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An unrestricted solution to predict the hydraulic conductivity of unsaturated soils

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An unrestricted solution to predict the hydraulic conductivity of unsaturated soils

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ABSTRACT

Several models to predict the hydraulic conductivity of unsaturated soils are found in the literature, however some of the most used in scientific reports, like those proposed by Burdine in 1953 and Mualem in 1976, depend on restrictions made when adjusting van Genuchten's model to experimental soil water retention data. A new analytical solution is here presented to predict the unsaturated soil hydraulic conductivity by using the combination of the models of general relative hydraulic conductivity proposed by Burdine and Mualem and the model of soil water retention proposed by van Genuchten in 1980. The relative hydraulic conductivity was derived eliminating all kinds of restrictions used in the past, resulting the best possible fitting between the predicted equation and observed soil water retention curve data, judged by the Akaike information criterion applied to data of different soils of large contrast in terms of granulometry.

Key words: Soil water diffusivity, soil water retention curve.

INTRODUCTION

Based on Richards' (1931), Burdine's (1953) and Mualem's (1976) contributions three main groups of hydraulic conductivity determination methods can be distinguished in terms of models for unsaturated soils. The first group (RK) involves Richards' (1931) contribution and the generalization based on Kozeny's approach for saturated and unsaturated porous media, which empirically assumes that the relative hydraulic conductivity Kr is a function of a dimensionless soil water content Se (Brutsaert, 1967; Mualem, 1976) or of the matric potential head h (L):

$$\mathrm{Kr} = \frac{\mathrm{K}}{\mathrm{Ks}} = \mathrm{Kr}(\mathrm{Se}) \text{ or } \mathrm{Kr} = \mathrm{Kr}(\mathrm{h}) \qquad [1]$$

$$\mathrm{Se} = \frac{\theta - \theta\mathrm{r}}{\theta\mathrm{s} - \theta\mathrm{r}} \qquad [2]$$

where K is the hydraulic conductivity (LT^{-1}), Ks its maximum value and θ is the soil water content ($L^3.L^{-3}$) (θr and θs are, respectively, the residual and saturated water contents).

In relation to the approach of this first group, for the case of $Kr = Se^{\varphi}$, Irmay (1954) determined theoretically the value of the constant $\varphi = 3.0$, although Averjanov (1950) had before proposed the value $\varphi = 3.5$, which was later taken as correct by Brooks and Corey (1964) and Boreli and Vachaud (1966).

The second group (BU) of methods (which follows Burdine's equation) is based on the models of Childs and Collis-George (1950) (including the modifications proposed by Marshall, 1958; Millington and Quirk, 1961; and Kunze; Uehara and Graham, 1968), Burdine (1953), Wyllie and Gardner (1958) and Farrell and Larson (1972)):

$$Kr\left(Se\right) = Se^2 \frac{\int_0^{Se} \frac{1}{\left[h\left(Se\right)\right]^2} dSe}{\int_0^{1} \frac{1}{\left[h\left(Se\right)\right]^2} dSe} \qquad [3]$$

The third group (MU) is represented by the equation derived by Mualem (1976), considering the capillary law and a homogeneous porous medium, applying a pore water distribution function and the probability theory. Mualem (1976) presented the following general model, changing the assumption of Childs and Collis-George (1950) concerning the hydraulic conductivity of the pore sequence in order to take into account the effect of the larger pore section, to obtain the relationship between Kr and h:

$$Kr\left(Se\right) = Se^{\lambda} \left[\frac{\int_0^{Se} \frac{1}{h\left(Se\right)} dSe}{\int_0^{1} \frac{1}{h\left(Se\right)} dSe}\right]^2 \qquad [4]$$

where λ is a dimensionless empirical parameter.

Based on the hypothesis that the best procedure to derive the relative soil hydraulic conductivity from water retention curves is a procedure that presents the best fitting of the van Genuchten's soil water content model to experimental data, a new general solution is proposed: A) for the models of Burdine (1953) and Mualem (1976), in relation to the relative hydraulic conductivity (Kr), and B) that of van Genuchten (1980), related to the soil water retention curve (SWRC), without imposing any restriction related to the choice of parameters.

THEORETICAL

A. The specific relative hydraulic conductivity equation derived by van Genuchten (1980) based on Mualem's (Kr) and van Genuchten's (SWR) models

The general soil water retention curve model relating Se to h, proposed by van Genuchten (1980) is:

$$\mathrm{Se} = \left(\frac{1}{1+(\alpha h)^{n}}\right)^{m} \qquad [5]$$

Mualem (1976) presented equation (4) using a specific value for the constant $(\lambda = 1/2)$, based on data from 45 different soils. Introducing equation [5] into [4] with restriction proposed by Mualem (1976), leads to:

$$\mathrm{Kr}(\mathrm{Se}) = \mathrm{Se}^{1/2}\left(\frac{\int_{0}^{\mathrm{Se}}\left(\frac{Se^{\frac{1}{m}}}{1-Se^{\frac{1}{m}}}\right)^{\frac{1}{n}}\mathrm{dSe}}{\int_{0}^{1}\left(\frac{Se^{\frac{1}{m}}}{1-Se^{\frac{1}{m}}}\right)^{\frac{1}{n}}\mathrm{dSe}}\right)^{2} \qquad [6]$$

Defining the auxiliary functions:

$$\mathrm{f}(\mathrm{Se}) = \int_{0}^{\mathrm{Se}}\left(\frac{\mathrm{Se}^{\frac{1}{m}}}{1-\mathrm{Se}^{\frac{1}{m}}}\right)^{\frac{1}{n}}\mathrm{dSe} \text{ and } \mathrm{f}(1) = \int_{0}^{1}\left(\frac{\mathrm{Se}^{\frac{1}{m}}}{1-\mathrm{Se}^{\frac{1}{m}}}\right)^{\frac{1}{n}}\mathrm{dSe} \qquad [7]$$

and substituting [7] into [6], we obtain:

$$\mathrm{Kr}(\mathrm{Se}) = \mathrm{Se}^{1/2}\left[\frac{\mathrm{f}(\mathrm{Se})}{\mathrm{f}(1)}\right]^{2} \qquad [8]$$

Taking 'y' as an auxiliary variable and substituting $Se = y^{m}$ into equations [7] leads to:

$$\mathrm{f}(\mathrm{Se}) = \mathrm{m}\int_{0}^{\mathrm{Se}^{1/m}} \mathrm{y}^{\mathrm{m}+\frac{1}{n}-1}(1-\mathrm{y})^{\frac{-1}{n}}\mathrm{dy} \text{ and } \mathrm{f}(\mathrm{Se}) = \mathrm{m}\int_{0}^{1} \mathrm{y}^{\mathrm{m}+\frac{1}{n}-1}(1-\mathrm{y})^{\frac{-1}{n}}\mathrm{dy} \qquad [9]$$

For the specific restriction $\mathrm{m} = 1 - 1/n$, equations [9] become:

$$\mathrm{f}(\mathrm{Se}) = \mathrm{m}\int_{0}^{\mathrm{Se}^{1/m}} (1-\mathrm{y})^{\frac{-1}{n}}\mathrm{dy} \text{ and } \mathrm{f}(\mathrm{Se}) = \mathrm{m}\int_{0}^{1}(1-\mathrm{y})^{\frac{-1}{n}}\mathrm{dy} \qquad [10]$$

and the integration of equations [10] yields:

$$\mathrm{f}(\mathrm{Se}) = 1-\left(1-Se^{1/m}\right)^{m} \text{ and } \mathrm{f}(1) = 1, \ (\mathrm{m} = 1 - 1/n) \qquad [11]$$

