The Abdus Salam
International Centre
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United Nations
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2444-24

College on Soil Physics - 30th Anniversary (1983-2013)

25 February - 1 March, 2013

The general equation to estimate the critical value of soil water content using matric flux potential with Van Genuchten hydraulic functions for agricultural and environmental purposes

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The general equation to estimate the critical value of soil water content using matric flux potential with Van Genuchten hydraulic functions for agricultural and environmental purposes

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1. Deriving an expression for the Matric Flux Potential based on the van Genuchten hydraulic function parameters

Soil hydraulic diffusivity (D, m^2 d^{-1}) is defined as the relationship between soil hydraulic conductivity (K, m d^{-1}) and the specific water content ($C = \mathrm{d}\theta/\mathrm{d}h$, m^{-1}):

$$D = \frac{K}{C} \quad [1]$$

and, using the Van Genuchten (1980) hydraulic functions:

$$K(\Theta) = K_s \Theta^{\lambda} \left\{ 1 - \left(1 - \Theta^{\frac{1}{m}} \right)^m \right\}^2 \quad (n > 1 \text{ and } 0 < m < 1) \quad [2]$$

$$C(\Theta) = \alpha(n-1)(\theta_s - \theta_r)\Theta^{\frac{1}{m}} \left(1 - \Theta^{\frac{1}{m}} \right)^m \quad [3]$$

with $m = 1 - \frac{1}{n}$, α (m^{-1}), m, n and λ are statistical parameters, and Θ is the effective degree of saturation (relative soil water content):

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} \quad [4]$$

where θ is the soil water content (m^3 m^{-3}), θ_s is the saturated volumetric water content (m^3 m^{-3}) and θ_r is the residual water content (m^3 m^{-3}).

Matric flux potential (M, m^2 d^{-1}) is defined as the integral of soil hydraulic diffusivity over soil water content (θ). Defining $M_w(\theta_a)$ as the matric flux potential at $\theta = \theta_a$, starting at a reference value θ_w, it follows that:

$$M_w(\theta_a) = \int_{\theta_w}^{\theta_a} D(\theta)d\theta = (\theta s - \theta r)\int_{\Theta_w}^{\Theta_a} D(\Theta)d\Theta \quad [5]$$

Substituting [1], [2] and [3] in [5]

$$M_w(\Theta_a) = \frac{K_s}{\alpha(n-1)} \int_{\Theta_w}^{\Theta_a} \frac{\Theta^{\lambda}\left\{1-\left(1-\Theta^{\frac{1}{m}}\right)^m\right\}^2}{\Theta^{1/m}\left(1-\Theta^{\frac{1}{m}}\right)^m} d\Theta \qquad [6]$$

To obtain an analytical solution for this integral, the following transformation is performed:

$$\Theta = \cos^{2m}(x) \qquad [7]$$

then:

$$x = \arccos\left(\Theta^{\frac{1}{2m}}\right) \qquad [8]$$

and

$$d\Theta = -\frac{2m\left(1-\Theta^{\frac{1}{m}}\right)^{\frac{1}{2}}}{\Theta^{\frac{1}{2m}-1}} dx \qquad [9]$$

Substituting [7] and [9] in [6]:

$$M_w(x_a) = -\frac{2mK_s}{\alpha(n-1)} \left\{ \begin{array}{l} \int_{x_w}^{x_a} \sin^{1-2m}(x)\cos^{2m(1+\lambda)-3}(x)dx - 2\int_{x_w}^{x_a} \sin(x)\cos^{2m(1+\lambda)-3}(x)dx + \\ + \int_{x_w}^{x_a} \sin^{1+2m}(x)\cos^{2m(1+\lambda)-3}(x)dx \end{array} \right\} \qquad [10]$$

Introducing Gauss's hypergeometric function ($_2F_1$), the three integrals from [10] can be solved respectively as:

$$\int_{x_w}^{x_a} \sin^{1-2m}(x)\cos^{2(\lambda+1)m-3}(x)dx = \left. \frac{\cos^{2(\lambda+1)m-2}(x)\,_2F_1\left[\lambda m+m-1, m; \lambda m+m; \cos^2(x)\right]\sin^{-2m}(x)\sin^2(x)^m}{2\text{-}2(\lambda+1)m} \right|_{x_w}^{x_a} \qquad [11]$$

$$\int_{x_w}^{x} \sin(x)\cos^{2(\lambda+1)m-3}(x)dx = \frac{\cos^{2(\lambda+1)m-2}(x)}{2\text{-}2(\lambda+1)m} \qquad [12]$$

$$\int_{x_w}^{x_a} \sin^{1+2m}(x)\cos^{2(\lambda+1)m-3}(x)dx =$$

$$\left| \frac{\cos^{2(\lambda+1)m-2}(x)\,{}_2F_1\left[\lambda m + m - 1,\text{-}m;\lambda\text{-} \ \ + m;\cos^2(x)\right]\sin^{2m}(x)\sin^2(x)^{-m}}{2\text{-}2(\lambda+1)m} \right|_{x_w}^{x_a} \qquad [13]$$

