\underline{\left[\nabla^2 + k^2\right]\vec{E}(\vec{r}) = \vec{0}},$$ Vector Helmholtz equation

where $k = \frac{\omega}{c}$ = wavenumber

and

$$\underline{\nabla \cdot \vec{E}(\vec{r}) = 0},$$ Transversality condition

The time-averaged measurable intensity is given by

$$I(\vec{r}) = \frac{1}{T}\int_{-T/2}^{T/2} \vec{\mathcal{E}}(\vec{r},t)\cdot \underbrace{\vec{\mathcal{E}}(\vec{r},t)}_{\frac{1}{2}(\vec{E}e^{-i\omega t} + \vec{E}^* e^{i\omega t})}\, dt \qquad \text{for } T\omega \gg 1$$

$$= \frac{1}{4T}\left[\vec{E}\cdot\vec{E}\cancel{\int_{-T/2}^{T/2} e^{-2i\omega t}\, dt} + 2\vec{E}^*\cdot\vec{E}\underbrace{\int_{-T/2}^{T/2} dt}_{T} + \vec{E}^*\cdot\vec{E}^*\cancel{\int_{-T/2}^{T/2} e^{2i\omega t}\, dt}^{\,0}\right]$$

$$= |\vec{E}(\vec{r})|^2$$

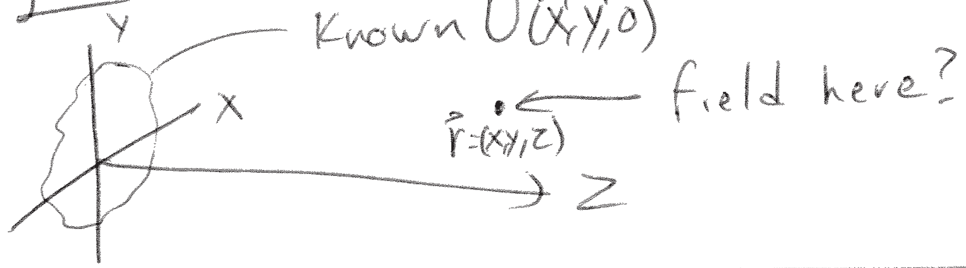
Let us, for simplicity, consider a scalar field $U(\vec{r})$, which could represent a Cartesian component of $\vec{E}(\vec{r})$. This field satisfies

$$\boxed{(\nabla^2+k^2)U(\vec{r})=0}$$ Scalar Helmholtz equation.

Suppose that we know:

- $U(x,y,0)$ i.e. the field at an initial plane $z=0$
- that the field is generated somewhere at $z<0$ so it traverses the $z=0$ plane towards $z>0$.

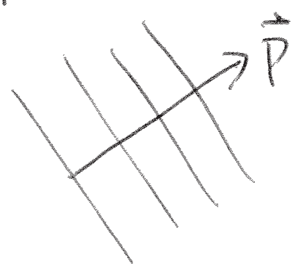
The goal is to find $U(x,y,z)$ for $z>0$.



Propose a plane wave solution

$$U_{pw} = e^{ik\vec{p}\cdot\vec{r}}$$

where $\vec{p}$ is the direction of propagation of the plane wave.



Substitute in Helmholtz

$$(\nabla^2+k^2)U_{pw} = (-k^2p_x^2-k^2p_y^2-k^2p_z^2+k^2)e^{ik\vec{p}\cdot\vec{r}}=0$$

so $|\vec{p}|^2=1$ (unit vector)

or $p_z=\sqrt{1-p_x^2-p_y^2}$

We should not choose a negative value for p_z, since this would represent a plane wave traveling from larger to smaller z.

The acceptable plane waves are:

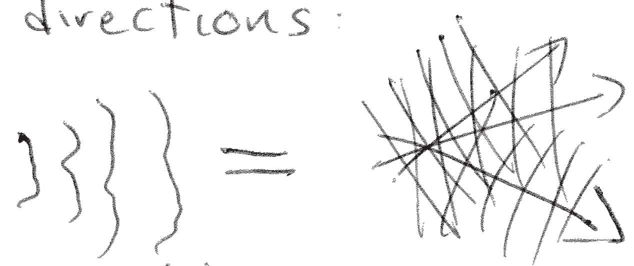
$$U_{pw}(\vec{r}; p_x, p_y) = e^{ik(x p_x + y p_y + z p_z)} \quad \text{with } p_z = \sqrt{1 - p_x^2 - p_y^2}$$

only p_x and p_y are independent variables.

