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Deriving an expression for the Matric Flux

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Deriving an expression for the Matric Flux Potential based on the van Genuchten hydraulic function parameters

Main references

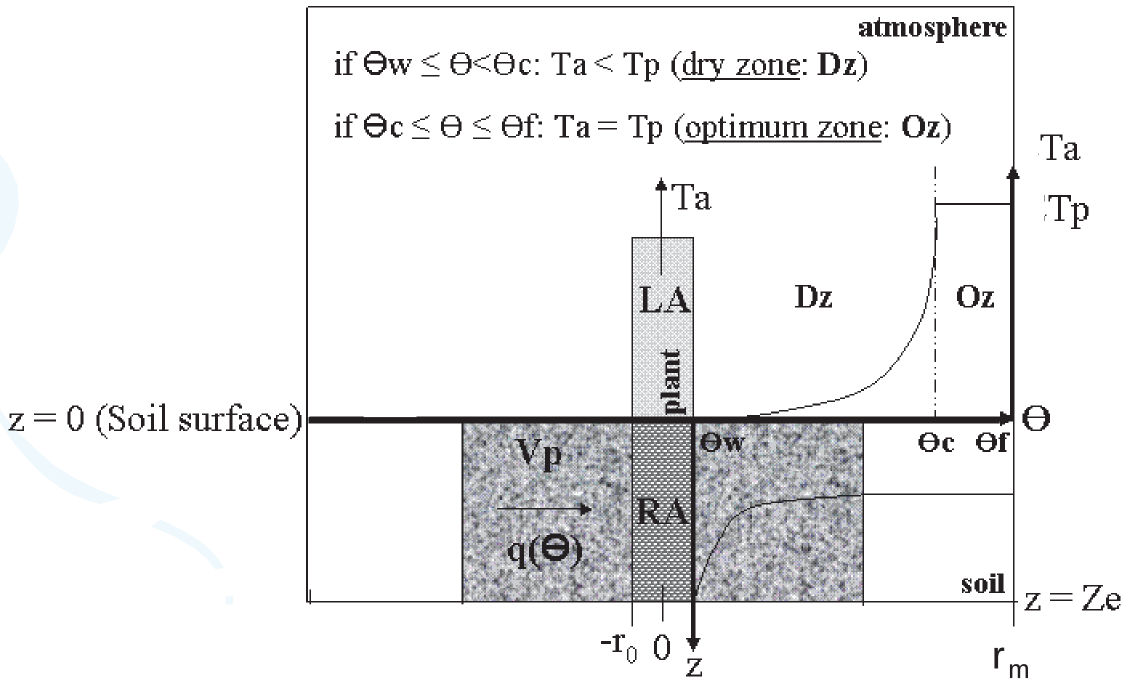
- Abramowitz, M.; Stegun, I. A. (Ed.). Handbook[1] of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, 1048p., 1972.
- Metselaar, K., and Q. De Jong van Lier. 2007. The shape of the transpiration reduction function under plant water stress. Vadose Zone J. 6:124-139.
- Van Genuchten, M. Th. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. **Soil Science Society of America Journal**, v.44, p.892-898. 1980.

[1] http://books.google.nl/books?id=MtU8uP7XMvoC&dq=Handbook+of+ Mathematical+Functions+with+Formulas, +Graphs,+and+Mathematical+Table+By+Irene+A.+Steg un,+Milton+Abramowitz&pg=PP1&ots=-BWPRtO 6Li&sig =7ZFRYldu EhZzfILo57ZVTSy0dNQ&prev=http://www.google.nl/search%3Fhl%3Den%26q%3DHandbook%2Bof %2BMathematical%2BFunctions%2Bwith%2BFormulas%252C%2BGraphs%252C%2Band%2BMathematical%2BTable %2BBy%2BIrene%2BA.%2BStegun%252C%2BMilto%2BAbramowitz%26btnG%3DGoogle%2BSearch%26meta %3D&sa=X&oi=print&ct=result&cd=1#PRA1-PR7,M1

- <u>Objective</u>

  The general equation to estimate the critical value of soil water content using matric flux potential with Van Genuchten hydraulic functions for agricultural and environmental purposes

Schematic representation of basic assumptions to characterize the water dynamic in the soil-plant-atmosphere system for agricultural and environmental purposes, where ‘q’ is the soil water flux density (Darcy-Buckingham, 1856-1907), ‘Ze’ is the effective root depth, ‘Vp’ is the effective soil volume explored by plant, ‘Θ’ is the soil water content (‘Θw’, ‘Θc’ and ‘Θf’ are wilting point, critical value and field capacity, respectively), ‘Ta’ is the actual transpiration, ‘Tp’ is the potential transpiration, ‘LA’ is the total leaf area, ‘r_0’ is the equivalent root diameter, ‘r_m’ is the radius of the root extraction zone and ‘RA’ is the total root area



Basic equations

Soil hydraulic diffusivity (D, m^2 d^{-1})

$$D = \frac{K}{C} \quad [1]$$

Soil hydraulic conductivity (K, m d^{-1})

$$K(\Theta) = K_s \Theta^{\lambda} \left\{ 1 - \left(1 - \Theta^{\frac{1}{m}} \right)^m \right\}^2$$

($n > 1$ and $0 < m < 1$)
[2]

Specific water content ($C = d\theta/dh$, m^{-1})

$$C(\Theta) = \alpha (n-1)(\theta_s - \theta_r) \Theta^{\frac{1}{m}} \left(1 - \Theta^{\frac{1}{m}} \right)^m \quad [3]$$

Effective degree of saturation (relative soil water content)

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} \quad [4]$$

Matric flux potential (M, m^2 d^{-1})

$$M_w(\theta_a) = \int_{\theta_w}^{\theta_a} D(\theta)\,d\theta = (\theta s - \theta r)\int_{\Theta_w}^{\Theta_a} D(\Theta)\,d\Theta$$ [5]

Substituting [1], [2] and [3] in [5]

$$M_w(\Theta_a) = \frac{K_s}{\alpha(n-1)}\int_{\Theta_w}^{\Theta_a} \frac{\Theta^{\lambda}\left\{1-\left(1-\Theta^{\frac{1}{m}}\right)^{m}\right\}^{2}}{\Theta^{1/m}\left(1-\Theta^{\frac{1}{m}}\right)^{m}}\,d\Theta$$ [6]

Analytical solution

$$\Theta = \cos^{2m}(x) \qquad [7]$$

$$x = \arccos\left(\Theta^{\frac{1}{2m}}\right) \qquad [8]$$

$$d\Theta = -\frac{2m\left(1-\Theta^{\frac{1}{m}}\right)^{\frac{1}{2}}}{\Theta^{\frac{1}{2m}-1}}dx \qquad [9]$$

Substituting [7] and [9] in [6]:

$$M_w(x_a) = -\frac{2mK_s}{\alpha(n-1)}\left\{\begin{aligned}&\int_{x_w}^{x_a}\sin^{1-2m}(x)\cos^{2m(1+\lambda)-3}(x)dx - 2\int_{x_w}^{x_a}\sin(x)\cos^{2m(1+\lambda)-3}(x)dx + \\ &+\int_{x_w}^{x_a}\sin^{1+2m}(x)\cos^{2m(1+\lambda)-3}(x)dx\end{aligned}\right\} \qquad [10]$$