Substituting [11], [12] and [13] in [10]:

$$M_w(x_a) = \frac{mK_s}{\alpha(n-1)(\lambda m + m\text{-}1)} \left| \begin{matrix} \cos^{2(\lambda+1)m-2}(x) \\ \left\{ \begin{matrix} {}_2F_1\left[\lambda m + m - 1, m; \lambda m + m; \cos^2(x)\right] \\ -2 + {}_2F_1\left[\lambda m + m - 1,\text{-}m;\lambda\text{-} \ \ + m;\cos^2(x)\right] \end{matrix} \right\} \end{matrix} \right|_{x_w}^{x_a} \qquad [14]$$

By definition, the hypergeometric function ($_2F_1$) can be written as:

$${}_2F_1\left[\lambda m + m - 1, m; \lambda m + m; \cos^2(x)\right] =$$

$$\sum_{k=0}^{\infty}\left\{ \frac{(\lambda m + m - 1)_k (m)_k}{(\lambda m + m)_k} \cdot \frac{\left[\cos^2(x)\right]^k}{k!} \right\} = \qquad [15]$$

$$1 + \sum_{k=1}^{\infty}\left[\frac{\cos^{2k}(x)}{k!} \prod_{j=1}^{k} \frac{(\lambda m + m + j - 2)(m + j - 1)}{(\lambda m + m + j - 1)} \right]$$

and, analogously:

$${}_2F_1\left[\lambda m + m - 1,\text{-}m;\lambda\text{-} \ \ + m;\cos^2(x)\right] =$$

$$1 + \sum_{k=1}^{\infty}\left[\frac{\cos^{2k}(x)}{k!} \prod_{j=1}^{k} \frac{(\lambda m + m + j - 2)(j\text{-}m - 1)}{(\lambda m + m + j - 1)} \right] \qquad [16]$$

Evaluating equations [15] for k = 4, we obtain:

$$1 + \sum_{k=1}^{4}\left[\frac{\cos^{2k}(x)}{k!} \prod_{j=1}^{k} \frac{(\lambda m + m + j - 2)(j + m - 1)}{(\lambda m + m + j - 1)} \right] = 1 + \frac{(\lambda m + m\text{-}1)m}{\lambda m + m}\cos^2(x)$$

$$\left[1 + \frac{(\lambda m + m)(1 + m)}{(\lambda m + m + 1)} \frac{\cos^2(x)}{2} \left\{ \begin{matrix} 1 + \dfrac{(\lambda m + m + 1)(2 + m)}{(\lambda m + m + 2)} \dfrac{\cos^2(x)}{3} \\ \left[1 + \dfrac{(\lambda m + m + 2)(3 + m)}{(\lambda m + m + 3)} \dfrac{\cos^2(x)}{4} \right] \end{matrix} \right\} \right] \qquad [17]$$

Analogously, equation [16] becomes

$$1 + \sum_{k=1}^{4}\left[\frac{\cos^{2k}(x)}{k!} \prod_{j=1}^{k} \frac{(\lambda m + m + j - 2)(j - m - 1)}{(\lambda m + m + j - 1)} \right] = 1 - \frac{(\lambda m + m\text{-}1)m}{\lambda m + m}\cos^2(x)$$

$$\left[1 + \frac{(\lambda m + m)(1 - m)}{(\lambda m + m + 1)} \frac{\cos^2(x)}{2} \left\{ \begin{matrix} 1 + \dfrac{(\lambda m + m + 1)(2 - m)}{(\lambda m + m + 2)} \dfrac{\cos^2(x)}{3} \\ \left[1 + \dfrac{(\lambda m + m + 2)(3 - m)}{(\lambda m + m + 3)} \dfrac{\cos^2(x)}{4} \right] \end{matrix} \right\} \right] \qquad [18]$$

Substituting [17] and [18] in [14] and defining:

$$\varphi = m(\lambda + 1) \qquad [19]$$

we obtain:

$$M_w(x_a) = \frac{m^2 K_s}{2\alpha(n-1)(1+\varphi)} \left| \cos^{2(1+\varphi)}(x) \left\{ \begin{array}{l} (1+m)\left\{ 1 + \frac{\cos^2(x)}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)} \left[1 + \frac{\cos^2(x)}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)} \right] \right\} - \\ (1-m)\left\{ 1 + \frac{\cos^2(x)}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)} \left[1 + \frac{\cos^2(x)}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)} \right] \right\} \end{array} \right\} \right|_{x_w}^{x_a} \qquad [20]$$

which can be rewritten as:

$$M_w(x_a) = \frac{m^2 K_s}{2\alpha(n-1)(1+\varphi)} \left[\Phi(x_a) - \Phi(x_w) \right] \qquad [21]$$

with:

$$\Phi(x) = \cos^{2(1+\varphi)}(x) \left\{ \begin{array}{l} (1+m)\left\{ 1 + \frac{\cos^2(x)}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)} \left[1 + \frac{\cos^2(x)}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)} \right] \right\} - \\ (1-m)\left\{ 1 + \frac{\cos^2(x)}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)} \left[1 + \frac{\cos^2(x)}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)} \right] \right\} \end{array} \right\} \qquad [22]$$