Due to the superposition principle, let us propose that $U(\vec{r})$ is a superposition of plane waves:

$$U(\vec{r}) = \underbrace{\frac{k}{2\pi}}_{\substack{\text{constant}\\ \text{inserted}\\ \text{for convenience}}} \iint_{-\infty}^{\infty} \underbrace{A(p_x, p_y)}_{\substack{\text{amplitude of}\\ \text{plane wave in}\\ \text{direction } \vec{p}}} U_{pw}(\vec{r}; p_x, p_y)\, dp_x\, dp_y \qquad (1)$$

That is, the field U is a sum of planewaves with different amplitudes traveling in many directions:



complicated field = superposition of plane waves

Note that, at $z = 0$:

$$U(x, y, 0) = \frac{k}{2\pi} \iint_{-\infty}^{\infty} A(p_x, p_y)\, e^{ik(x p_x + y p_y)}\, dp_x\, dp_y$$

$$= \hat{\mathcal{F}}^{-1}_{\substack{p_x \to x\\ p_y \to y}} A(p_x, p_y)$$

That is, $A(p_x, p_y)$ is the 2D Fourier transform of the initial field $U(x, y, 0)$, with $\underline{K = k = \frac{\omega}{c}}$

That is:

$$A(p_x,p_y) = \hat{\mathcal{F}}_{\substack{x\to p_x \\ y\to p_y}} U(x,y,0). \qquad (2)$$

Since we know $U(x,y,0)$, we know $A(p_x,p_y)$.

A is called the "angular spectrum".

In fact, note that Eq. (1) can be written as

$$U(x,y,z) = \frac{k}{2\pi} \iint_{-\infty}^{\infty} \left[A(p_x,p_y)\, e^{ikp_z z}\right] e^{ik(xp_x+yp_y)}\, dp_x dp_y$$

$$= \hat{\mathcal{F}}^{-1}_{\substack{p_x\to x \\ p_y\to y}} \left[A(p_x,p_y)\, e^{ikzp_z}\right] \qquad (1')$$

Note however, that this integral involves all real values of p_x & p_y, even those for which $p_x^2+p_y^2>1$. For these values,

$$p_z = \sqrt{1-p_x^2-p_y^2} = \pm i\sqrt{p_x^2+p_y^2-1}$$

that is, p_z is purely imaginary. Therefore

$$U_{pw}(\vec{r};p_x,p_y) = e^{ik(xp_x+yp_y)}\, e^{\mp z\sqrt{p_x^2+p_y^2-1}}$$

we must choose the top sign, since otherwise the exponential in z diverges!

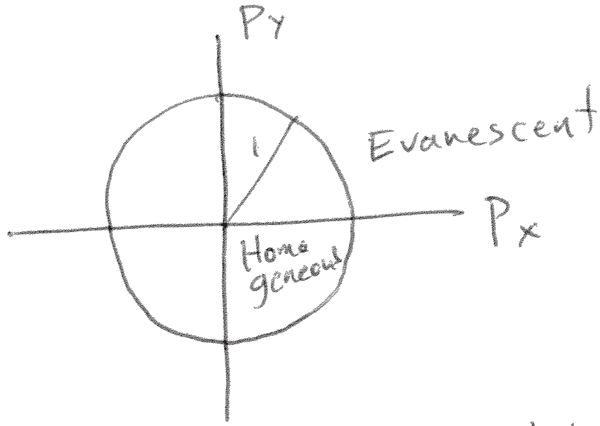
That is, for $p_x^2+p_y^2 \geq 1$

$$U_{pw}(\vec{r};p_x,p_y) = e^{ik(xp_x+yp_x)}\, e^{-z\sqrt{p_x^2+p_y^2-1}}$$

This is an <u>evanescent wave</u>

Therefore

$$p_z = \begin{cases} \sqrt{1-p_x^2-p_y^2}\,, & p_x^2+p_y^2 \le 1 \quad \text{(homogeneous plane waves)} \\ i\sqrt{p_x^2+p_y^2-1}\,, & p_x^2+p_y^2 > 1 \quad \text{(evanescent waves)} \end{cases}$$