Hypergeometric function ($_2F_1$)

$$\int_{x_w}^{x_a} \sin^{1-2m}(x)\cos^{2(\lambda+1)m-3}(x)\,dx = \left|\frac{\cos^{2(\lambda+1)m-2}(x)\,{}_2F_1\left[\lambda m+m-1, m; \lambda m+m; \cos^2(x)\right]\sin^{-2m}(x)\sin^2(x)^m}{2-2(\lambda+1)m}\right|_{x_w}^{x_a} \quad [11]$$

$$\int_{x_w}^{x} \sin(x)\cos^{2(\lambda+1)m-3}(x)\,dx = \frac{\cos^{2(\lambda+1)m-2}(x)}{2\text{-}2(\lambda+1)m} \quad [12]$$

$$\int_{x_w}^{x_a} \sin^{1+2m}(x)\cos^{2(\lambda+1)m-3}(x)\,dx = \left|\frac{\cos^{2(\lambda+1)m-2}(x)\,{}_2F_1\left[\lambda m+m-1, \text{-}m; \lambda m+m; \cos^2(x)\right]\sin^{2m}(x)\sin^2(x)^{-m}}{2-2(\lambda+1)m}\right|_{x_w}^{x_a} \quad [13]$$

Substituting [11], [12] and [13] in [10]:

$$M_w(x_a) = \frac{mK_s}{\alpha(n-1)(\lambda m+m\text{-}1)}\left|\cos^{2(\lambda+1)m-2}(x)\left\{{}_2F_1\left[\lambda m+m-1, m; \lambda m+m; \cos^2(x)\right]-2+{}_2F_1\left[\lambda m+m-1, \text{-}m; \lambda m+m; \cos^2(x)\right]\right\}\right|_{x_w}^{x_a} \quad [14]$$

The hypergeometric function ($_2F_1$)

$$ {}_2F_1\left[\lambda m+m-1, m; \lambda m+m; \cos^2(x)\right] = \sum_{k=0}^{\infty}\left\{\frac{(\lambda m+m-1)_k (m)_k}{(\lambda m+m)_k}\cdot\frac{\left[\cos^2(x)\right]^k}{k!}\right\} = 1+\sum_{k=1}^{\infty}\left[\frac{\cos^{2k}(x)}{k!}\prod_{j=1}^{k}\frac{(\lambda m+m+j-2)(m+j-1)}{(\lambda m+m+j-1)}\right] \quad [15] $$

and, analogously:

$$ {}_2F_1\left[\lambda m+m-1, -m; \lambda m+m; \cos^2(x)\right] = 1+\sum_{k=1}^{\infty}\left[\frac{\cos^{2k}(x)}{k!}\prod_{j=1}^{k}\frac{(\lambda m+m+j-2)(j-m-1)}{(\lambda m+m+j-1)}\right] \quad [16] $$

$$ 1+\sum_{k=1}^{4}\left[\frac{\cos^{2k}(x)}{k!}\prod_{j=1}^{k}\frac{(\lambda m+m+j-2)(j+m-1)}{(\lambda m+m+j-1)}\right] = 1+\frac{(\lambda m+m-1)m}{\lambda m+m}\cos^2(x)\left[1+\frac{(\lambda m+m)(1+m)}{(\lambda m+m+1)}\frac{\cos^2(x)}{2}\left\{1+\frac{(\lambda m+m+1)(2+m)}{(\lambda m+m+2)}\frac{\cos^2(x)}{3}\left[1+\frac{(\lambda m+m+2)(3+m)}{(\lambda m+m+3)}\frac{\cos^2(x)}{4}\right]\right\}\right] \quad [17] $$

$$ 1+\sum_{k=1}^{4}\left[\frac{\cos^{2k}(x)}{k!}\prod_{j=1}^{k}\frac{(\lambda m+m+j-2)(j-m-1)}{(\lambda m+m+j-1)}\right] = 1-\frac{(\lambda m+m-1)m}{\lambda m+m}\cos^2(x)\left[1+\frac{(\lambda m+m)(1-m)}{(\lambda m+m+1)}\frac{\cos^2(x)}{2}\left\{1+\frac{(\lambda m+m+1)(2-m)}{(\lambda m+m+2)}\frac{\cos^2(x)}{3}\left[1+\frac{(\lambda m+m+2)(3-m)}{(\lambda m+m+3)}\frac{\cos^2(x)}{4}\right]\right\}\right] \quad [18] $$

$$ \varphi = m(\lambda+1) \quad [19] $$

The analytical solution

$$M_w(x_a)=\frac{m^2K_s}{2\alpha(n-1)(1+\varphi)}\left|\cos^{2(1+\varphi)}(x)\left\{(1+m)\left\{1+\frac{\cos^2(x)}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)}\left[1+\frac{\cos^2(x)}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)}\right]\right\}-(1-m)\left\{1+\frac{\cos^2(x)}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)}\left[1+\frac{\cos^2(x)}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)}\right]\right\}\right\}\right|_{x_w}^{x_a} \qquad [20]$$

which can be rewritten as:

$$M_w(x_a)=\frac{m^2K_s}{2\alpha(n-1)(1+\varphi)}\left[\Phi(x_a)-\Phi(x_w)\right] \qquad [21]$$

with:

$$\Phi(x)=\cos^{2(1+\varphi)}(x)\left\{(1+m)\left\{1+\frac{\cos^2(x)}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)}\left[1+\frac{\cos^2(x)}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)}\right]\right\}-(1-m)\left\{1+\frac{\cos^2(x)}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)}\left[1+\frac{\cos^2(x)}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)}\right]\right\}\right\} \qquad [22]$$

Matric flux potential (M, m^2 d^{-1})

$$M_w(\Theta_a) = \frac{m^2 K_s}{2\alpha(n-1)(1+\varphi)}\left[\Lambda(\Theta_a) - \Lambda(\Theta_w)\right] \quad [23]$$

$$\Lambda(\Theta) = \Theta^{\frac{1+\varphi}{m}}\left\{(1+m)\left(1+\frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)}\left[1+\frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)}\right]\right)-(1-m)\left(1+\frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)}\left[1+\frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)}\right]\right)\right\} \quad [24]$$

The rhizosphere...