The substitution of equations [11] into [8] leads to:

$$\mathrm{Kr(Se)} = \mathrm{Se}^{1/2}\left[1-\left(1-Se^{1/m}\right)^{m}\right]^{2},\ (\mathrm{m}=1-1/n),\ (0<\mathrm{m}<1) \text{ and } (\lambda=1/2) \qquad [12]$$

Using another restriction $\mathrm{m}=2-1/n$, van Genuchten (1980) simplified the equations [9] to obtain:

$$\mathrm{f(Se)} = \mathrm{m}\int_0^{\mathrm{Se}^{1/m}} \mathrm{y}(1-\mathrm{y})^{\frac{-1}{n}}\,\mathrm{dy} \text{ and } \mathrm{f(Se)} = \mathrm{m}\int_0^{1} \mathrm{y}(1-\mathrm{y})^{\frac{-1}{n}}\,\mathrm{dy},\ (\mathrm{m}=2-1/n) \qquad [13]$$

and the following solution:

$$\mathrm{Kr(Se)} = \mathrm{Se}^{1/2}\left[1-m\left(1-Se^{1/m}\right)^{m-1}+(m-1)\left(1-Se^{1/m}\right)^{m}\right]^{2},\ (\mathrm{m}=2-1/n) \qquad [14]$$

Van Genuchten (1980) comments that equation [14] does not present any attractive alternative to equation [12].

A new general equation is here proposed, following Mualem's (Kr) and van Genuchten's (SWR) models, starting with the substitution of equation [5] into [4]:

$$\mathrm{Kr(Se)} = \mathrm{Se}^{\lambda}\left(\frac{\int_0^{\mathrm{Se}}\left(\frac{Se^{\frac{1}{m}}}{1-Se^{\frac{1}{m}}}\right)^{\frac{1}{n}}\mathrm{dSe}}{\int_0^{1}\left(\frac{Se^{\frac{1}{m}}}{1-Se^{\frac{1}{m}}}\right)^{\frac{1}{n}}\mathrm{dSe}}\right)^{2} = \mathrm{Se}^{\lambda}\left[\frac{\mathrm{f(Se)}}{\mathrm{f(1)}}\right]^{2} \qquad [15]$$

To obtain the general analytical solution, the integration of equations [9] yields:

$$\mathrm{f(Se)} = \left.\frac{mn.y^{m+\frac{1}{n}}.HGf\left(m+\frac{1}{n},\frac{1}{n};m+\frac{1}{n}+1;y\right)}{m.n+1}\right|_0^{Se^{1/m}} \text{ and } \mathrm{f(1)} = \left.\frac{mn.HGf\left(m+\frac{1}{n},\frac{1}{n};m+\frac{1}{n}+1;1\right)}{m.n+1}\right|_0^{1} \qquad [16]$$

where 'HGf' is the hypergeometric function (Abramowitz and Stegun, 1972).

The development of the equations [16] leads to:

$$\mathrm{f(Se)} = \frac{\mathrm{m}.n.Se^{1+\frac{1}{m.n}}.HGf_1\left(m+\frac{1}{n},\frac{1}{n};m+\frac{1}{n}+1;Se^{\frac{1}{m}}\right)}{m.n+1} \text{ and } \mathrm{f(1)} = \frac{\mathrm{m}.n.HGf\left(m+\frac{1}{n},\frac{1}{n};m+\frac{1}{n}+1;1\right)}{m.n+1} \qquad [17]$$

The hypergeometric functions, for the general solution, can be applied as follows:

$$\mathrm{Kr(Se)} = \left[\frac{Se^{\frac{\lambda}{2}+\frac{1}{m.n}+1}\left[1+\eta(Se)\right]}{1+\beta}\right]^{2} \qquad [18]$$

where

$$\eta(\mathrm{Se}) = (mn+1)\sum_{k=1}^{\infty}\left\{\frac{Se^{\frac{k}{m}}}{k!}\prod_{i=1}^{k}\frac{\left[1+(i-1)n\right]}{n^{k}(mn+kn+1)}\right\} \approx (mn+1)\sum_{k=1}^{4}\left\{\frac{Se^{\frac{k}{m}}}{k!}\prod_{i=1}^{k}\frac{\left[1+(i-1)n\right]}{n^{k}(mn+kn+1)}\right\} \qquad [19]$$

and

$$\beta = \frac{(mn+1)}{n^4}\left[\frac{n^3}{(mn+n+1)} + \frac{(1+n)n^2}{2(mn+2n+1)} + \frac{(1+n)(1+2n)n}{6(mn+3n+1)} + \frac{(1+n)(1+2n)(1+3n)}{24(mn+4n+1)}\right] \quad [20]$$

Equation [19] can be used with four terms in the model of Mualem (1976), without imposing the restriction for 'λ' and the model of van Genuchten (1980), without the restrictions made by the author for 'm' and 'n':

$$\mathrm{Kr(Se)} = \left[\frac{Se^{\frac{\lambda}{2}+\frac{1}{m.n}+1}\left(1+\eta_1 Se^{\frac{1}{m}} + \eta_2 Se^{\frac{2}{m}} + \eta_3 Se^{\frac{3}{m}} + \eta_4 Se^{\frac{4}{m}}\right)}{1+\beta}\right]^2 \quad [21]$$

where

$$\eta_1 = \frac{(mn+1)}{n(mn+n+1)} \quad [22]$$

$$\eta_2 = \frac{(mn+1)(1+n)}{2n^2(mn+2n+1)} \quad [23]$$

$$\eta_3 = \frac{(mn+1)(1+n)(1+2n)}{6n^3(mn+3n+1)} \quad [24]$$

$$\eta_4 = \frac{(mn+1)(1+n)(1+2n)(1+3n)}{24n^4(mn+4\mathrm{n}+1)} \quad [25]$$

B. The specific relative hydraulic conductivity equation derived by van Genuchten (1980) based on Burdine's (Kr) and van Genuchten's (SWR) models

The substitution of equation [5] into [3] leads to the Burdine (1953) model written in a more convenient form:

$$\mathrm{Kr(Se)} = \mathrm{Se}^2 \frac{\int_0^{\mathrm{Se}} \frac{1}{\mathrm{h(Se)}^2}\mathrm{dSe}}{\int_0^1 \frac{1}{\mathrm{h(Se)}^2}\mathrm{dSe}} = \mathrm{Se}^2\frac{\mathrm{g(Se)}}{\mathrm{g(1)}} \quad [26]$$

Van Genuchten (1980) obtained the following equations similar to equations [7]:

$$\mathrm{g(Se)} = \int_0^{\mathrm{Se}}\left(\frac{\mathrm{Se}^{\frac{1}{\mathrm{m}}}}{1-\mathrm{Se}^{\frac{1}{\mathrm{m}}}}\right)^{\frac{2}{\mathrm{n}}}\mathrm{dSe} \text{ and } \mathrm{g(1)} = \int_0^1\left(\frac{\mathrm{Se}^{\frac{1}{\mathrm{m}}}}{1-\mathrm{Se}^{\frac{1}{\mathrm{m}}}}\right)^{\frac{2}{\mathrm{n}}}\mathrm{dSe} \quad [27]$$

The substitution of $Se = y^m$ into equations [27] yields:

$$\mathrm{g(Se)} = \int_0^{\mathrm{Se}^{1/m}} y^{m-1+\frac{2}{n}}(1-y)^{-\frac{2}{n}}\mathrm{dy} \text{ and } \mathrm{g(1)} = \int_0^1 y^{m-1+\frac{2}{n}}(1-y)^{-\frac{2}{n}}\mathrm{dy} \quad [28]$$

Assuming $\mathrm{m} = 1 - 2/\mathrm{m}$, the equations [28] reduce to:

$$\mathrm{g(Se)} = \left[1-\left(1-Se^{\frac{1}{m}}\right)^m\right] \text{ and } \mathrm{g(1)} = 1 \quad [29]$$

Therefore, van Genuchten (1980) obtained:

$$\mathrm{Kr}(\mathrm{Se}) = \mathrm{Se}^2\left[1-\left(1-Se^{\frac{1}{m}}\right)^m\right];\text{ with }\left(m = 1-\frac{2}{m}\right),\ (0 < m < 1)\text{ and }(n > 2) \quad [30]$$

A new general equation is also here proposed, following Burdine's (Kr) and van Genuchten's (SWR) models, eliminating the restrictions of equation [30].