So to propagate a field from $z=0$ to $z>0$:

i) Find angular spectrum from initial field:

$$A(p_x, p_y) = \hat{f}_{\substack{x\to p_x \\ y\to p_y}}\, U(x,y,0)$$

ii) Multiply angular spectrum by e^{ikzp_z}

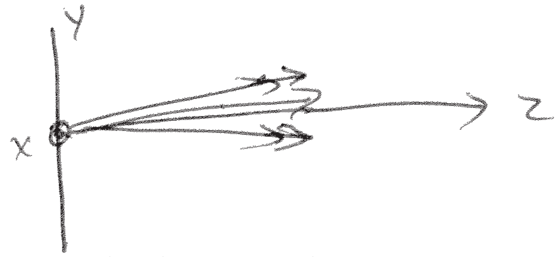
$$A(p_x,p_y)\, e^{ikzp_z}$$

iii) Take inverse Fourier transform

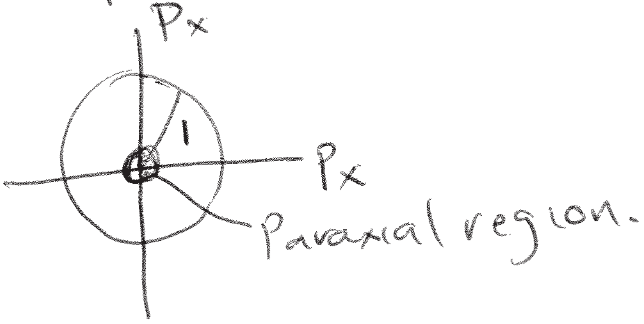
$$U(x,y,z) = \hat{f}^{-1}_{\substack{p_x\to x \\ p_y\to y}} \left[A(p_x,p_y)\, e^{ikzp_z} \right]$$

Paraxial approximation

Suppose that the field is composed only of plane waves travelling at very small angles with respect to the z axis:



That is $A(p_x, p_y)$ is negligible except for $p_x^2 + p_y^2 << 1$



We can then approximate:

$$p_z = \left(1 - (p_x^2 + p_y^2)\right)^{1/2} \approx 1 - \frac{p_x^2 + p_y^2}{2}$$

Equation (1) then becomes

$$U(\vec{r}) = \frac{k}{2\pi} \iint_{-\infty}^{\infty} A(p_x, p_y)\, e^{ik\left(x p_x + y p_y + z - z\frac{p_x^2 + p_y^2}{2}\right)} dp_x\, dp_y$$

$$= e^{ikz} \frac{k}{2\pi} \iint_{-\infty}^{\infty} A(p_x, p_y)\, e^{-ikz\frac{p_x^2 + p_y^2}{2}}\, e^{ik(xp_x + yp_y)}\, dp_x\, dp_y \qquad (3)$$

Note that the field now satisfies: instead of Helmholtz $\frac{1}{ik}\frac{\partial}{\partial z}\left[U(\vec{r})e^{-ikz}\right] = \frac{-1}{2k^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[U(\vec{r})e^{-ikz}\right]$

that is

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-2ik\frac{\partial}{\partial z}\right)\left[U(\vec{r})e^{-ikz}\right]=0 \quad \text{Paraxial wave eq.}}$$

Substitute now (2) in (3):

$$U(\vec{r})=e^{ikz}\frac{k}{2\pi}\iint_{-\infty}^{\infty}\frac{k}{2\pi}\iint_{-\infty}^{\infty}U(x',y',0)\,e^{-ik(x'p_x+y'p_y)}dx'dy'\,e^{-ikz\frac{p_x^2+p_y^2}{2}}e^{ik(xp_x+yp_y)}dp_xdp_y$$

$$=\frac{k^2}{(2\pi)^2}e^{ikz}\iint_{-\infty}^{\infty}U(x',y',0)\left[\int_{-\infty}^{\infty}e^{-i\frac{kz}{2}\left(p_x^2+2p_x\frac{(x'-x)}{z}\right)}dp_x\right]$$