Finally, substituting [8] in [21] and [22] yields an expression for $M_w(\Theta_a)$:

$$M_w(\Theta_a) = \frac{m^2 K_s}{2\alpha(n-1)(1+\varphi)} \left[\Lambda(\Theta_a) - \Lambda(\Theta_w) \right] \qquad [23]$$

with

$$\Lambda(\Theta) = \Theta^{\frac{1+\varphi}{m}} \left\{ \begin{array}{l} (1+m)\left(1 + \frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)} \left[1 + \frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)} \right] \right) - \\ (1-m)\left(1 + \frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)} \left[1 + \frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)} \right] \right) \end{array} \right\} \qquad [24]$$

2. Deriving an expression for obtaining analytical solution

To obtain an analytical solution, the following assumptions were made: (i) the equivalent root system can be represented by a cylinder with radius r_0 and height z_e; (ii) there is a value of soil water content (θ_l) defining the limit between potential and sub-potential transpiration rate; (iii) the effective soil volume explored per plant (rhizosphere) depends on the radius (r_m) of the root extraction zone and on z_e; (iv) the radial water flux density (q) in the rhizosphere is governed by the actual transpiration (T_a); (v) the radial water flux density in the rhizosphere, the water uptake and T_a are equivalent during the drying process; (vi) there is no water recharge, i.e., there is no ascendant flux density bellow the lower limit of the root zone; (vii) the total water transpiration from plant to atmosphere is defined by T_a and the total surface A_p and (viii) the relative transpiration is equivalent to the relative matric flux potential in the entire rhizosphere (V_p) (Figure 1).

The equivalent root volume per plant (V_e, m^3) is defined as follows:

$$V_e = \pi r_0^2 Z_e \qquad [25]$$

where r_0 is the equivalent root diameter (m) and z_e is the effective root depth (m), from the soil surface to the lower border of the root zone.

For a transpiring plant extracting water from a soil, hydraulic conditions can be classified according to the soil water availability in a non-limiting water range and a limiting water range. While the soil water content is in the non-limiting water range, transpiration occurs at the potential rate and relative transpiration is constant and equal to 1. When soil water content falls below a certain threshold value (θ_l, m^3 m^{-3}), relative transpiration starts to diminish. Therefore, this limiting water range ($\theta < \theta_c$) is also called the falling rate phase. During the constant rate phase, from t_0 to t_l, the flux density (q, m d^{-1}) in the rhizosphere can be calculated by:

$$q_p = -\frac{T_p A_p}{A_r} = -K(\theta)\frac{\partial h}{\partial r}\bigg|_{\substack{t_0 \le t \le t_l \\ r_0}} \qquad [26]$$

where T_p is the potential transpiration rate (m d^{-1}), A_p is the surface area (m^2), A_r is the root area (m^2), and θ is the soil water content.

In the falling rate phase ($\theta_w \le \theta < \theta_c$):

$$q_a = -\frac{T_a A_p}{A_r} = -K(\theta)\frac{\partial h}{\partial r}\bigg|_{\substack{t > t_l \\ r_0}} \qquad [27]$$

where T_a is the actual transpiration rate (m d^{-1}) and θ_w is the soil water content at permanent wilting.

The soil volume explored per plant (V_p, m^3), representing the rhizosphere, is defined as follows:

$$V_p = \pi z_e \left(r_m^2 - r_0^2 \right) \quad [28]$$

where r_m is the radius (m) of the root extraction zone.

It has been shown (Metselaar & De Jong van Lier, 2007) that it is reasonable to assume that

$$\frac{M_{r_0}}{M_{l,r_0}} = \frac{\overline{M}}{\overline{M}_l} = \frac{q_a}{q_p} \quad [29]$$

where $\overline{M}$ and M_{r_0} are the mean matric flux potential and the matric flux potential at the root surface, respectively, and $\overline{M}_l$ and M_{l,r_0} are the mean matric flux potential and the matric flux potential at the root surface at the onset of the falling rate phase (corresponding to $\theta = \theta_l$). Substituting equations [26] and [27] in [29]:

$$\frac{\overline{M}}{\overline{M}_l} = \frac{T_a}{T_p} = T_r \quad [30]$$

where T_r is the (dimensionless) relative transpiration. Substituting the expression derived for M (equation [23]) in equation [30] yields:

$$T_r = \frac{M}{M_l} = \frac{\Lambda(\overline{\Theta}) - \Lambda(\Theta_w)}{\Lambda(\overline{\Theta}_l) - \Lambda(\Theta_w)} \quad [31]$$