The equivalent root volume per plant (V_e, m^3) is defined as follows:

$$V_e = \pi r_0^2 Z_e \qquad [25]$$

During the constant rate phase, from t_0 to t_l, the flux density (q, m d^{-1}) in the rhizosphere can be calculated by:

$$q_p = -\frac{T_p A_p}{A_r} = -K(\theta)\frac{\partial h}{\partial r}\bigg|_{r_0}^{t_0 \le t \le t_l} \qquad [26]$$

In the falling rate phase ($q_w \le q < q_c$):

$$q_a = -\frac{T_a A_p}{A_r} = -K(\theta)\frac{\partial h}{\partial r}\bigg|_{r_0}^{t > t_l} \qquad [27]$$

The soil volume explored per plant (V_p, m^3), representing the rhizosphere, is defined as follows:

$$V_p = \pi z_e \left(r_m^2 - r_0^2\right) \qquad [28]$$

Matric flux potential (root surface and falling rate phase)

It has been shown (Metselaar & De Jong van Lier, 2007) that it is reasonable to assume that

$$\frac{M_{r_0}}{M_{l,r_0}} = \frac{\overline{M}}{\overline{M}_l} = \frac{q_a}{q_p} \qquad [29]$$

Substituting equations [26] and [27] in [29]:

$$\frac{\overline{M}}{\overline{M}_l} = \frac{T_a}{T_p} = T_r \qquad [30]$$

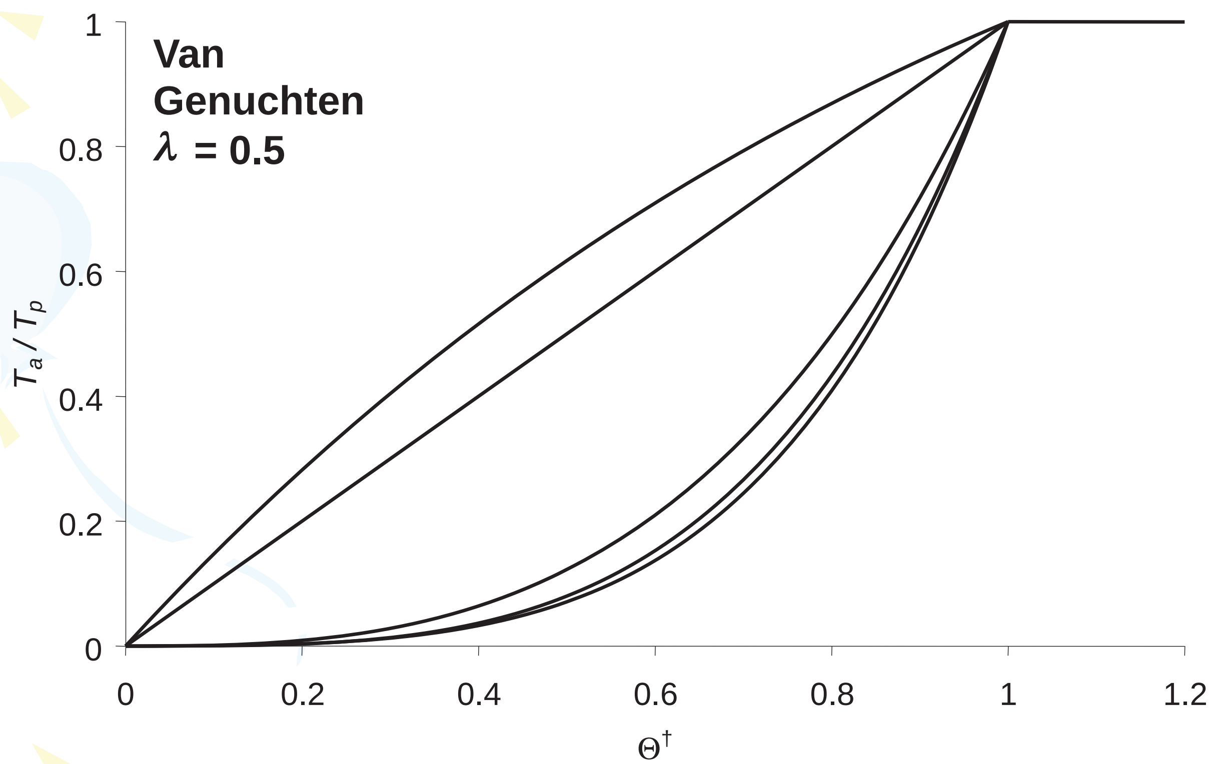
where T_r is the (dimensionless) relative transpiration. Substituting the expression derived for M (equation [23]) in equation [30] yields:

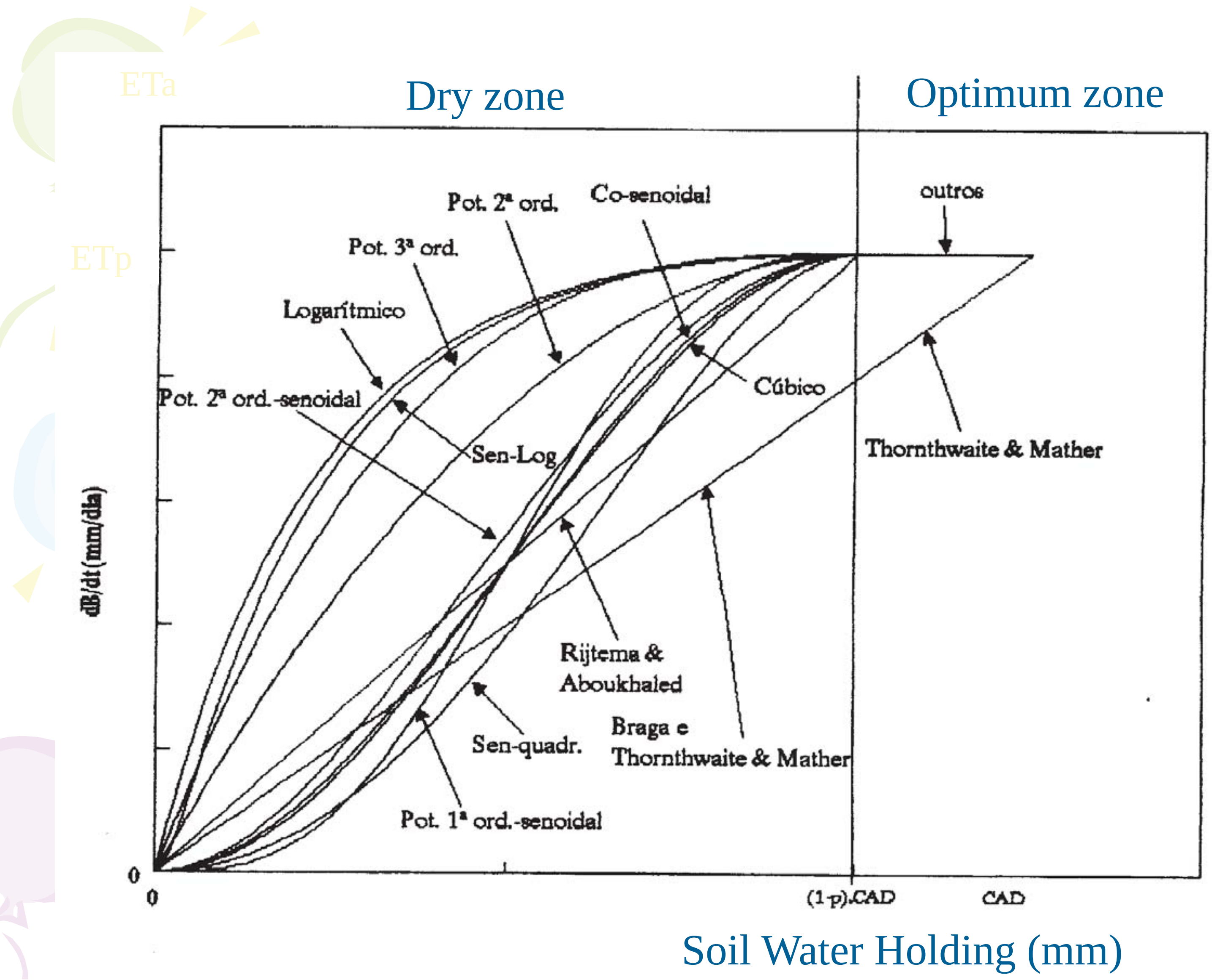
$$T_r = \frac{M}{M_l} = \frac{\Lambda(\overline{\Theta}) - \Lambda(\Theta_w)}{\Lambda(\overline{\Theta}_l) - \Lambda(\Theta_w)} \qquad [31]$$