$$\left[\int_{-\infty}^{\infty}e^{-i\frac{kz}{2}\left(p_y^2+2p_y\frac{(y'-y)}{z}\right)}dp_y\right]dx'dy'$$

but $\displaystyle\int_{-\infty}^{\infty}e^{-i\frac{kz}{2}\left(p_x^2+2p_x\frac{(x'-x)}{z}\right)}dp_x=\underbrace{\int_{-\infty}^{\infty}e^{-i\frac{kz}{2}\left(p_x+\frac{(x'-x)}{z}\right)^2}dp}_{\sqrt{\frac{2\pi}{ikz}}}\,e^{i\frac{kz}{2}\frac{(x'-x)^2}{z^2}}$

$$=\sqrt{\frac{2\pi}{ikz}}\;e^{i\frac{k(x'-x)^2}{2z}}$$ & same for the other integral.

Therefore

$$U(x,y,z)=\frac{k}{2\pi iz}e^{ikz}\iint_{-\infty}^{\infty}U(x',y',0)\,e^{ik\frac{(x'-x)^2+(y'-y)^2}{2z}}dx'dy'$$

Fresnel Propagation formula.

Gaussian Beam

$$U^G(x,y,0) = u_0\, e^{-\frac{x^2+y^2}{2w_0^2}}$$

Substituting in Fresnel formula

$$U^G(x,y,z) = \frac{k e^{ikz}}{2\pi i z} \iint_{-\infty}^{\infty} u_0 e^{-\frac{x'^2+y'^2}{2w_0^2}} e^{ik\frac{(x'-x)^2+(y'-y)^2}{2z}} dx'dy'$$

$$= \frac{k u_0}{2\pi i z} e^{ikz} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2w_0^2}+ik\frac{(x'-x)^2}{2z}} dx'}_{\text{solve this integral}} \int_{-\infty}^{\infty} e^{-\frac{y'^2}{2w_0^2}+ik\frac{(y'-y)^2}{2z}} dy'.$$

$$\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2w_0^2}+ik\frac{(x'-x)^2}{2z}} dx' = \int_{-\infty}^{\infty} e^{-\frac{1}{2w_0^2}\underbrace{\left[1-ik\frac{w_0^2}{z}\right]}_{a} x'^2 - ik\frac{xx'}{z}} dx'\, e^{\frac{ikx^2}{2z}}$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-\frac{a}{2w_0^2}\left[\underbrace{x'^2 + 2i\frac{kw_0^2}{az}xx' + \left(\frac{ikw_0^2}{az}x\right)^2}_{\text{complete square}}\right]} dx'}_{\sqrt{\frac{2\pi}{a}}\, w_0} e^{-\frac{a}{2w_0^2}\left(\frac{kw_0^2 x}{az}\right)^2 + \frac{ikx^2}{2z}}$$

$$= \sqrt{\frac{2\pi}{a}}\, w_0\, e^{-\left(\frac{(kw_0^2)^2}{2w_0^2 a z^2} - i\frac{k}{2z}\right)x^2}$$

$$= \sqrt{\frac{2\pi}{a}}\, w_0\, e^{-\frac{kw_0^2 x^2}{2w_0^2 a z^2}\big(\underbrace{kw_0^2 - iaz}_{kw_0^2 - iz - kw_0^2}\big)} = \sqrt{\frac{2\pi}{a}}\, w_0\, e^{\frac{ikx^2}{2az}}$$

so

$$U^G(x,y,z) = \frac{ku_0}{2\pi i z} e^{ikz} \frac{2\pi w_0^2}{a} e^{\frac{ik(x^2+y^2)}{2az}}$$

$$= \frac{kw_0^2 u_0}{iz\left(1-ik\frac{w_0^2}{z}\right)} e^{ikz + \frac{ik(x^2+y^2)}{2z\left(1-ik\frac{w_0^2}{z}\right)}} = \frac{u_0}{1+i\frac{z}{z_R}} e^{ikz + ik\frac{(x^2+y^2)}{2(z-iz_R)}}$$

$$\boxed{U^G(x,y,z) = \frac{u_0}{1+i\frac{z}{z_R}}\, e^{ikz + ik\frac{(x^2+y^2)}{2(z-iz_R)}}}$$

where $z_R = kW_0^2$ is called the Rayleigh range.