In Figure **Error! Bookmark not defined.**, T_r calculated for Van Genuchten type of soils by equation [31] is shown as a function of relative soil water content $\Theta^\dagger$ with:

$$\Theta^\dagger = \frac{\theta - \theta_w}{\theta_l - \theta_w} \quad [32]$$

Figure **Error! Bookmark not defined.**a shows $T_r(\Theta^\dagger)$ for different values of m for $\lambda = 0.5$. Reported values of m in literature range from 0.1 to 0.8. As can be observed, $m = 0.1134$ (corresponding to $n = 1.1279$) corresponds to a straight line and, therefore, to a constant diffusivity. Van Genuchten diffusivity can be shown to be equal to

$$D = K\frac{dh}{d\theta} = \frac{K_s}{\alpha(n-1)(\theta_s - \theta_r)} \Theta^{\lambda - \frac{1-2n}{1-n}} \left[1 - \left(1 - \Theta^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \right]^2 \left[\Theta^{\frac{n}{1-n}} - 1 \right]^{\frac{1-n}{n}} \quad [33]$$

For values smaller than $m = 0.1134$, the reduction curve shows to be convex, which did not occur for any of the soils analyzed by Metselaar and de Jong van Lier (2007). Figure **Error! Bookmark not defined.**a also shows that convexity increases with increasing m, until reaching a maximum at $m = 0.352$. For values higher than 0.352, convexity decreases with increasing m.

Figure **Error! Bookmark not defined.**b shows $T_r(\Theta^{\dagger})$ for different values of λ at $m = 0.5$. Reported values in literature range from -5 to +2. As can be seen, low values of λ correspond to convex curves; high values of λ result in concave curves; $\lambda = -2.1$ corresponds to a more or less straight line.

For the special case of $\Theta_w = 0$ (or: $\theta_w = \theta_r$), equation [31] reduces to:

$$T_r = \frac{\Lambda(\overline{\Theta})}{\Lambda(\overline{\Theta}_l)} \qquad [34]$$

Substituting [24] in [34] yields:

$$T_r = \left(\frac{\overline{\Theta}}{\overline{\Theta}_l}\right)^{\frac{\varphi+1}{m}} \left\{ \frac{(1+m)\left\{1+\frac{\overline{\Theta}^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2+m)}{(\varphi+2)}\left[1+\frac{\overline{\Theta}^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3+m)}{(\varphi+3)}\right]\right\} - (1-m)\left\{1+\frac{\overline{\Theta}^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2-m)}{(\varphi+2)}\left[1+\frac{\overline{\Theta}^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3-m)}{(\varphi+3)}\right]\right\}}{(1+m)\left\{1+\frac{\overline{\Theta}_l^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2+m)}{(\varphi+2)}\left[1+\frac{\overline{\Theta}_l^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3+m)}{(\varphi+3)}\right]\right\} - (1-m)\left\{1+\frac{\overline{\Theta}_l^{\frac{1}{m}}}{3}\frac{(\varphi+1)(2-m)}{(\varphi+2)}\left[1+\frac{\overline{\Theta}_l^{\frac{1}{m}}}{4}\frac{(\varphi+2)(3-m)}{(\varphi+3)}\right]\right\}} \right\} \qquad [35]$$

It can easily be seen that:

$$\frac{\partial\overline{\theta}}{\partial \mathrm{t}} = -\frac{T_a}{z_e} = -\frac{T_p}{z_e}T_r = -\frac{T_p}{z_e}T_r \qquad [36]$$

or, substituting equation [34] in [36]:

$$\frac{\partial\overline{\theta}}{\partial \mathrm{t}} = -\frac{T_p}{z_e}\frac{\Lambda(\overline{\Theta})}{\Lambda(\overline{\Theta}_l)} \qquad [37]$$

which can be reorganized as:

$$\frac{1}{\Lambda(\overline{\Theta})}\partial\overline{\theta} = -\frac{T_p}{z_e\Lambda(\overline{\Theta}_l)}dt \qquad [38]$$

Substituting $d\overline{\theta} = (\theta_s - \theta_r)d\overline{\Theta}$ and integrating yields

$$\left(\theta_s-\theta_r\right)\int_{\overline{\Theta}_l}^{\overline{\Theta}}\frac{1}{\Lambda\left(\overline{\Theta}\right)}\partial\overline{\Theta}=-\frac{T_p}{z_e\Lambda\left(\overline{\Theta}_l\right)}\int_{t_l}^{t}dt=-\frac{T_p}{z_e\Lambda\left(\overline{\Theta}_l\right)}\left(t-t_l\right) \quad [39]$$

Substituting equation [24] in [39] and performing the following transformation:

$$\overline{\Theta}^{\frac{1}{m}}=\cos^2\left(z\right) \quad [40]$$

$$z=\arccos\left(\overline{\Theta}^{\frac{1}{2m}}\right) \quad [41]$$

$$d\overline{\Theta}=-2m\cos^{2m-1}\left(z\right)\sin\left(z\right)dz \quad [42]$$