In Figure 1, T_r calculated for Van Genuchten type of soils by equation [31] is shown as a function of relative soil water content $\Theta^\dagger$ with:

$$\Theta^\dagger = \frac{\theta - \theta_w}{\theta_l - \theta_w} \qquad [32]$$

Relative transpiration *Ta/Tp* as a function of relative soil water content $\Theta^{\dagger}$ [$\Theta^{\dagger} = (\theta - \theta w)/(\theta l - \theta w)$] for Van Genuchten type of soils at different values of *m* with $\lambda = 0.5$




ETa
ETp
Dry zone
Optimum zone
Pot. 2ª ord.
Co-senoidal
outros
Pot. 3ª ord.
Logarítmico
Pot. 2ª ord.-senoidal
Cúbico
Sen-Log
Thornthwaite & Mather
dB/dt (mm/dia)
Rijtema &
Aboukhaled
Braga e
Thornthwaite & Mather
Sen-quadr.
Pot. 1ª ord.-senoidal
0
0
(1-p).CAD
CAD
Soil Water Holding (mm)

Van Genuchten diffusivity

$$D = K\frac{dh}{d\theta} = \frac{K_s}{\alpha(n-1)(\theta_s-\theta_r)}\Theta^{\lambda-\frac{1-2n}{1-n}}\left[1-\left(1-\Theta^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\right]^2\left[\Theta^{\frac{n}{1-n}}-1\right]^{\frac{1-n}{n}}$$ [33]

The relative transpiration

For the special case of $\Theta_w = 0$ (or: $\theta_w = \theta_r$), equation [31] reduces to

$$T_r = \frac{\Lambda\left(\bar{\Theta}\right)}{\Lambda\left(\bar{\Theta}_l\right)} \qquad [34]$$

Substituting [24] in [34] yields:

$$T_r = \left(\frac{\bar{\Theta}}{\Theta_l}\right)^{\frac{\varphi+1}{m}} \left\{ \frac{(1+m)\left\{1+\frac{\bar{\Theta}^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2+m)}{(\varphi+2)}\left[1+\frac{\bar{\Theta}^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3+m)}{(\varphi+3)}\right]\right\}-(1-m)\left\{1+\frac{\bar{\Theta}^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2-m)}{(\varphi+2)}\left[1+\frac{\bar{\Theta}^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3-m)}{(\varphi+3)}\right]\right\}}{(1+m)\left\{1+\frac{\bar{\Theta}_l^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2+m)}{(\varphi+2)}\left[1+\frac{\bar{\Theta}_l^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3+m)}{(\varphi+3)}\right]\right\}-(1-m)\left\{1+\frac{\bar{\Theta}_l^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2-m)}{(\varphi+2)}\left[1+\frac{\bar{\Theta}_l^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3-m)}{(\varphi+3)}\right]\right\}} \right\} \qquad [35]$$

It can easily be seen that:

$$\frac{\partial \bar{\theta}}{\partial t} = -\frac{T_a}{z_e} = -\frac{T_p}{z_e}T_r = -\frac{T_p}{z_e}T_r \qquad [36]$$

or, substituting equation [34] in [36]:

$$\frac{\partial \bar{\theta}}{\partial t} = -\frac{T_p}{z_e}\frac{\Lambda(\bar{\Theta})}{\Lambda(\bar{\Theta}_l)} \qquad [37]$$

which can be reorganized as:

$$\frac{1}{\Lambda(\bar{\Theta})}\partial\bar{\theta} = -\frac{T_p}{z_e\Lambda(\bar{\Theta}_l)}dt \qquad [38]$$

Substituting $d\bar{\theta} = (\theta_s - \theta_r)d\bar{\Theta}$ and integrating yields

$$(\theta_s - \theta_r)\int_{\bar{\Theta}_l}^{\bar{\Theta}}\frac{1}{\Lambda(\bar{\Theta})}\partial\bar{\Theta} = -\frac{T_p}{z_e\Lambda(\bar{\Theta}_l)}\int_{t_l}^{t}dt = -\frac{T_p}{z_e\Lambda(\bar{\Theta}_l)}(t - t_l) \qquad [39]$$

Substituting equation [24] in [39] and performing the following transformation:

$$\bar{\Theta}^{\frac{1}{m}} = \cos^2(z) \qquad [40]$$

$$z = \arccos\left(\bar{\Theta}^{\frac{1}{2m}}\right) \qquad [41]$$

$$d\bar{\Theta} = -2m\cos^{2m-1}(z)\sin(z)dz \qquad [42]$$

results in

$$\frac{-\tau}{2}=\int_{z_l}^{z}\cos^{c}(z)\sin(z)\begin{bmatrix}(1+m)\left\{1+\frac{\cos^2(z)}{3}\frac{(\varphi+1)(2+m)}{(\varphi+2)}\left[1+\frac{\cos^2(z)}{4}\frac{(\varphi+2)(3+m)}{(\varphi+3)}\right]\right\}- \\ (1-m)\left\{1+\frac{\cos^2(z)}{3}\frac{(\varphi+1)(2-m)}{(\varphi+2)}\left[1+\frac{\cos^2(z)}{4}\frac{(\varphi+2)(3-m)}{(\varphi+3)}\right]\right\}\end{bmatrix}^{-1}dz \quad [43]$$

which can be rewritten as

$$\frac{-\tau}{2}=\int_{z_1}^{z}\frac{\cos^{2m-2\varphi-3}(z)\sin(z)}{g\left\{1+a.\cos^2(z)\left[1+b.\cos^2(z)\right]\right\}-h\left\{1+c.\cos^2(z)\left[1+d.\cos^2(z)\right]\right\}}dz \quad [44]$$

where

$$\tau=-\frac{T_p(t-t_l)}{m(\theta_s-\theta_r)\Lambda(\overline{\Theta}_l)z_e} \quad [45]$$

$$a=\frac{(\varphi+1)(2+m)}{3(\varphi+2)} \quad [46]$$

$$b=\frac{(\varphi+2)(3+m)}{4(\varphi+3)} \quad [47]$$

$$c=\frac{(\varphi+1)(2-m)}{3(\varphi+2)} \quad [48]$$

$$d=\frac{(\varphi+2)(3-m)}{4(\varphi+3)} \quad [49]$$

$$g=1+m \quad [50]$$

$$h=1-m \quad [51]$$

Analytical solution

Transforming equation [44] according to

$$y = \cos^2(z) \qquad [52] \qquad\qquad dz = -\frac{1}{2\cos(z)\sin(z)}dy \qquad [53]$$