[Note: some references (including Wikipedia) use the initial field $U_0^G = u_0\, e^{-\frac{x^2+y^2}{W_0^2}}$, without the "2" in the denominator. That is, their W_0 is $\sqrt{2}$ times the W_0 used here. Therefore, in those references, $z_R = kW_0^2/2$.]

In terms of amplitude and phase

$$1 + i\frac{z}{z_R} = \sqrt{1+\frac{z^2}{z_R^2}}\, e^{i\zeta(z)}, \quad \zeta = \arctan\left(\frac{z}{z_R}\right), \text{ so}$$

$$U^G = \frac{u_0}{\sqrt{1+\frac{z^2}{z_R^2}}}\, e^{i(kz-\zeta(z))}\, e^{ik\frac{(x^2+y^2)(z+iz_R)}{2(z_R^2+z^2)}}$$

$$= \frac{u_0}{\sqrt{1+\frac{z^2}{z_R^2}}}\, e^{-\frac{kz_R(x^2+y^2)}{2(z_R^2+z^2)}}\, e^{i\left[kz-\zeta(z)+kz\frac{(x^2+y^2)}{2(z_R^2+z^2)}\right]}$$

$$= \frac{u_0}{\sqrt{1+\frac{z^2}{z_R^2}}}\, e^{-\frac{x^2+y^2}{2W^2(1+z^2/z_R^2)}}\, e^{i\left[kz-\zeta(z)+\frac{z\,(x^2+y^2)}{2z_R W_0^2(1+z^2/z_R^2)}\right]}$$

$$= \frac{u_0}{\sqrt{1+\frac{z^2}{z_R^2}}}\, e^{-\frac{x^2+y^2}{2W^2(z)}}\, e^{i\left[kz-\zeta(z)+\frac{z}{2z_R}\frac{(x^2+y^2)}{W^2(z)}\right]}$$

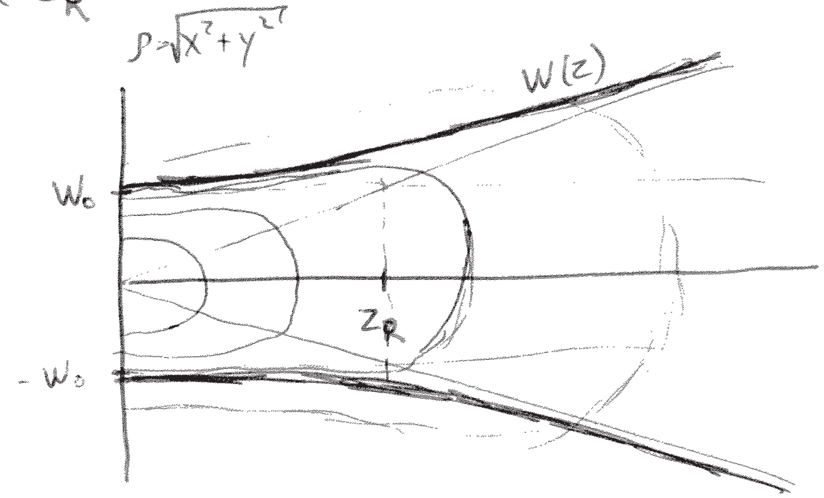
where $W(z) = W_0\sqrt{1+z^2/z_R^2}$

Intensity:

$$I^G(x,y,z) = \underbrace{\frac{|u_0|^2}{1+z^2/z_R^2}} \underbrace{e^{-\frac{x^2+y^2}{w^2(z)}}}$$

Lorenzian in z, with width z_R (on axis)

Gaussian in x & y with width $w(z)$



Phase:

$$\Phi(x,y,z) = kz - \zeta(z) + \frac{z}{2z_R}\frac{(x^2+y^2)}{w^2(z)}$$

On-axis, $\Phi(0,0,z) = kz - \zeta(z)$

rate of change $\frac{\partial \Phi}{\partial z}(0,0,z) = k - \frac{1}{z_R}\frac{1}{1+z^2/z_R^2}$

spacing of wavefronts $\approx \frac{2\pi}{\partial\Phi/\partial z} = \frac{2\pi}{\underset{\frac{2\pi}{\lambda}}{k} - \frac{1}{z_R}(1+z^2/z_R^2)^{-1}}$

$$= \frac{\lambda}{1 - \frac{1}{kz_R}\left(1+\frac{z^2}{z_R^2}\right)^{-1}}$$

Wavefronts more spaced near waist: Gouy phase shift.