In a similar way, as made for equation [9], the integration of equations [24] results in the general solution:

$$\mathrm{g}(\mathrm{Se}) = \left.\frac{n.y^{m+\frac{2}{n}}HGf\left(m+\frac{2}{n},\frac{2}{n};m+\frac{2}{n}+1;y\right)}{m.n+2}\right|_0^{Se^{1/m}} \text{ and } \mathrm{g}(1) = \left.\frac{n.HGf\left(m+\frac{2}{n},\frac{2}{n};m+\frac{2}{n}+1;1\right)}{m.n+2}\right|_0^1 \quad [31]$$

The development of equations [31] yields:

$$\mathrm{g}(\mathrm{Se}) = \left.\frac{n.Se^{1+\frac{2}{m.n}}HGf\left(m+\frac{2}{n},\frac{2}{n};m+\frac{2}{n}+1;Se^{1/m}\right)}{m.n+2}\right|_0^{Se^{1/m}} \text{ and } \mathrm{g}(1) = \frac{n.HGf\left(m+\frac{2}{n},\frac{2}{n};m+\frac{2}{n}+1;1\right)}{m.n+2} \quad [32]$$

By expansion of the hypergeometric functions using the similar procedure used above, the general equation for predicting Kr can be expressed as follows:

$$\mathrm{Kr}(\mathrm{Se}) = \frac{Se^{3+\frac{2}{m.n}}\left[1+\xi(Se)\right]}{1+\xi} \quad [33]$$

where

$$\xi(\mathrm{Se}) = (mn+2)\sum_{k=1}^{\infty}\left\{\frac{Se^{\frac{k}{m}}}{k!}\prod_{i=1}^{k}\frac{\left[2+(i-1)n\right]}{n^k(mn+kn+2)}\right\} \approx (mn+2)\sum_{k=1}^{4}\left\{\frac{Se^{\frac{k}{m}}}{k!}\prod_{i=1}^{k}\frac{\left[2+(i-1)n\right]}{n^k(mn+kn+2)}\right\} \quad [34]$$

$$\xi = \frac{(mn+2)}{n^4}\left[\frac{n^3}{(mn+n+2)}+\frac{(2+n)n^2}{2(mn+2n+2)}+\frac{(2+n)(2+2n)n}{6(mn+3n+2)}+\frac{(2+n)(2+2n)(2+3n)}{24(mn+4n+2)}\right] \quad [35]$$

Equation [34] can also be used with four terms without the restriction for 'λ', 'm' and 'n':

$$\mathrm{Kr}(\mathrm{Se}) = \frac{Se^{3+\frac{2}{m.n}}\left(1+\zeta_1 Se^{\frac{1}{m}}+\zeta_2 Se^{\frac{2}{m}}+\zeta_3 Se^{\frac{3}{m}}+\zeta_4 Se^{\frac{4}{m}}\right)}{1+\xi} \quad [36]$$

where

$$\zeta_1 = \frac{2(mn+2)}{n(mn+n+2)} \quad [37]$$

$$\zeta_2 = \frac{(mn+2)(2+n)}{n^2(mn+2n+2)} \quad [38]$$

$$\zeta_3 = \frac{(mn+2)(2+n)(2+2n)}{3n^3(mn+3n+2)} \quad [39]$$

$$\zeta_4 = \frac{(mn+2)(2+n)(2+2n)(2+3n)}{12n^4(mn+4n+2)} \quad [40]$$

A summary of van Genuchten's model and the new solution for predicting Kr function of Se and h, and the soil water diffusivity $D(\theta) = K(\theta)\frac{dh}{d\theta}$ ($L^2.T^{-1}$) as function of Se, using Mualem's and Burdine's model, is shown in Tables 2, 3 and 4.

Comparison with experimental data

To compare the NS (unrestricted) with the restricted models, twelve procedures were used to calculate the parameters for the soil water retention curve following van Genuchten's (1980) model, which are shown below (Table 5).

These procedures were applied to data of a Rhodic Kanhapludalf soil using the SWRC software (Dourado-Neto et al., 2000), consisting of eight experimental points [pressure head (cm)/soil water content ($cm^3.cm^{-3}$): 5/0.315, 10/0.314, 60/0.278, 100/0.267, 300/0.252, 800/0.235, 3000/0.208 and 15000/0.199], which represent the global mean of 250 soil water retention curves (Moraes et al., 1993). The result is shown below, in Table 6. The Akaike Information Criterion (AIC) values (Akaike, 1974) were used in this comparison (Figure 1) and they indicate that the best procedure is number 4 (Table 1), in which 'm' is independent of 'n' and the residual and saturated soil water contents are defined by non linear regression, i.e., without any restriction.

For other four soils of large contrast in terms of granulometry (Macedo, 1991), comprising forty soil water retention curves corresponding to three layers (0-15, 15-30 and 30-50 cm), Figure 2 also compares criteria used to obtain the SWRC parameters. Note that for the case of the parameter 'm', that is restricted to the range 0-1 for the restricted models, the new solution shows values larger than 1, of the order of 2 to 3, for soils 2, 3, 5, 7, 31, 32, 33, 35, 37 and 40 which are sandy (Figure 2A), indicating a significant superiority of NS. This affects the regression procedures and, as shown in Figure 2B, the AIC values were the lowest for all 40 soils.

Some of the results are shown in Figures 3 to 6 in relation to the performance of the methods in relation to the adjustment of the SWRC and the respective Kr equations.