results in

$$\tau = \int_{y_1}^{y} \frac{y^f}{g\left[a.y\left(b.y+1\right)+1\right]-h\left[c.y\left(d.y+1\right)+1\right]}dy \qquad [54]$$

$$f = c-1 \qquad [55]$$

Equation [54] can be solved as follows:

$$\tau = \frac{y^f}{fS_A}\left\{\frac{{}_2\mathrm{F}_1\left(-\mathrm{f},-\mathrm{f},1-\mathrm{f},\frac{S_B - S_D}{-\Gamma+\mathrm{S_A}-S_E y}\right)}{\left[\frac{y}{y-\frac{S_C-S_D}{S_E}}\right]^f} - \frac{{}_2\mathrm{F}_1\left(-\mathrm{f},-\mathrm{f},1-\mathrm{f},\frac{S_B + S_D}{\Gamma+\mathrm{S_A}+2S_E y}\right)}{\left[\frac{y}{y+\frac{S_C+S_D}{S_E}}\right]^f}\right\}\Bigg|_{y_l}^{y} \qquad [56]$$

where the auxiliary variables (S_A, S_B, S_C, S_D and S_E) are defined as follows:

$$S_A = \sqrt{a^2 g^2 - 2ag\left[2b(g-h) + ch\right] + ch\left[4d(g-h) + ch\right]} \quad [57]$$

$$S_B = \sqrt{(ag - ch)^2 - 4(g-h)(abg - cdh)} \quad [58]$$

$$S_C = \sqrt{a^2 g^2 + 4ab\left(gh - g^2\right) - 2acgh + 4cd\left(gh - h^2\right) + c^2 h^2} \quad [59]$$

$$S_D = ag - ch \quad [60]$$

$$S_E = 2(abg - cdh) \quad [61]$$

Expanding equation [56] results in:

$$\tau = \frac{1}{fS_A}\left[y^{f}\left\{ \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B - S_D}{-\Gamma + S_A - S_E y}\right)}{\left[\dfrac{y}{y - \dfrac{S_C - S_D}{S_E}}\right]^{f}} - \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B + S_D}{\Gamma + S_A + S_E y}\right)}{\left[\dfrac{y}{y + \dfrac{S_C + S_D}{S_E}}\right]^{f}} \right\} - y_l^{f}\left\{ \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B - S_D}{-\Gamma + S_A - 2S_E y_l}\right)}{\left[\dfrac{y_1}{y_l - \dfrac{S_C - S_D}{S_E}}\right]^{f}} - \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B + S_D}{\Gamma + S_A + S_E y_l}\right)}{\left[\dfrac{y_1}{y_1 + \dfrac{S_C + S_D}{S_E}}\right]^{f}} \right\} \right] \quad [62]$$

Substituting [52] in [62]:

$$\tau = \frac{1}{fS_A}\left[\cos^{2f}(z)\left\{\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B\text{-}S_D}{\text{-}S_D+S_A\text{-}S_E\cos^2(z)}\right)}{\left[\dfrac{\cos^2(z)}{\cos^2(z)-\dfrac{S_C-S_D}{S_E}}\right]^f}-\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B+S_D}{S_D+S_A+S_E\cos^2(z)}\right)}{\left[\dfrac{\cos^2(z)}{\cos^2(z)+\dfrac{S_C+S_D}{S_E}}\right]^f}\right\}-\cos^{2f}(z_l)\left\{\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B\text{-}S_D}{\text{-}S_D+S_A\text{-}S_E\cos^2(z_l)}\right)}{\left[\dfrac{\cos^2(z_l)}{\cos^2(z_l)-\dfrac{S_C-S_D}{S_E}}\right]^f}-\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B+S_D}{S_D+S_A+S_E\cos^2(z_l)}\right)}{\left[\dfrac{\cos^2(z_l)}{\cos^2(z_l)+\dfrac{S_C+S_D}{S_E}}\right]^f}\right\}\right] \quad [63]$$

Substituting [41] in [63]:

$$\tau = \frac{1}{fS_A}\left[\Theta^{\frac{f}{m}}\left\{\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B-S_D}{-S_D+S_A-S_E\Theta^{\frac{1}{m}}}\right)}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}}-\dfrac{S_C-S_D}{S_E}}\right]^f}-\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B+S_D}{S_D+S_A+S_E\Theta^{\frac{1}{m}}}\right)}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}}+\dfrac{S_C+S_D}{S_E}}\right]^f}\right\}-\Theta_l^{\frac{f}{m}}\left\{\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B-S_D}{-S_D+S_A-S_E\Theta_l^{\frac{1}{m}}}\right)}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}}-\dfrac{S_C-S_D}{S_E}}\right]^f}-\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B+S_D}{S_D+S_A+S_E\Theta_l^{\frac{1}{m}}}\right)}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}}+\dfrac{S_C+S_D}{S_E}}\right]^f}\right\}\right] \qquad [64]$$

Expanding Gauss's hypergeometric functions for *k* = 4:

$$\tau = \frac{1}{f S_A}\left[\Theta^{\frac{f}{m}}\left\{\frac{1+\dfrac{(S_B - S_D)\Omega}{S_A - S_D - S_E\Theta^{\frac{1}{m}}}}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}} - \dfrac{S_C - S_D}{S_E}}\right]^f} - \frac{1+\dfrac{(S_B + S_D)\Omega}{S_A + S_D + S_E\Theta^{\frac{1}{m}}}}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}} + \dfrac{S_C + S_D}{S_E}}\right]^f}\right\} - \Theta_l^{\frac{f}{m}}\left\{\frac{1+\dfrac{(S_B - S_D)\Omega}{S_A - S_D - S_E\Theta_l^{\frac{1}{m}}}}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}} - \dfrac{S_C - S_D}{S_E}}\right]^f} - \frac{1+\dfrac{(S_B + S_D)\Omega}{S_A + S_D + S_E\Theta_l^{\frac{1}{m}}}}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}} + \dfrac{S_C + S_D}{S_E}}\right]^f}\right\}\right] \quad [65]$$

with

$$\Omega = \left[\frac{1}{(1-f)} + \left(\frac{1-f}{2(2-f)}\right) + \frac{(1-f)(2-f)}{6(3-f)} + \frac{(1-f)(2-f)(3-f)}{24(4-f)}\right] \quad [66]$$