Hermite-Gaussian Beams

$$U_{m,n}^{HG} = u_0 H_m\left(\frac{x}{w_0}\right) H_n\left(\frac{y}{w_0}\right) e^{-\frac{x^2+y^2}{2w_0^2}}$$

where

$H_n(\tau)$ = Hermite polynomial $= e^{\frac{\tau^2}{2}}\left(\tau - \frac{d}{d\tau}\right)^n e^{-\tau^2/2}$

$$H_0(\tau) = 1$$
$$H_1(\tau) = 2\tau$$
$$H_2(\tau) = 4\tau^2 - 2$$
$$H_3(\tau) = 8\tau^3 - 12\tau$$
$$\vdots$$

Plugging into Fresnel formula:

$$U_{m,n}^{HG}(x,y,z) = \frac{u_0 H_m\left(\frac{x}{w(z)}\right) H_n\left(\frac{y}{w(z)}\right) e^{-\frac{x^2+y^2}{2w^2(z)}}}{\sqrt{1+z^2/z_R^2}} e^{i\left[kz-(1+m+n)\zeta(z)+\frac{z(x^2+y^2)}{2z_R w^2(z)}\right]}$$

Their intensity is always of the same shape in x,y except for a scaling

$$I_{m,n}^{HG}(x,y,z) = \left|U_{m,n}^{HG}(x,y,z)\right|^2 = \frac{|u_0|^2 H_m^2\left(\frac{x}{w(z)}\right) H_n^2\left(\frac{y}{w(z)}\right) e^{-\frac{x^2+y^2}{w^2(z)}}}{1+z^2/z_R^2}$$

The phase is

$$\Phi_{m,n}(x,y,z) = kz - (1+m+n)\zeta(z) + \frac{z}{2z_R}\frac{(x^2+y^2)}{w^2(z)}$$

so the wavefronts have the same shape as for U^G; there is only an extra phase $-(m+n)\zeta(z)$.

Far field approximation (paraxial)

Assume that the initial field $U(x,y,0)$ differs significantly from zero only within a region of radius a, so

$$U(x,y,0) \simeq 0 \quad \text{for } x^2+y^2 > a^2.$$

We can then simplify the Fresnel propagation formula:

$$U(x,y,z) = \frac{k}{2\pi i z} e^{ikz} \iint_{-\infty}^{\infty} U(x',y',0)\, e^{\frac{ik}{2z}\left[x'^2+y'^2-2xx'-2yy'+x^2+y^2\right]} dx'dy'$$

$$= \frac{k}{2\pi i z} e^{ik\left[\frac{x^2+y^2}{2z}+z\right]} \iint_{-\infty}^{\infty} U(x',y',0)\, \underbrace{e^{ik\frac{x'^2+y'^2}{2z}}}\, e^{-ik\left(x'\frac{x}{z}+y'\frac{y}{z}\right)} dx'dy'$$

Note, $k\dfrac{x'^2+y'^2}{2z} \leq \dfrac{ka^2}{2z} = \dfrac{2\pi a^2}{2\lambda z}$

for the values of x', y' that contribute to the integral

If z is large enough that $\dfrac{2\pi a^2}{2\lambda z} \ll 2\pi$,

then $e^{ik\frac{x'^2+y'^2}{2z}} \approx 1$ and

$$U(x,y,z) \approx \frac{k}{2\pi i z} e^{ik\left[\frac{x^2+y^2}{2z}+z\right]} \iint_{-\infty}^{\infty} U(x',y',0)\, e^{-ik\left(x'\frac{x}{z}+y'\frac{y}{z}\right)} dx'dy'$$

$$= \underbrace{\frac{1}{iz} e^{ik\left[\frac{x^2+y^2}{2z}+z\right]}}_{\text{Paraxial approximation of a spherical wavefront}} \hat{f}\, U(x',y',0)\Big|_{\substack{x'\to x/z \\ y'\to y/z}}, \quad \text{valid when } \frac{a^2}{2\lambda z} \ll 1$$