Comparison with numerical integration

The comparison of the analytical solution with a numerical integration was made in two ways: (i) considering seven computed values of the dimensionless water content as function of the pressure head (0, 60, 100, 333, 1000, 5000 and 15000 cm) (Table 7) using all forty empirical parameters 'α' (cm^{-1}), 'm', 'n', Θ_r and Θ_s ($cm^3.cm^{-3}$) calculated according to the criterion 4 ('m' independent of 'n'), for the three layers of the four soils of Macedo (1991) (Table 8); and (ii) considering the analytical solution, equation [16], $\frac{f(y)}{m} = \frac{n}{(m.n+1)}\left[b^{m+\frac{1}{n}}.HGf\left(m+\frac{1}{n},\frac{1}{n};m+\frac{1}{n}+1;b\right) - a^{m+\frac{1}{n}}.HGf\left(m+\frac{1}{n},\frac{1}{n};m+\frac{1}{n}+1;a\right)\right]$, using Mualem's approach (Figure 7-A) compared with equation [9], $\frac{f(y)}{m} = m\int_a^b y^{m+\frac{1}{n}-1}(1-y)^{\frac{-1}{n}}dy$; and equation [31], $g(y) = \frac{n}{m.n+2}\left[b^{m+\frac{2}{n}}HGf\left(m+\frac{2}{n},\frac{2}{n};m+\frac{2}{n}+1;b\right) - a^{m+\frac{2}{n}}HGf\left(m+\frac{2}{n},\frac{2}{n};m+\frac{2}{n}+1;a\right)\right]$, using Burdine's approach (Figure 7-B) to compare with the equation [28], $g(y) = \int_a^b y^{m-1+\frac{2}{n}}(1-y)^{-\frac{2}{n}}dy$, according to the numerical integration by Simpson sums' method (a=0.10 and b=0.94). This comparison between the analytical solution for Mualem's approach (equation 16: f(Se)/m) and a numerical integration (equation 9: f(Se)/m) (Figure 7A) and the comparison between the analytical solution for Burdines's approach (equation 31), and the numerical integration (equation 28) (Figure 7B), indicate an excellent adjustment showing that the NS is in agreement with the theoretical aspects of the methods.

FINAL CONSIDERATIONS

The new general model for predicting the unsaturated hydraulic conductivity was based on a combination of the general relative hydraulic conductivity of Burdine (1953) and Mualem (1976) with the soil water retention model of van Genuchten (1980). The relative hydraulic conductivity was derived without any kind of restriction (the model can be used for any values of all empirical parameters as 'λ', 'm' and 'n', mainly), resulting in the highest possible flexibility, allowing to obtain the minimum deviations between predicted and measured values of soil water hydraulic conductivity.

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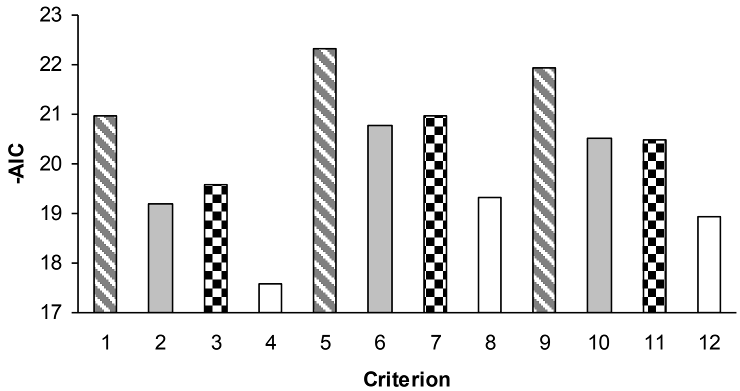

Figure 1. Akaike information criterion (AIC) (Akaike, 1971) used to compare twelve fitting procedures (Table 5) of the water retention curve of the van Genuchten's (1980) model to the observed values for the Rhodic Kanhapludalf soil.

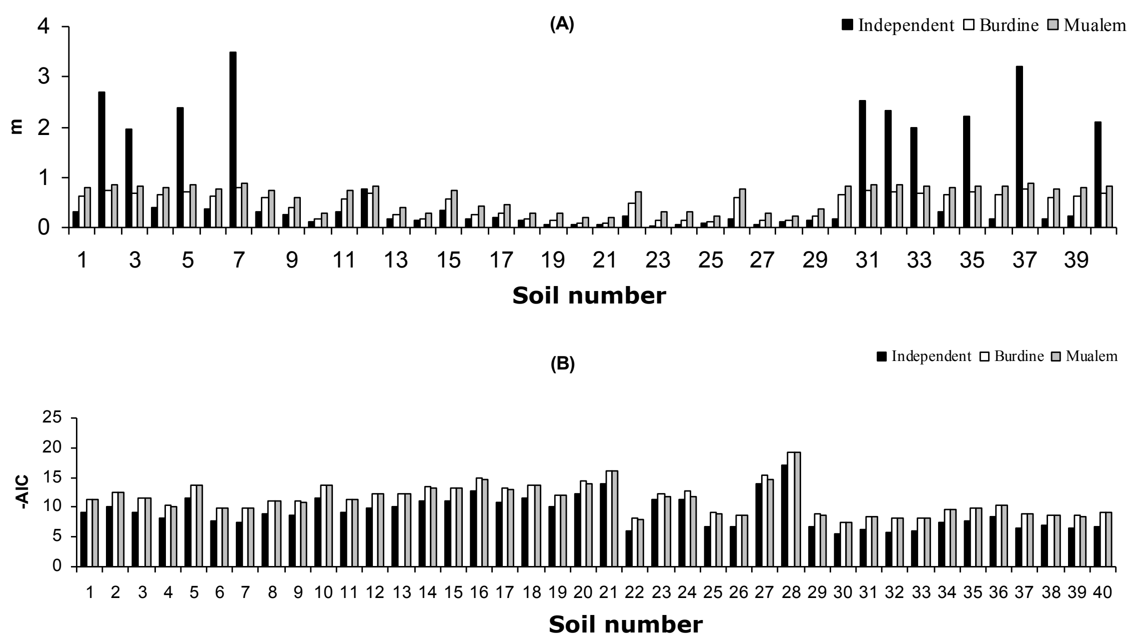

Figure 2. (A) Empirical parameter 'm' and (B) Akaike Information criterion (AIC) (Akaike, 1971) according to the following criteria: (4) 'm' independent of 'n', (8) m=1-2/n (Burdine's approach) (12) m=1-1/n (Mualem's approach) for four Brazilian soils with contrast in terms of granulometry (Macedo, 1991).