results in

$$\frac{-\tau}{2}=$$

$$\int_{z_l}^{z}\cos^c\left(z\right)\sin\left(z\right)\left[\begin{array}{l}\left(1+m\right)\left\{1+\frac{\cos^2(z)}{3}\frac{\left(\varphi+1\right)\left(2+m\right)}{\left(\varphi+2\right)}\left[1+\frac{\cos^2(z)}{4}\frac{\left(\varphi+2\right)\left(3+m\right)}{\left(\varphi+3\right)}\right]\right\}-\\ \left(1-m\right)\left\{1+\frac{\cos^2(z)}{3}\frac{\left(\varphi+1\right)\left(2-m\right)}{\left(\varphi+2\right)}\left[1+\frac{\cos^2(z)}{4}\frac{\left(\varphi+2\right)\left(3-m\right)}{\left(\varphi+3\right)}\right]\right\}\end{array}\right]^{-1}dz \quad [43]$$

which can be rewritten as

$$\frac{-\tau}{2}=\int_{z_1}^{z}\frac{\cos^{2\mathrm{m}-2\varphi-3}\left(z\right)\sin\left(z\right)}{g\left\{1+a\cos^2\left(\mathrm{z}\right)\left[1+b\cos^2\left(\mathrm{z}\right)\right]\right\}-h\left\{1+c\cos^2\left(\mathrm{z}\right)\left[1+d\cos^2\left(\mathrm{z}\right)\right]\right\}}dz \quad [44]$$

where

$$\tau=-\frac{T_p\left(t-t_l\right)}{m\left(\theta_s-\theta_r\right)\Lambda\left(\overline{\Theta}_l\right)z_e} \quad [45]$$

$$a=\frac{\left(\varphi+1\right)\left(2+m\right)}{3\left(\varphi+2\right)} \quad [46]$$

$$b=\frac{\left(\varphi+2\right)\left(3+m\right)}{4\left(\varphi+3\right)} \quad [47]$$

$$c=\frac{\left(\varphi+1\right)\left(2-m\right)}{3\left(\varphi+2\right)} \quad [48]$$

$$d=\frac{\left(\varphi+2\right)\left(3-m\right)}{4\left(\varphi+3\right)} \quad [49]$$

$$g=1+m \quad [50]$$

$$h=1-m \quad [51]$$

Transforming equation [44] according to

$$y = \cos^2(z) \qquad [52]$$

$$dz = -\frac{1}{2\cos(z)\sin(z)}dy \qquad [53]$$

results in

$$\tau = \int_{y_1}^{y} \frac{y^f}{g\left[a.y(b.y+1)+1\right]-h\left[c.y(d.y+1)+1\right]}dy \qquad [54]$$

$$f = c-1 \qquad [55]$$

Equation [54] can be solved as follows:

$$\tau = \frac{y^f}{fS_A}\left\{\frac{{}_2\mathrm{F}_1\left(-\mathrm{f},-\mathrm{f},1-\mathrm{f},\frac{S_B - S_D}{-\Gamma + \mathrm{S_A} - S_E y}\right)}{\left[\frac{y}{y-\frac{S_C - S_D}{S_E}}\right]^f} - \frac{{}_2\mathrm{F}_1\left(-\mathrm{f},-\mathrm{f},1-\mathrm{f},\frac{S_B + S_D}{\Gamma + \mathrm{S_A} + 2S_E y}\right)}{\left[\frac{y}{y+\frac{S_C + S_D}{S_E}}\right]^f}\right\}\Bigg|_{y_I}^{y} \qquad [56]$$

where the auxiliary variables (S_A, S_B, S_C, S_D and S_E) are defined as follows:

$$S_A = \sqrt{a^2g^2 - 2ag\left[2b(g-h)+ch\right] + ch\left[4d(g-h)+ch\right]} \qquad [57]$$

$$S_B = \sqrt{(ag-ch)^2 - 4(g-h)(abg-cdh)} \qquad [58]$$

$$S_C = \sqrt{a^2g^2 + 4ab\left(gh-g^2\right) - 2acgh + 4cd\left(gh-h^2\right) + c^2h^2} \qquad [59]$$

$$S_D = ag - ch \qquad [60]$$

$$S_E = 2(abg - cdh) \qquad [61]$$

Expanding equation [56] results in:

$$\tau = \frac{1}{fS_A}\left[y^f \left\{ \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B - S_D}{-\Gamma + S_A - S_E y}\right)}{\left[\dfrac{y}{y - \dfrac{S_C - S_D}{S_E}}\right]^f} - \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B + S_D}{\Gamma + S_A + S_E y}\right)}{\left[\dfrac{y}{y + \dfrac{S_C + S_D}{S_E}}\right]^f} \right\} - y_l^f \left\{ \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B - S_D}{-\Gamma + S_A - 2S_E y_l}\right)}{\left[\dfrac{y_1}{y_l - \dfrac{S_C - S_D}{S_E}}\right]^f} - \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B + S_D}{\Gamma + S_A + S_E y_l}\right)}{\left[\dfrac{y_1}{y_1 + \dfrac{S_C + S_D}{S_E}}\right]^f} \right\} \right] \quad [62]$$