M^2 beam quality factor

For a Gaussian beam, the intensity's standard deviation width in x is:

$$I^G(x,y,0) = \left| u_0 e^{-\frac{x^2+y^2}{2w_0^2}} \right|^2 = |u_0|^2 e^{-\frac{x^2+y^2}{w_0^2}}$$

$$\Delta_x^G = \left[\frac{\iint x^2 I^G(x,y,0)\,dx\,dy}{\iint I^G(x,y,0)\,dx\,dy} \right]^{1/2}$$

$$= \left[\frac{\cancel{|u_0|^2}}{\cancel{|u_0|^2}} \frac{\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{w_0^2}} dx}{\int_{-\infty}^{\infty} e^{-\frac{x^2}{w_0^2}} dx} \cancel{\frac{\int_{-\infty}^{\infty} e^{-\frac{y^2}{w_0^2}} dy}{\int_{-\infty}^{\infty} e^{-\frac{y^2}{w_0^2}} dy}} \right]^{1/2}$$

$$= \left[\frac{\int_{-\infty}^{\infty} x \, \underbrace{x e^{-x^2/w_0^2} dx}_{-\frac{w_0^2}{2} d e^{-x^2/w_0^2}}}{\sqrt{\pi}\, w_0} \right]^{1/2} \underset{\text{integrate by parts}}{=} \left[\frac{\cancel{-x\frac{w_0^2}{2} e^{-\frac{x^2}{w_0^2}} \Big|_{-\infty}^{\infty}}^{\,0}}{\sqrt{\pi} w_0} + \frac{w_0^2}{2} \frac{\sqrt{\pi} w_0}{\sqrt{\pi} w_0} \right]^{1/2}$$

$$= \frac{w_0}{\sqrt{2}}$$

& Δ_y^G gives the same result.

On the other hand, the angular spectrum is:

$$A^G(p_x,p_y) = \hat{\mathcal{F}}_{\substack{x\to p_x \\ y \to p_y}} U(x,y,0) = u_0 \hat{\mathcal{F}}_{x\to p_x} e^{-\frac{x^2}{2w_0^2}} \hat{\mathcal{F}}_{y\to p_y} e^{-\frac{y^2}{2w_0^2}}$$

$$= u_0 k w_0^2 e^{-k^2 \frac{p_x^2+p_y^2}{2} w_0^2}$$

The "radiant intensity" (the squared modulus of the angular spectrum) is

$$|A^G|^2 = |u_0|^2 k^2 w_0^4 e^{-k^2 w_0^2 (p_x^2+p_y^2)}$$

The directional standard deviation is defined as

$$\Delta_{p_x}^{G} = \left[\frac{\iint p_x^2 |A^G(p_x,p_y)|^2 \, dp_x \, dp_y}{\iint |A^G(p_x,p_y)|^2 \, dp_x \, dp_y} \right]^{1/2},$$

which can be found to be

$$\Delta_{p_x}^{G} = \frac{1}{\sqrt{2}\, k w_0}, \quad \text{and same for } \Delta_{p_y}^{G}.$$

These results are in agreement with the uncertainty relation since the field is Gaussian:

$$\Delta_x^G \Delta_{p_x}^G = \frac{1}{2k}, \quad \Delta_y^G \Delta_{p_y}^G = \frac{1}{2k}$$

For any field that is not Gaussian:

$$\Delta_x \Delta_{p_x} \geq \frac{1}{2k}, \quad \Delta_y \Delta_{p_y} \geq \frac{1}{2k}$$

The M^2 "beam propagation factors" (or beam quality factors) are defined as

$$M_x^2 = \frac{\Delta_x \Delta_{p_x}}{\Delta_x^G \Delta_{p_x}^G} = 2k \Delta_x \Delta_{p_x}, \quad M_y^2 = \frac{\Delta_y \Delta_{p_y}}{\Delta_y^G \Delta_{p_y}^G} = 2k \Delta_y \Delta_{p_y}.$$

Notice that, due to the uncertainty relation,

$$M_x^2 \geq 1, \quad M_y^2 \geq 1,$$ with $M_x^2 = M_y^2 = 1$ only for Gaussian beams.

These factors then estimate how much a beam differs from a Gaussian, i.e., how much faster than a Gaussian it spreads.