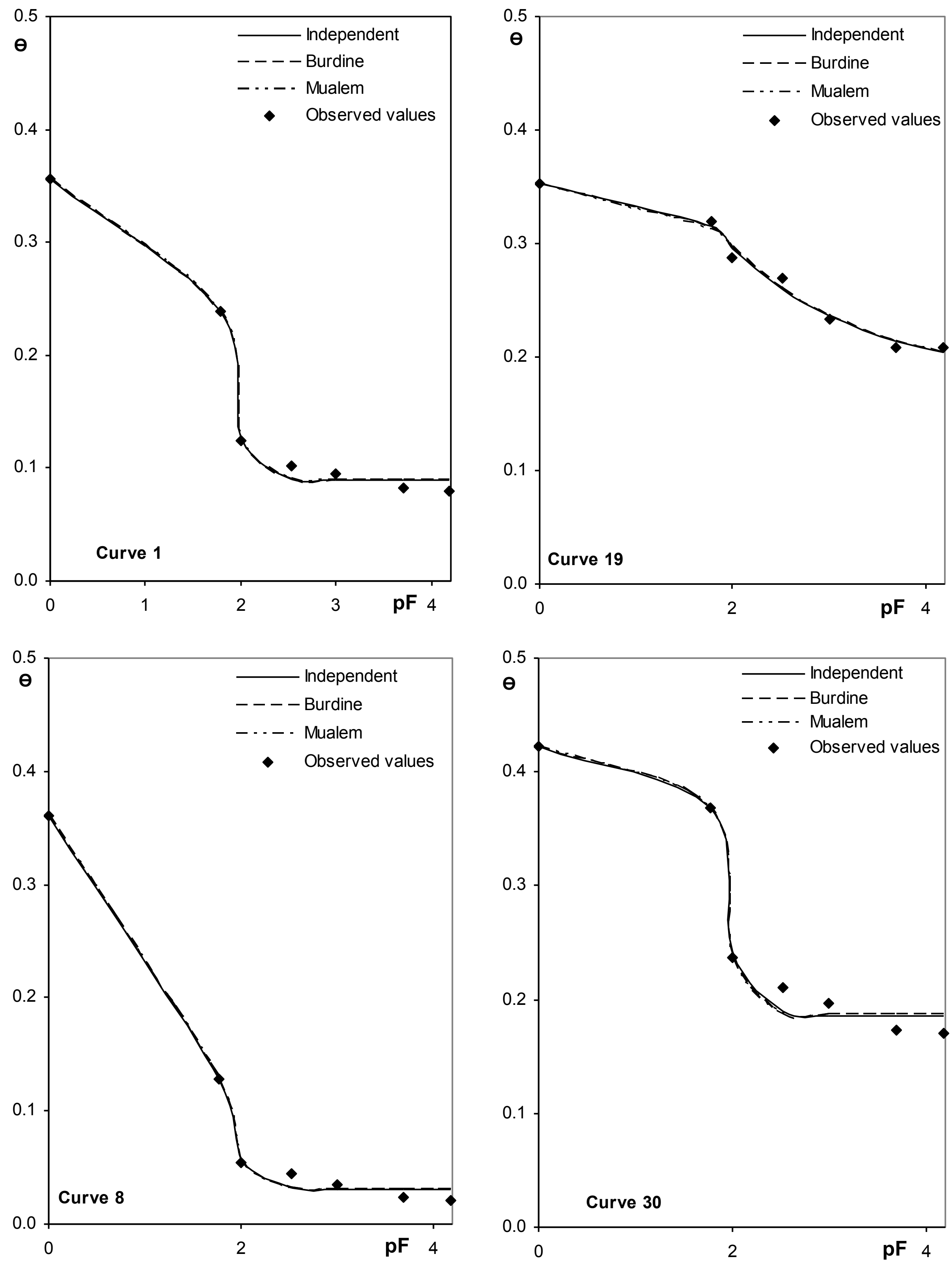

Figure 3. Soil water content (Θ, $cm^3.cm^{-3}$) as function of pF (layer 0-15 cm) for soils 1, 8, 19, and 30 of Macedo (1991).

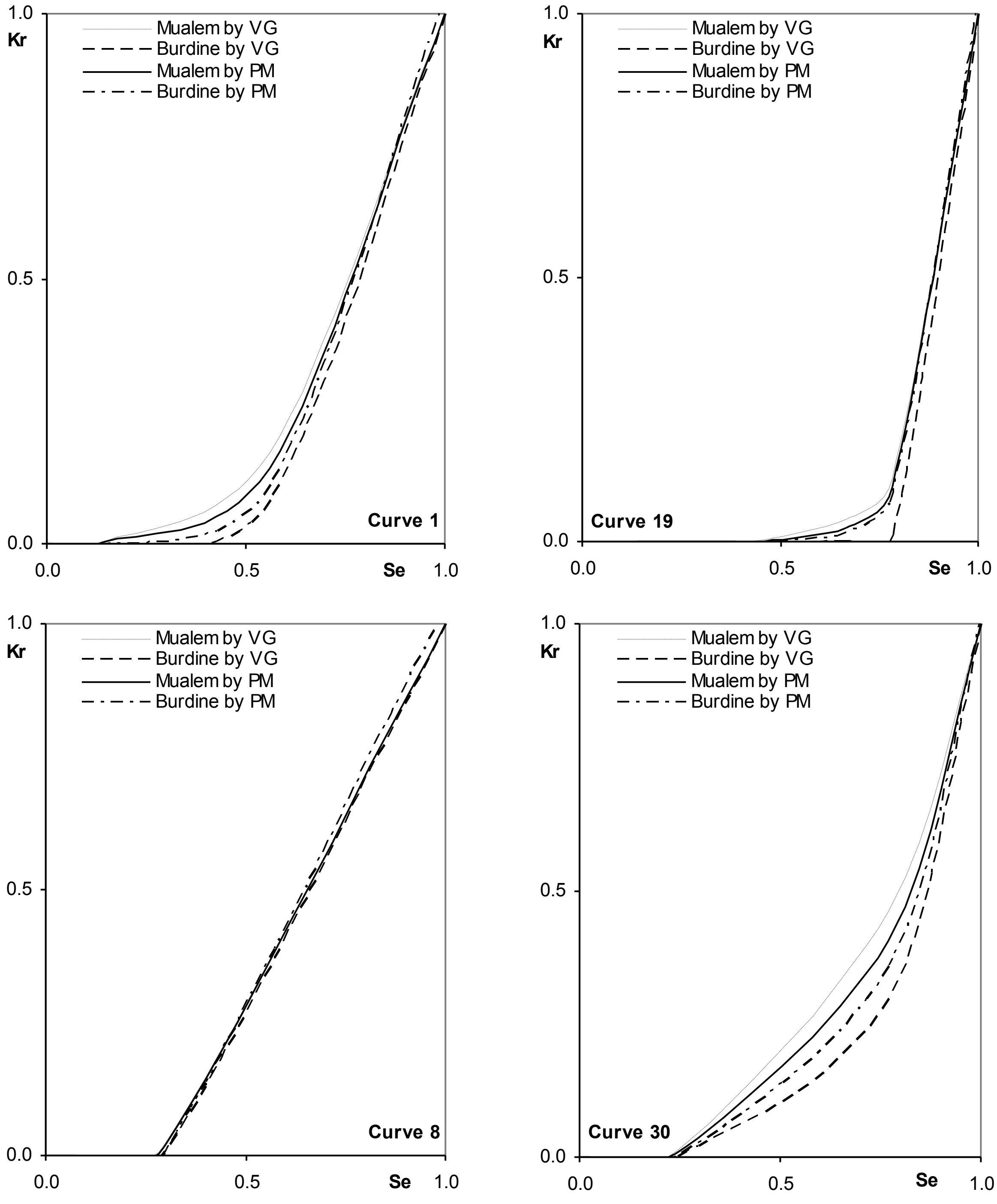

Figure 4. Relative soil hydraulic conductivity (Kr) as function of dimensionless water content (layer 0-15 cm) for soils 1, 8, 19, and 30 of Macedo (1991).