Substituting [52] in [62]:

$$\tau = \frac{1}{fS_A}\left[\cos^{2f}(z)\left\{\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B - S_D}{-S_D + S_A - S_E\cos^2(z)}\right)}{\left[\dfrac{\cos^2(z)}{\cos^2(z) - \dfrac{S_C - S_D}{S_E}}\right]^f} - \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B + S_D}{S_D + S_A + S_E\cos^2(z)}\right)}{\left[\dfrac{\cos^2(z)}{\cos^2(z) + \dfrac{S_C + S_D}{S_E}}\right]^f}\right\} - \cos^{2f}(z_l)\left\{\frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B - S_D}{-S_D + S_A - S_E\cos^2(z_l)}\right)}{\left[\dfrac{\cos^2(z_l)}{\cos^2(z_l) - \dfrac{S_C - S_D}{S_E}}\right]^f} - \frac{{}_2F_1\left(-f,-f,1-f,\dfrac{S_B + S_D}{S_D + S_A + S_E\cos^2(z_l)}\right)}{\left[\dfrac{\cos^2(z_l)}{\cos^2(z_l) + \dfrac{S_C + S_D}{S_E}}\right]^f}\right\}\right] \quad [63]$$

Substituting [41] in [63]:

$$\tau = \frac{1}{fS_A}\left[\Theta^{\frac{f}{m}}\left\{\frac{{}_2F_1\left(-f,-f,1-f,\frac{S_B-S_D}{-S_D+S_A-S_E\Theta^{\frac{1}{m}}}\right)}{\left[\frac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}}-\frac{S_C-S_D}{S_E}}\right]^f}-\frac{{}_2F_1\left(-f,-f,1-f,\frac{S_B+S_D}{S_D+S_A+S_E\Theta^{\frac{1}{m}}}\right)}{\left[\frac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}}+\frac{S_C+S_D}{S_E}}\right]^f}\right\}-\Theta_l^{\frac{f}{m}}\left\{\frac{{}_2F_1\left(-f,-f,1-f,\frac{S_B-S_D}{-S_D+S_A-S_E\Theta_l^{\frac{1}{m}}}\right)}{\left[\frac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}}-\frac{S_C-S_D}{S_E}}\right]^f}-\frac{{}_2F_1\left(-f,-f,1-f,\frac{S_B+S_D}{S_D+S_A+S_E\Theta_l^{\frac{1}{m}}}\right)}{\left[\frac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}}+\frac{S_C+S_D}{S_E}}\right]^f}\right\}\right] \quad [64]$$

Expanding Gauss's hypergeometric functions for $k = 4$, the final equation can be written as follows:

$$\tau = \frac{1}{fS_A}\left[\Theta^{\frac{f}{m}}\left\{\frac{1+\dfrac{(S_B\text{-}S_D)\Omega}{S_A\text{-}S_D\text{-}S_E\Theta^{\frac{1}{m}}}}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}}-\dfrac{S_C-S_D}{S_E}}\right]^f}-\frac{1+\dfrac{(S_B+S_D)\Omega}{S_A+S_D+S_E\Theta^{\frac{1}{m}}}}{\left[\dfrac{\Theta^{\frac{1}{m}}}{\Theta^{\frac{1}{m}}+\dfrac{S_C+S_D}{S_E}}\right]^f}\right\}-\Theta_l^{\frac{f}{m}}\left\{\frac{1+\dfrac{(S_B\text{-}S_D)\Omega}{S_A\text{-}S_D\text{-}S_E\Theta_l^{\frac{1}{m}}}}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}}-\dfrac{S_C-S_D}{S_E}}\right]^f}-\frac{1+\dfrac{(S_B+S_D)\Omega}{S_A+S_D+S_E\Theta_l^{\frac{1}{m}}}}{\left[\dfrac{\Theta_l^{\frac{1}{m}}}{\Theta_l^{\frac{1}{m}}+\dfrac{S_C+S_D}{S_E}}\right]^f}\right\}\right] \quad [65]$$

with

$$\Omega = \left[\frac{1}{(1-f)}+\left(\frac{1-f}{2(2-f)}\right)+\frac{(1-f)(2-f)}{6(3-f)}+\frac{(1-f)(2-f)(3-f)}{24(4-f)}\right] \quad [66]$$

The mean soil water content (θ, m^3 m^{-3}) can be calculated as the weighted average of two soil volumes as follows:

$$\bar{\theta} = \frac{\dfrac{\int_{r_0}^{r_m}\theta(r)dr}{\int_{r_0}^{r_m} r\,dr}\left(r_m^2 - r_0^2\right)+\theta(r_m)\left(\dfrac{e^2}{4}-r_0^2\right)}{r_m^2 - 2r_0^2 + \dfrac{e^2}{4}} \quad [67]$$

where $\theta(r_m)$ is the soil water content (m^3 m^{-3}) at $r = r_m$.

REFERENCES

Abramowitz, M.; Stegun, I. A. (Ed.). Handbook[1] of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, 1048p., 1972.

Metselaar, K., and Q. De Jong van Lier. 2007. The shape of the transpiration reduction function under plant water stress. Vadose Zone J. 6:124-139.

Van Genuchten, M. Th. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. **Soil Science Society of America Journal**, v.44, p.892-898. 1980.