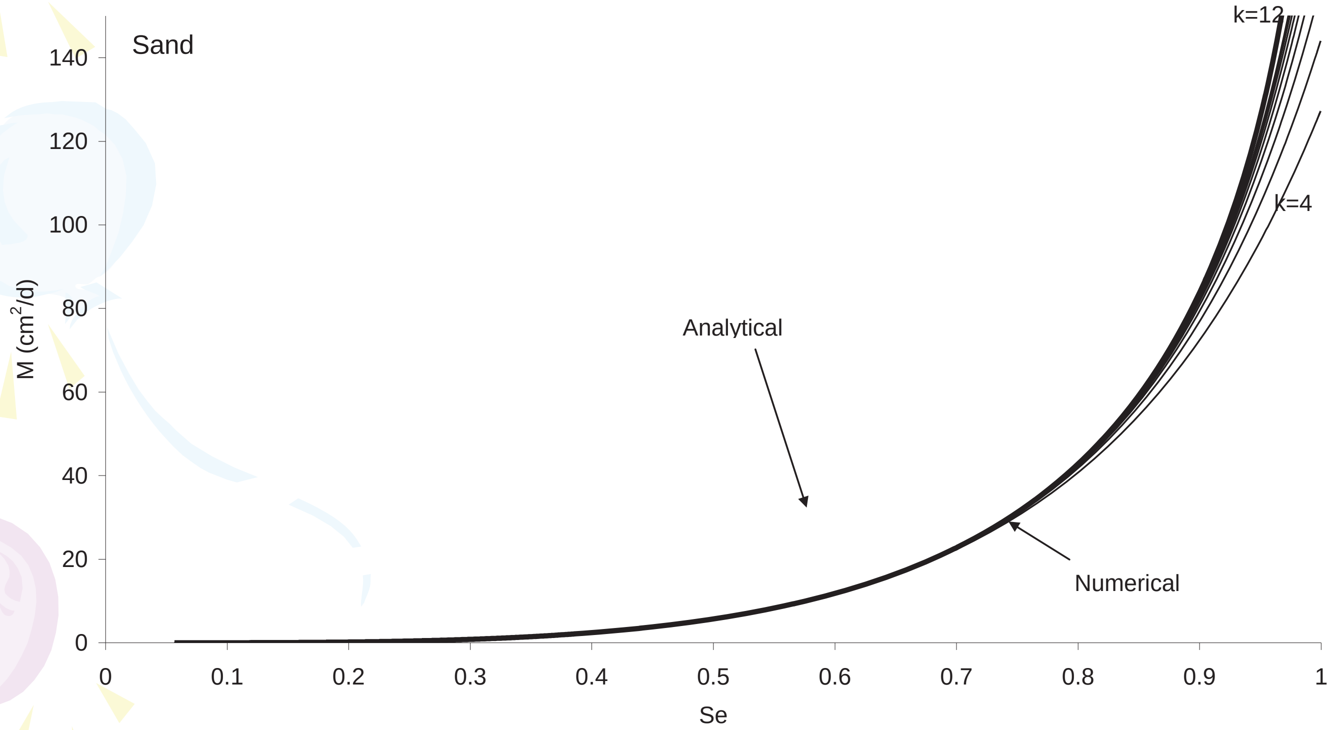
Summary: M

Matric flux potential (M, m^2 d^{-1})

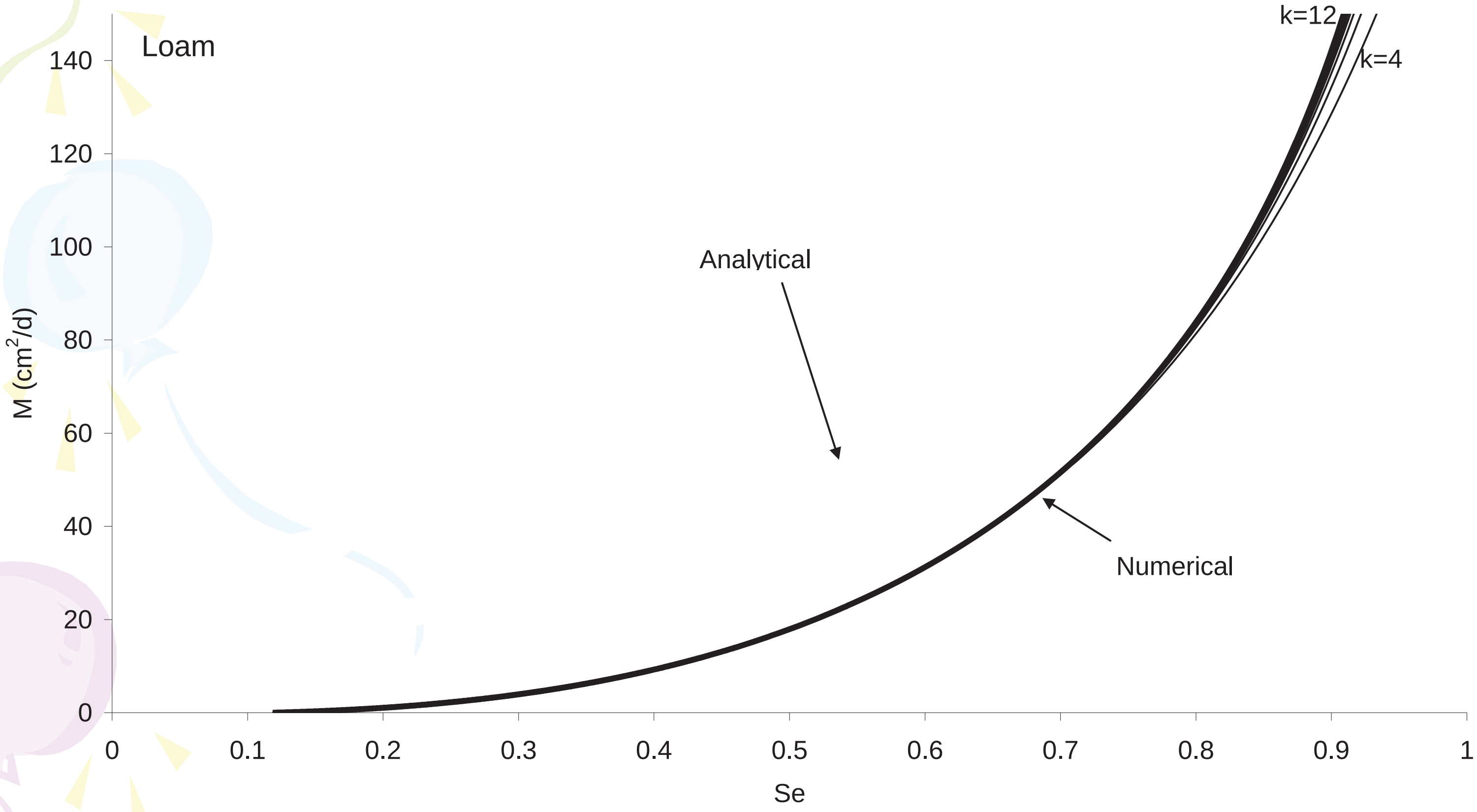
$$M_w(\Theta_a) = \frac{m^2 K_s}{2\alpha(n-1)(1+\varphi)}\left[\Lambda(\Theta_a) - \Lambda(\Theta_w)\right] \quad [23]$$

$$\Lambda(\Theta) = \Theta^{\frac{1+\varphi}{m}}\left\{(1+m)\left(1+\frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)}\left[1+\frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)}\right]\right)-(1-m)\left(1+\frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)}\left[1+\frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)}\right]\right)\right\} \quad [24]$$