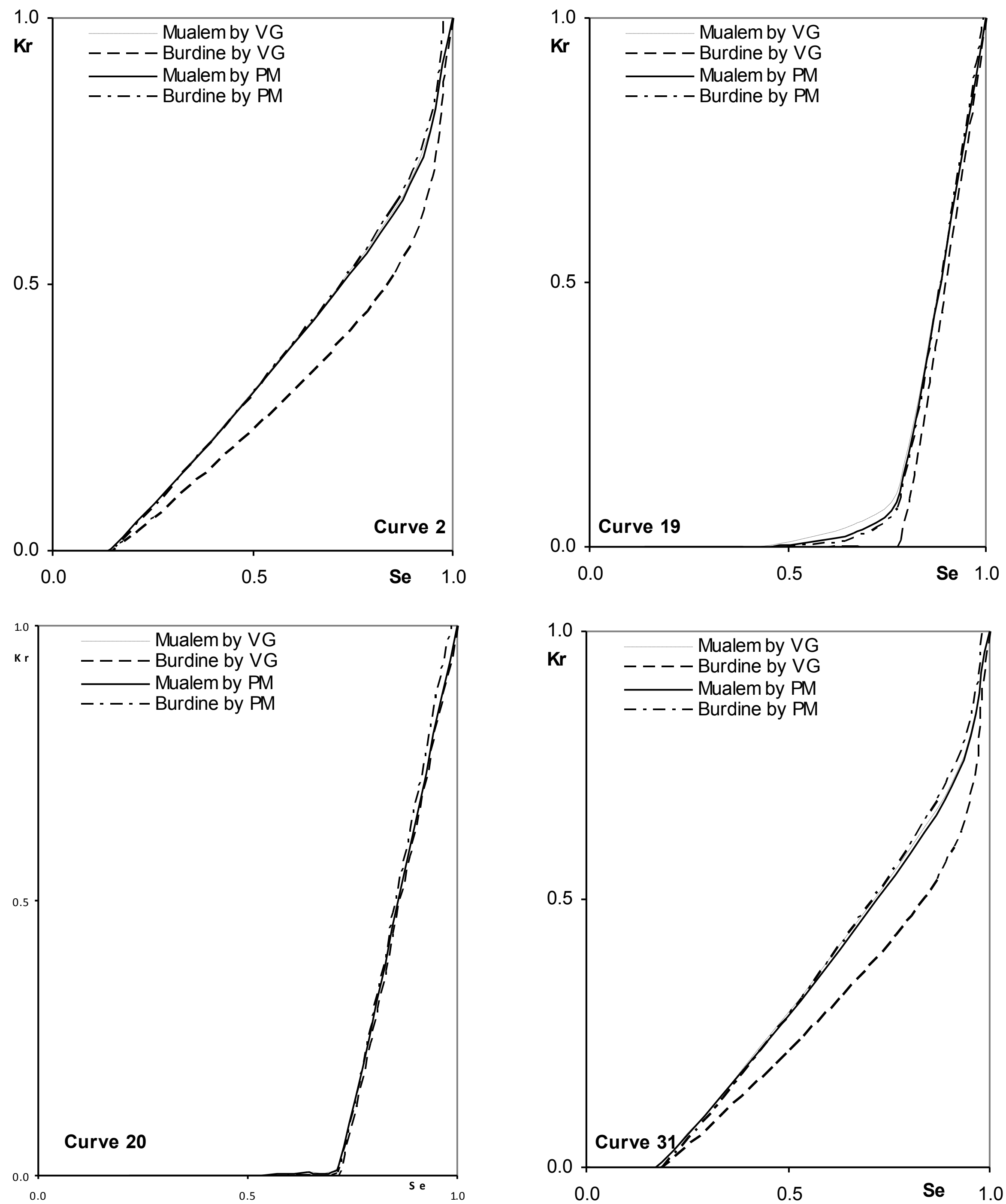

Figure 5. Relative soil hydraulic conductivity (Kr) as function of dimensionless water content (layer 0-15 cm) for soils 2, 9, 20, and 31 of Macedo (1991).

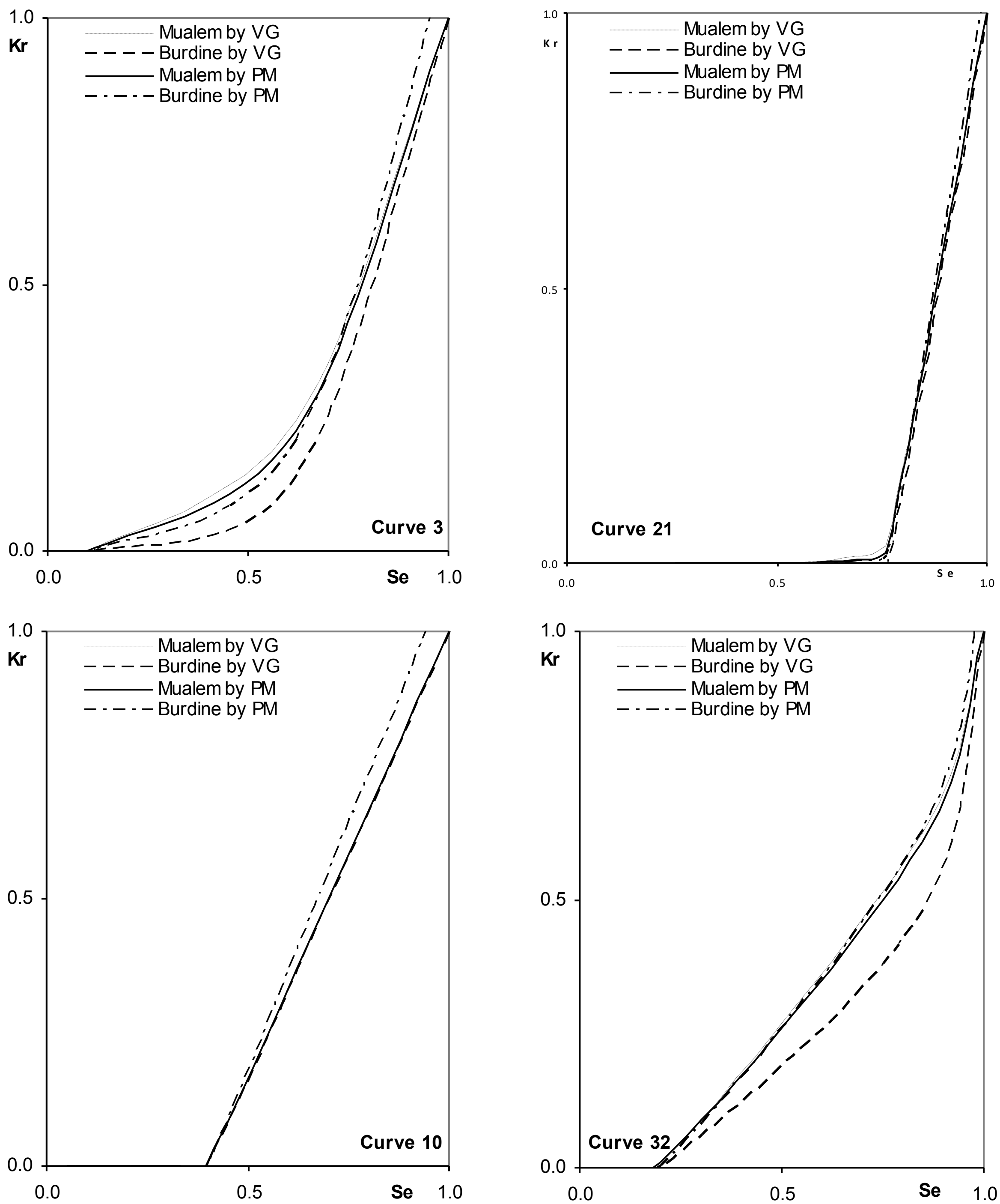

Figure 6. Relative soil hydraulic conductivity (Kr) as function of dimensionless water content (layer 0-15 cm) for soils 3, 10, 21, and 32 of Macedo (1991).