[1] http://books.google.nl/books?id=MtU8uP7XMvoC&dq=Handbook+of+Mathematical+Functions+with+Formulas,+Graphs,+and+Mathematical+Table+By+Irene+A.+Stegun,+Milton+Abramowitz&pg=PP1&ots=-BWPRtO6Li&sig=7ZFRYlduEhZzfILo57ZVTSy0dNQ&prev=http://www.google.nl/search%3Fhl%3Den%26q%3DHandbook%2Bof%2BMathematical%2BFunctions%2Bwith%2BFormulas%252C%2BGraphs%252C%2Band%2BMathematical%2BTable%2BBy%2BIrene%2BA.%2BStegun%252C%2BMilto%2BAbramowitz%26btnG%3DGoogle%2BSearch%26meta%3D&sa=X&oi=print&ct=result&cd=1#PRA1-PR7,M1

APPENDIX

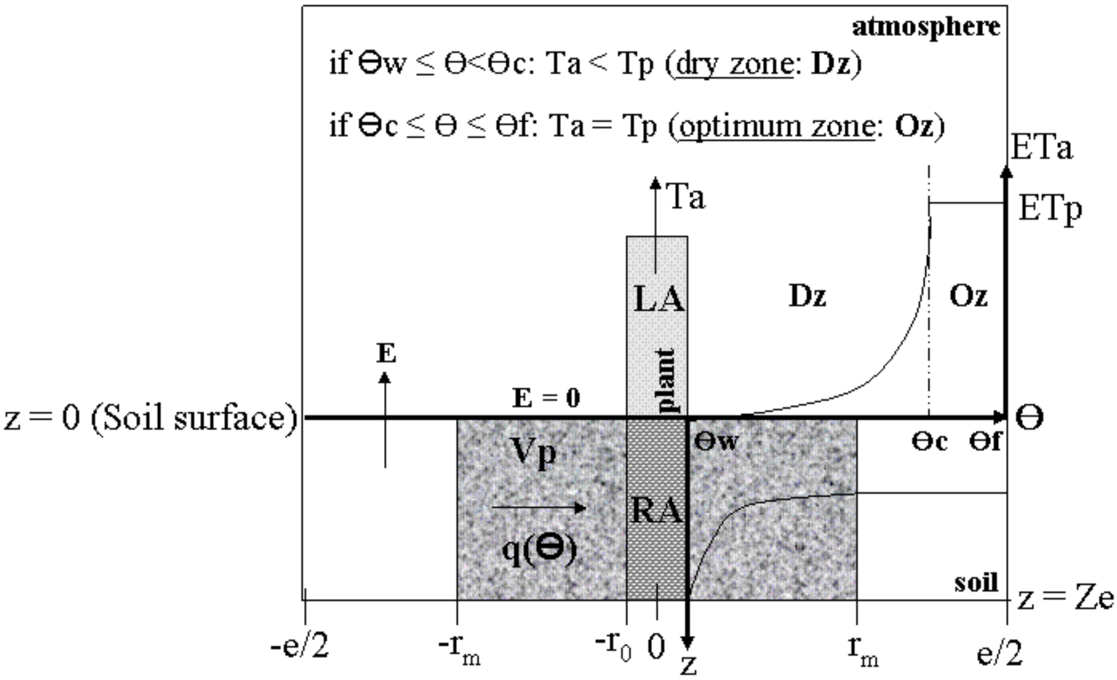


Figure 1. Schematic representation of basic assumptions to characterize the water dynamic in the soil-plant-atmosphere system for agricultural and environmental purposes, where ‘q’ is the soil water flux density (Darcy-Buckingham, 1856-1907), ‘Ze’ is the effective root depth, ‘Vp’ is the effective soil volume explored by plant, ‘Θ’ is the soil water content (‘Θw’, ‘Θc’ and ‘Θf’ are wilting point, critical value and field capacity, respectively), ‘E’ is the evaporation, ‘Ta’ is the actual transpiration, ‘Tp’ is the potential transpiration, ‘ETa’ is the actual evapotranspiration, ‘ETp’ is the potential evapotranspiration, ‘LA’ is the total leaf area, ‘r_0’ is the equivalent root diameter, ‘r_m’ is the radius of the root extraction zone and ‘RA’ is the total root area.