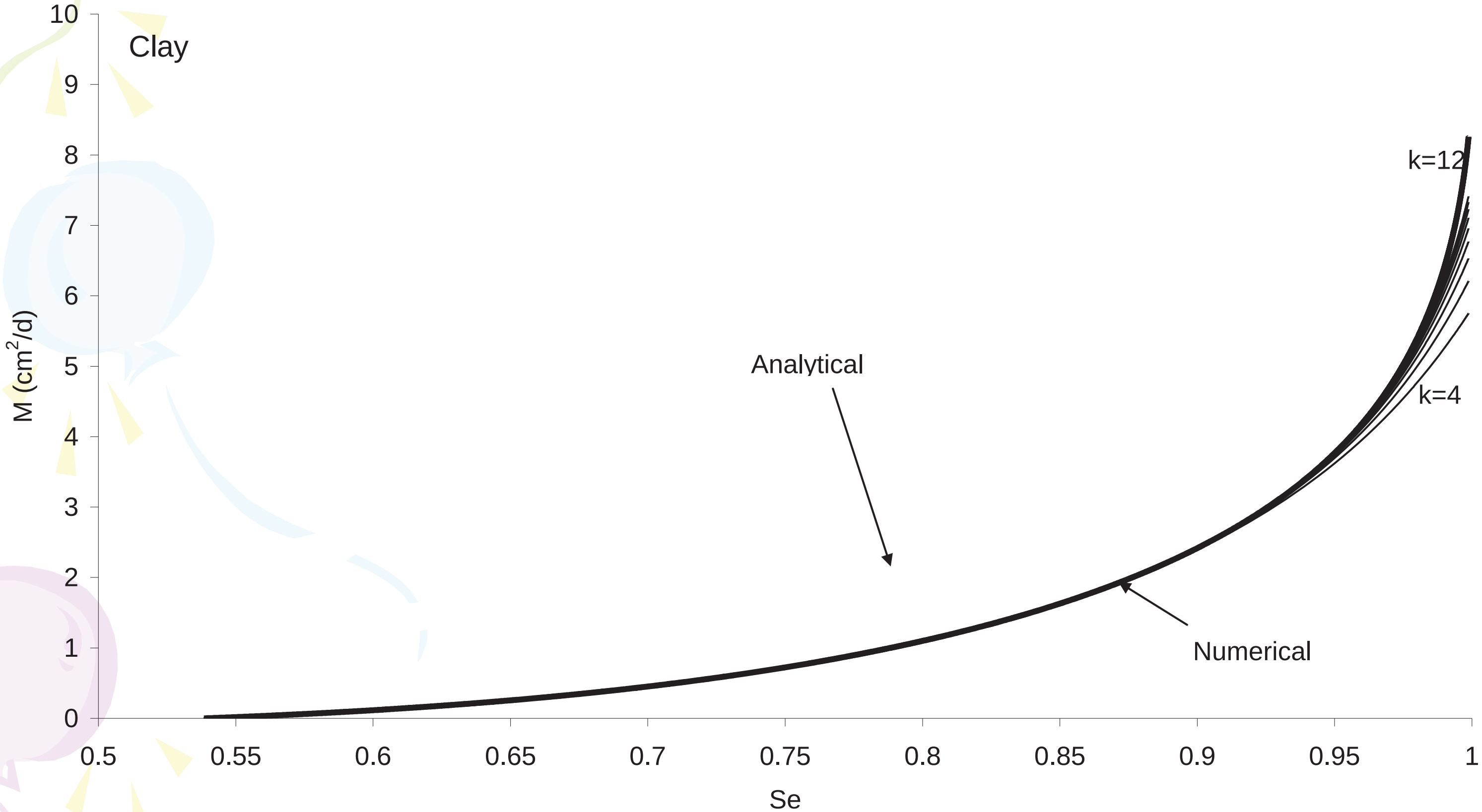
Comparison between analytical (results for **sand soil** according to equation 22 using different values of ‘k’ - from 4 to 12) and numerical solution showing the matric flux potential (M, $cm^2.d^{-1}$) as function of effective degree of saturation (Se), where α = 0.0144 cm^{-1}; m = 0.348109518; n = 1.534; θs = 0.46 $m^3.m^{-3}$; θr = 0.02 $m^3.m^{-3}$; λ = -0.215 and Ks = 15.42 $cm.d^{-1}$.



Comparison between analytical (results for **loam soil** according to equation 22 using different values of 'k' - from 4 to 12) and numerical solution showing the matric flux potential (M, $cm^2.d^{-1}$) as function of effective degree of saturation (Se), where $\alpha = 0.0084\ cm^{-1}$; $m = 0.306037474$; $n = 1.441$; $\theta s = 0.42\ m^3.m^{-3}$; $\theta r = 0.01\ m^3.m^{-3}$; $\lambda = -1.497$ and $Ks = 12.98\ cm.d^{-1}$



Comparison between analytical (results for **clay soil** according to equation 22 using different values of 'k' - from 4 to 12) and numerical solution showing the matric flux potential (M, cm2.d-1) as function of effective degree of saturation (Se), where α = 0.0195 cm^{-1}; m = 0.098286745; n = 1.109; Θs = 0.59 $m^3.m^{-3}$; Θr = 0.01 $m^3.m^{-3}$; λ = - 5.901 and Ks = 4.53 $cm.d^{-1}$



Summary: $\frac{\mathrm{Ta}}{\mathrm{Tp}} = \frac{\mathrm{Ta}}{\mathrm{Tp}}\left(\Theta^{\dagger}\right)$

From [1] to [32]

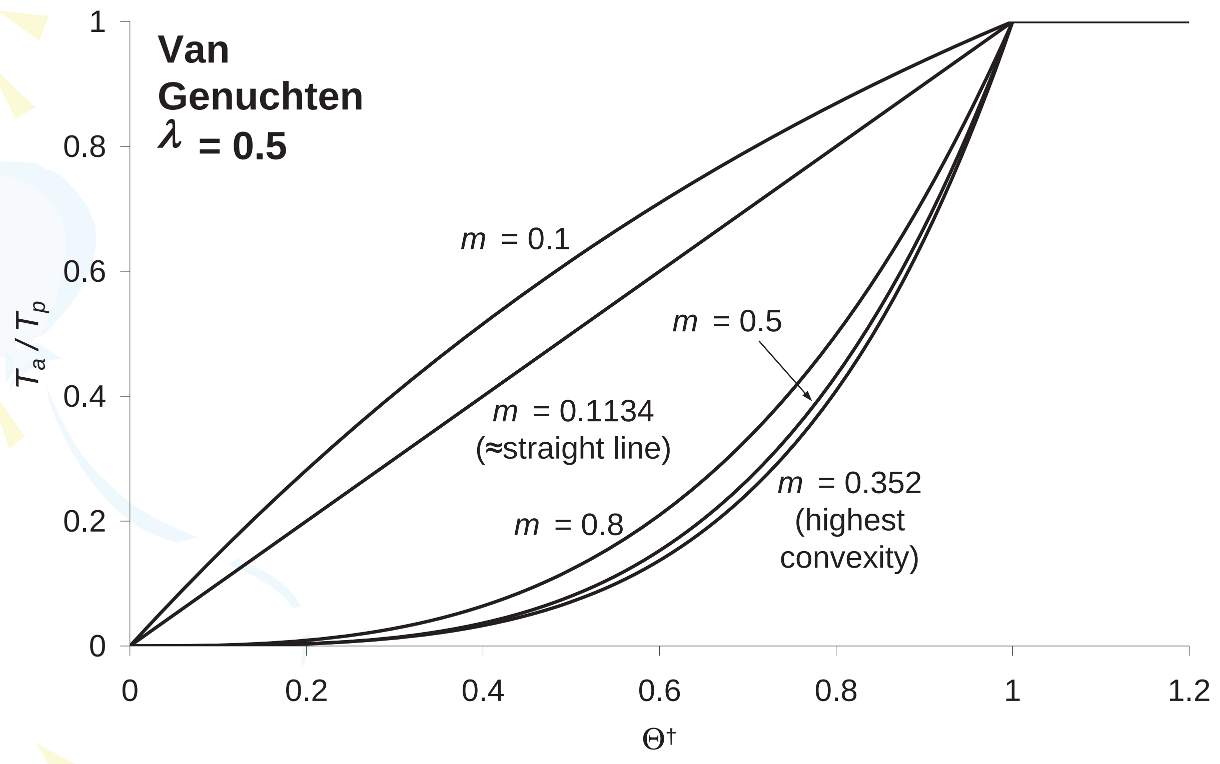
Substituting [30] and [32] in [34], the final equation can be written as follows:

$$\Theta^{\dagger} = \frac{\theta - \theta_w}{\theta_l - \theta_w} \quad [32]$$

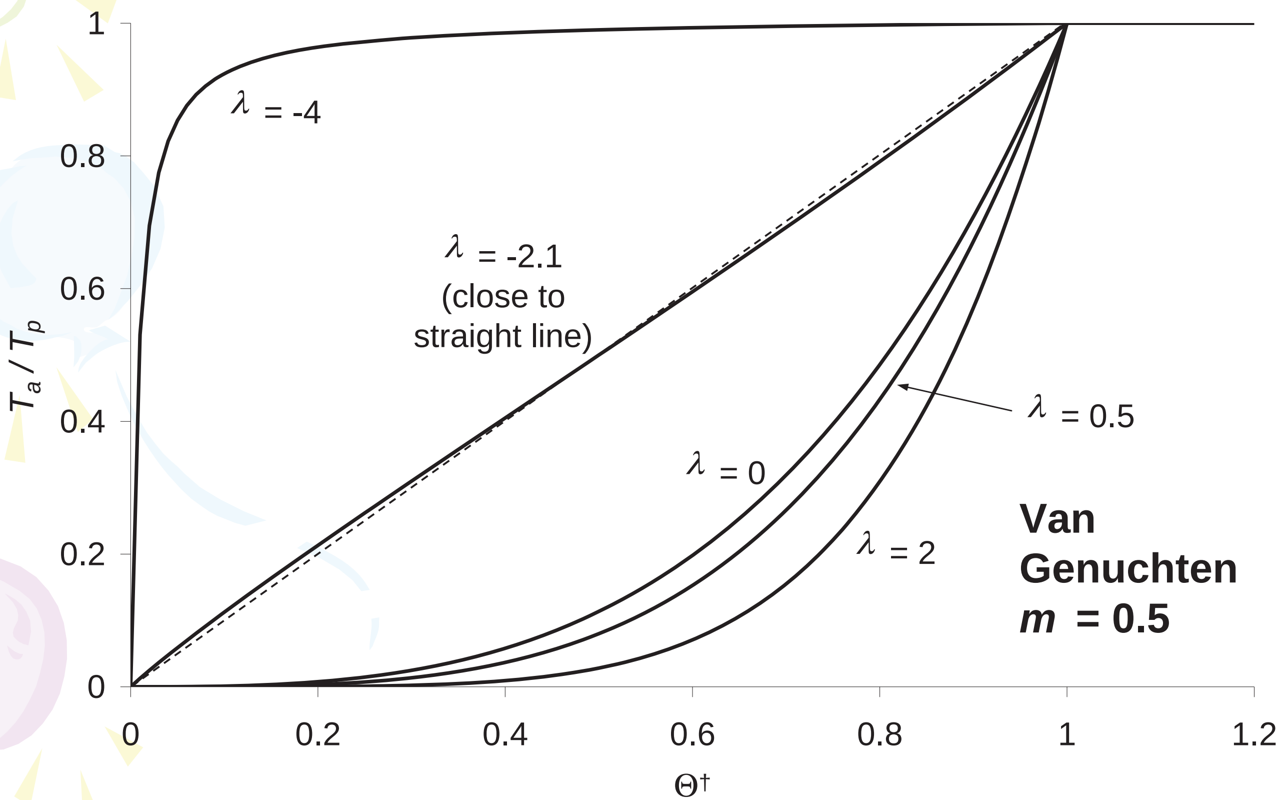
$$\mathrm{T_r} = \frac{\mathrm{M}}{\mathrm{M_l}} = \frac{\mathrm{Ta}}{\mathrm{Tp}} = \frac{\Lambda\left(\bar{\Theta}\right)}{\Lambda\left(\bar{\Theta}_{\mathrm{l}}\right)}\left(for \quad \Theta^{\dagger} = \bar{\Theta} \right) \quad [34]$$

$$\Lambda(\Theta) = \Theta^{\frac{1+\varphi}{m}} \left\{ \begin{aligned} &(1+m)\left(1 + \frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)}\left[1 + \frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)}\right]\right) - \\ &(1-m)\left(1 + \frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)}\left[1 + \frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)}\right]\right) \end{aligned} \right\} \quad [24]$$

Relative transpiration *Ta*/*Tp* as a function of relative soil water content $\Theta^\dagger$ [$\Theta^\dagger = (\theta-\theta w)/(\theta l- \theta w)$] for Van Genuchten type of soils at different values of *m* with $\lambda = 0.5$



Relative transpiration *Ta/Tp* as a function of relative soil water content $\Theta^\dagger$ [$\Theta^\dagger = (\theta - \theta w)/(\theta l - \theta w)$] for Van Genuchten type of soils at different values of λ with m = 0.5



Summary: $\Theta = \Theta(t)$ From [33] to [66]

Substituting [45] in [65], the final equation can be written as follows:

$$\left[\Theta^{\frac{f}{m}} \left\{ \frac{1 + \dfrac{(S_B - S_D)\Omega}{S_A - S_D - S_E \Theta^{\frac{1}{m}}}}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}} - \dfrac{S_C - S_D}{S_E}} \right]^f} - \frac{1 + \dfrac{(S_B + S_D)\Omega}{S_A + S_D + S_E \Theta^{\frac{1}{m}}}}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}} + \dfrac{S_C + S_D}{S_E}} \right]^f} \right\} - \Theta_l^{\frac{f}{m}} \left\{ \frac{1 + \dfrac{(S_B - S_D)\Omega}{S_A - S_D - S_E \Theta_l^{\frac{1}{m}}}}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}} - \dfrac{S_C - S_D}{S_E}} \right]^f} - \frac{1 + \dfrac{(S_B + S_D)\Omega}{S_A + S_D + S_E \Theta_l^{\frac{1}{m}}}}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}} + \dfrac{S_C + S_D}{S_E}} \right]^f} \right\} \right] = -\frac{T_p (t - t_l) f . S_A}{m(\theta_s - \theta_r) \Lambda(\bar{\Theta}_l) z_e} \qquad [67]$$