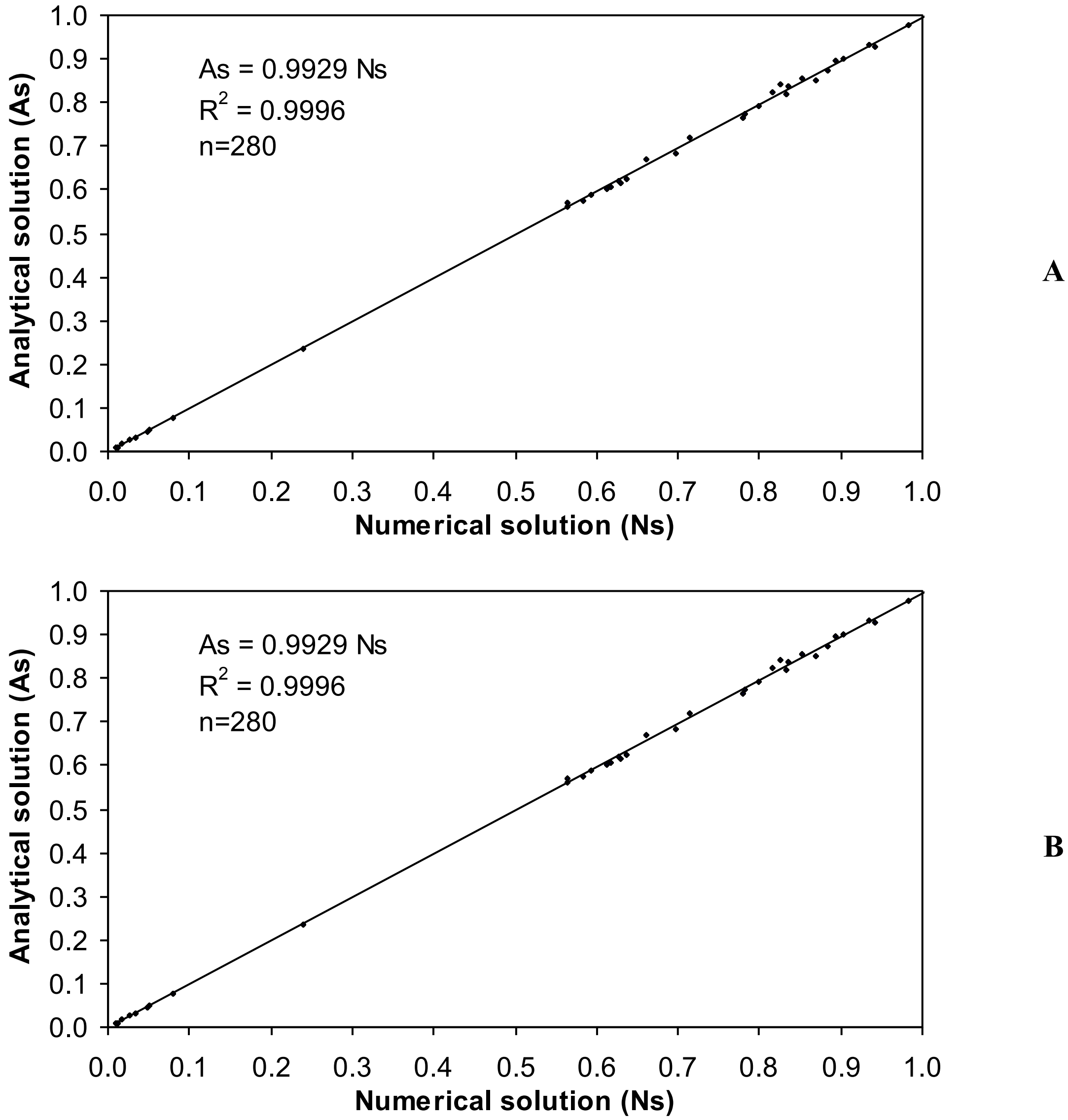

Figure 7. Comparison between the analytical solution for (A) Mualem's and (B) Burdine's approach and numerical integration.

Table 1. Summary of the main models developed by different authors for predicting the hydraulic conductivity (K or Kr) as function of the dimensionless water content (Se) or of the matric potential head (h).

HC model	Group[1]	Author
$K = ah + b$	RK	Richards (1931)
$Kr = Se^{3.5}$	RK	Averjanov (1950)
$Kr(Se) = Se^2 \left[\int_0^{Se} \frac{1}{h(Se)^2} dSe\right] / \left[\int_0^1 \frac{1}{h(Se)^2} dSe\right]$	BU	Burdine (1953)
$Kr = Se^3$	RK	Irmay (1954)
$K = \alpha h^{-n}$	RK	Wind (1955)
$Kr = Se^4$	RK	Corey (1957)
$Kr = e^{-\alpha h}$ or $Kr = \frac{1}{ah^n + 1}$	RK	Gardner (1958)
$Kr = [1 - n(1 - Se)]$	RK	Scott (1963)[5]
$Kr = 1$ if $h < h_e$; $Kr = (h/h_e)^{-n}$ if $h \geq h_e$	RK	Brooks and Corey (1964)
$Kr = 1$ if $h < h_e$; $Kr = e^{-\alpha(h_e - h)}$ if $h_e \leq h < h_1$; $K = K_1 (h/h_1)^{-n}$ if $h \geq h_1$	RK	Ritjema (1965)[5]
$Kr = Se^n$	RK	Brutsaert (1967)
$Kr = \left(\frac{\theta}{\theta s}\right)^n$	RK	Campbell (1973)
$Kr(Se) = Se^{\lambda} \left\{\left[\int_0^{Se} \frac{1}{h(Se)} dSe\right] / \left[\int_0^1 \frac{1}{h(Se)} dSe\right]\right\}^2$, $\lambda = 1/2$	MU	Mualem (1976)
$Kr(Se) = Se^{\lambda + 2 + 2/\gamma}$	MU	Mualem (1976)[2]
$Kr(Se) = \frac{Se^n (e^{2.\alpha.Se} - 2.e^{\alpha.Se} + 1)}{(e^{2.\alpha} - 2.e^{\alpha} + 1)}$	MU	Mualem (1976)[3]
$Kr(Se) = Se^{3 + 2/\gamma}$	BU	van Genuchten (1980)[2]
$Kr(Se) = Se^{5/2 + 2/\gamma}$, $\lambda = 1/2$	MU	van Genuchten (1980)[2]
$Kr(Se) = Se^{1/2} [1 - (1 - Se^{1/m})]^2$	MU	van Genuchten (1980)[4]
$Kr(Se) = Se^2 \left[1 - \left(1 - Se^{\frac{1}{m}}\right)^m\right]$	BU	van Genuchten (1980)[4]
$Kr(Se) = Se^n [1 - (1 - Se^{1/m})]^2$	MU	Nielsen et al. (1986)[4,5]

1 Relative hydraulic conductivity groups: (RK) Richard's (1931) and Kozeny's approach (cited by Mualem, 1976); (BU) Burdines's and (MU) Mualem's models.
2 Deriving Kr using the Brooks and Corey's (1964) soil water retention model: $Se = (\psi/\psi_{crit})^{-\gamma}$
3 Deriving Kr using the Farrel and Larson's (1972) soil water retention model: $\psi = \psi_{crit}.e^{\alpha(1-Se)}$.
4 Deriving Kr using the van Genuchten's (1980) soil water retention model (see equation 5).
5 Cited by Silva (2005).

Table 2. Summary of van Genuchten's (1980) (VG) and new solution (NS) for predicting the relative hydraulic conductivity (Kr) as function of the dimensionless water content (Se).

Model	Restriction	Group[1]	Equation	Author
$\mathrm{Kr}(\mathrm{Se}) = \mathrm{Se}^{1/2}\left[1-\left(1-Se^{1/m}\right)\right]^2$	$\mathrm{m} = 1-1/n$, $0 < \mathrm{m} < 1$ and $\lambda = 1/2$	MU	12	VG
$\mathrm{Kr}(\mathrm{Se}) = \mathrm{Se}^{1/2}\left[1-m\left(1-Se^{1/m}\right)^{m-1}+(m-1)\left(1-Se^{1/m}\right)^m\right]^2$	$\mathrm{m} = 2-1/n$ and $\lambda = 1/2$	MU	14	VG
$\mathrm{Kr}(\mathrm{Se}) = \left[\dfrac{Se^{\frac{\lambda}{2}+\frac{1}{m.n}+1}\left[1+\eta(Se)\right]}{1+\beta}\right]^2$	no	MU	18[2]	NS
$\mathrm{Kr}(\mathrm{Se}) = \mathrm{Se}^2\left[1-\left(1-Se^{\frac{1}{m}}\right)^m\right]$	$m = 1-2/n$, $0 < m < 1$ and $n > 2$	BU	26	VG
$\mathrm{Kr}(\mathrm{Se}) = \dfrac{Se^{3+\frac{2}{m.n}}\left[1+\xi(Se)\right]}{1+\xi}$	no	BU	33[3]	NS

[1] Relative hydraulic conductivity groups: (BU) Burdines's model and (MU) Mualem's model. [2] See equations [19] and [20]. [3] See equations [34] and [35].

Table 3. Summary of van Genuchten's (1980) (VG) and the new solution (NS) for predicting the relative hydraulic conductivity (Kr) as function of the pressure head (h).

Model	Restriction	Group[1]	Equation	Author
$\mathrm{Kr}(\mathrm{h}) = \dfrac{\left\{1-(\alpha h)^{n-1}\left[1+(\alpha h)^n\right]^{-m}\right\}^2}{\left[1+(\alpha h)^n\right]^{m/2}}$	$\mathrm{m} = 1-1/n$, $0 < \mathrm{m} < 1$ and $\lambda = 1/2$	MU	12	VG
$\mathrm{Kr}(\mathrm{h}) = \left[\dfrac{1+\eta(h)}{(1+\beta)\left[1+(\alpha h)^n\right]^{\left(\frac{\lambda}{2}+\frac{1}{m.n}+1\right)m}}\right]^2$	No	MU	18[2]	NS
$\mathrm{Kr}(\mathrm{h}) = \dfrac{1-(\alpha h)^{n-2}\left[1+(\alpha h)^n\right]^{-m}}{\left[1+(\alpha h)^n\right]^{2m}}$	$m = 1-2/n$, $0 < m < 1$ and $n > 2$	BU	26	VG
$\mathrm{Kr}(\mathrm{h}) = \dfrac{1+\xi(h)}{(1+\xi)\left[1+(\alpha h)^n\right]^{\left(3+\frac{2}{m.n}\right)m}}$	No	BU	33[3]	NS

[1] Relative hydraulic conductivity groups: (BU) Burdines's model and (MU) Mualem's model.

[2] $$\eta(\mathrm{h}) = (mn+1)\sum_{k=1}^{4}\left\{\frac{1}{k!\left[1+(\alpha h)^n\right]^k}\prod_{i=1}^{k}\frac{\left[1+(i-1)n\right]}{n^k(mn+kn+1)}\right\}$$

[3] $$\xi(\mathrm{h}) = (mn+2)\sum_{k=1}^{4}\left\{\frac{1}{k!\left[1+(\alpha h)^n\right]^k}\prod_{i=1}^{k}\frac{\left[2+(i-1)n\right]}{n^k(mn+kn+2)}\right\}$$

Table 4. Summary of van Genuchten's (1980) (VG) and the new solution (NS) for predicting the diffusivity (D, $L.T^{-2}$) as function of dimensionless soil water content (Se).

Model	Restriction	Group[1]	Equation	Author
$D(Se) = \frac{(1-m)Ks}{\alpha(\theta s-\theta r)m} Se^{\frac{1}{2}-\frac{1}{m}}\left[(1-Se^{1/m})^{-m}+(1-Se^{1/m})^{m}-2\right]$	$m = 1-1/n$, $0 < m < 1$ and $\lambda = 1/2$	MU	12	VG
$D(Se) = \frac{-(1-m)Se^{\lambda+2\left(\frac{1}{m.n}+1\right)-\frac{1}{m}}\left[1+\eta(Se)\right]^{2}}{\alpha(\theta s-\theta r)m(1+\beta)^{2}(1-Se^{1/m})^{m}}$	No	MU	18[2]	NS
$D(Se) = \frac{(1-m).Ks}{2\alpha(\theta s-\theta r)m} Se^{\frac{8-\frac{1}{m}}{2}}\left[\left(1-Se^{\frac{1}{m}}\right)^{\frac{-(m+1)}{2}}\right]-\left(1-Se^{\frac{1}{m}}\right)^{\frac{m-1}{2}}$	$m = 1-2/n$, $0 < m < 1$ and $n > 2$	BU	24	VG
$D(Se) = \frac{-(1-m)Se^{3+\frac{2}{m.n}-\frac{1}{n}}\left[1+\xi(Se)\right]}{\alpha(\theta s-\theta r)m(1+\xi)(1-Se^{1/m})^{m}}$	No	BU	33[3]	NS

1 Relative hydraulic conductivity groups: (BU) Burdines's model and (MU) Mualem's model. **2** See equations [19] and [20]. **3** See equations [34] and [35].

Table 5. Procedures used to calculate the soil physical parameters ['α' (cm^{-1}), 'm', 'n', Θ_r and Θ_s] for the soil water retention curve following van Genuchten's (1980) model.

Procedure	Statistical procedure of calculating 'α' and 'n': Regression		
	'm'	Θ_r ($cm^3.cm^{-3}$)	Θ_s ($cm^3.cm^{-3}$)
1	Unrestricted[1]	Extrapolation[4]	Extrapolation[4]
2			Regression
3		Regression	Extrapolation
4			Regression
5	Restricted (m=1-2/n)[2]	Extrapolation	Extrapolation
6			Regression
7		Regression	Extrapolation
8			Regression
9	Restricted (m=1-1/n)[3]	Extrapolation	Extrapolation
10			Regression
11		Regression	Extrapolation
12			Regression

1 Criteria that allow the derivation of the new solution for predicting the relative hydraulic conductivity.
2 Criteria that allow the derivation of the van Genuchten's model for predicting the relative hydraulic conductivity using Burdine's (1953) approach.
3 Criteria that allow the derivation of the van Genuchten's model for predicting the relative hydraulic conductivity using Mualem's (1976) approach.
4 Jong van Lier and Dourado-Neto (1993).

Table 6. Soil physical parameters [α (cm^{-1}), m, n, Θ_r ($cm^3.cm^{-3}$) and Θ_s ($cm^3.cm^{-3}$)] of the water retention adjusted by van Genuchten's (1980) model using twelve procedures for a Rhodic Kanhapludalf soil and the Akaike Information Criterion (AIC) (Akaike, 1971).

Procedure	α	m	n	Θ_r	Θ_s
1	0.0730	0.0335	4.7543	0.142	0.315
2	0.0732	0.0346	4.6000	0.142	0.315
3	0.0806	0.0229	5.5983	0.116	0.315
4	0.0800	0.0318	4.1557	0.119	0.316
5	0.0662	0.0816	2.1778	0.150	0.316
6	0.0710	0.0801	2.1743	0.149	0.318
7	0.0710	0.0710	2.1529	0.134	0.316
8	0.0773	0.0692	2.1486	0.131	0.318
9	0.0534	0.1784	1.2172	0.162	0.320
10	0.0626	0.1722	1.2081	0.160	0.323
11	0.0580	0.1613	1.1923	0.151	0.321
12	0.0697	0.1539	1.1818	0.147	0.324