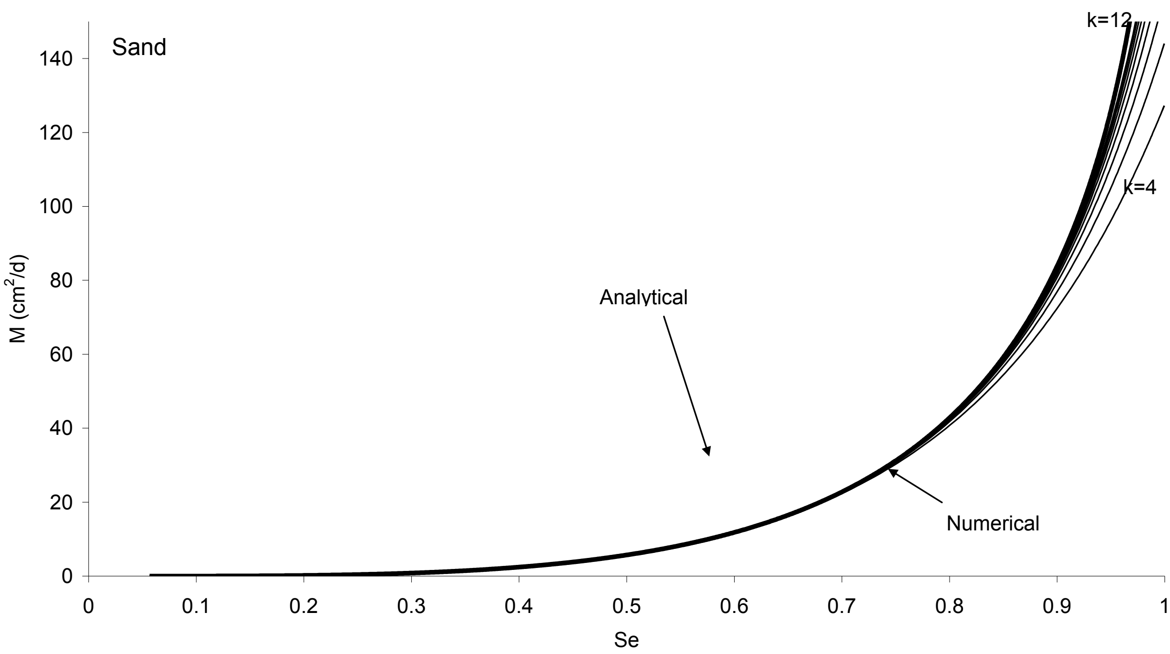


Figure 2. Comparison between analytical (results for **sand soil** according to equation 22 using different values of 'k' - from 4 to 12) and numerical solution showing the matric flux potential (M, $cm^2.d^{-1}$) as function of effective degree of saturation (Se), where $\alpha = 0.0144$ cm^{-1}; m = 0.348109518; n = 1.534; Ɵs = 0.46 $m^3.m^{-3}$; Ɵr = 0.02 $m^3.m^{-3}$; λ = -0.215 and Ks = 15.42 $cm.d^{-1}$.

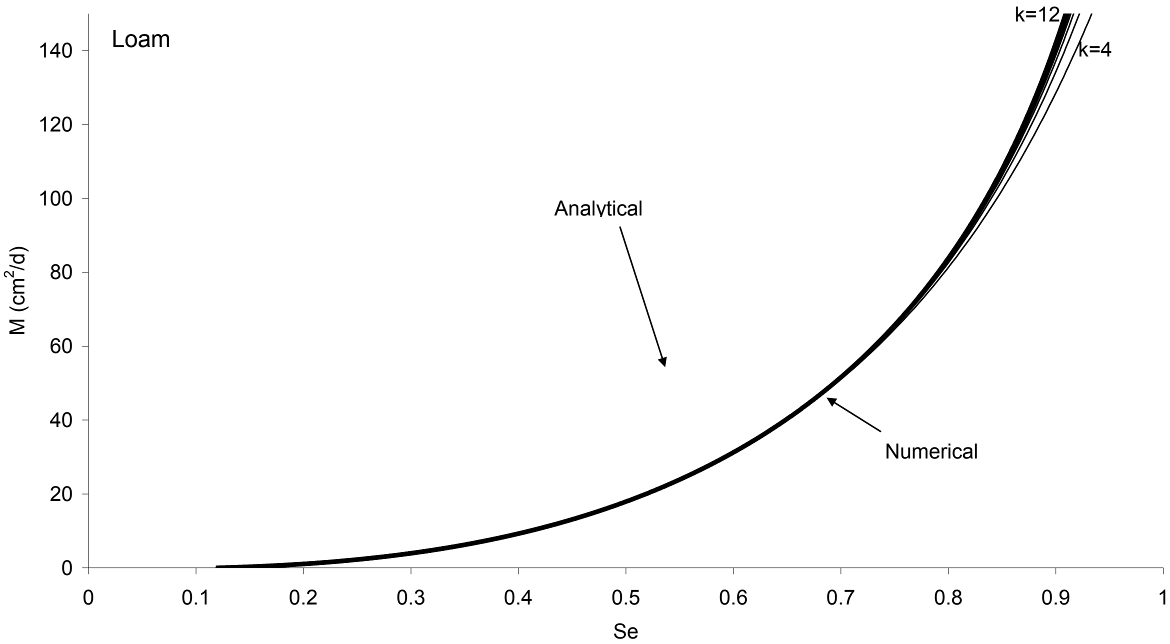


Figure 3. Comparison between analytical (results for **loam soil** according to equation 22 using different values of 'k' - from 4 to 12) and numerical solution showing the matric flux potential (M, $cm^2.d^{-1}$) as function of effective degree of saturation (Se), where $\alpha = 0.0084\ cm^{-1}$; m = 0.306037474; n = 1.441; $\Theta s = 0.42\ m^3.m^{-3}$; $\Theta r = 0.01\ m^3.m^{-3}$; $\lambda = -1.497$ and $Ks = 12.98\ cm.d^{-1}$.

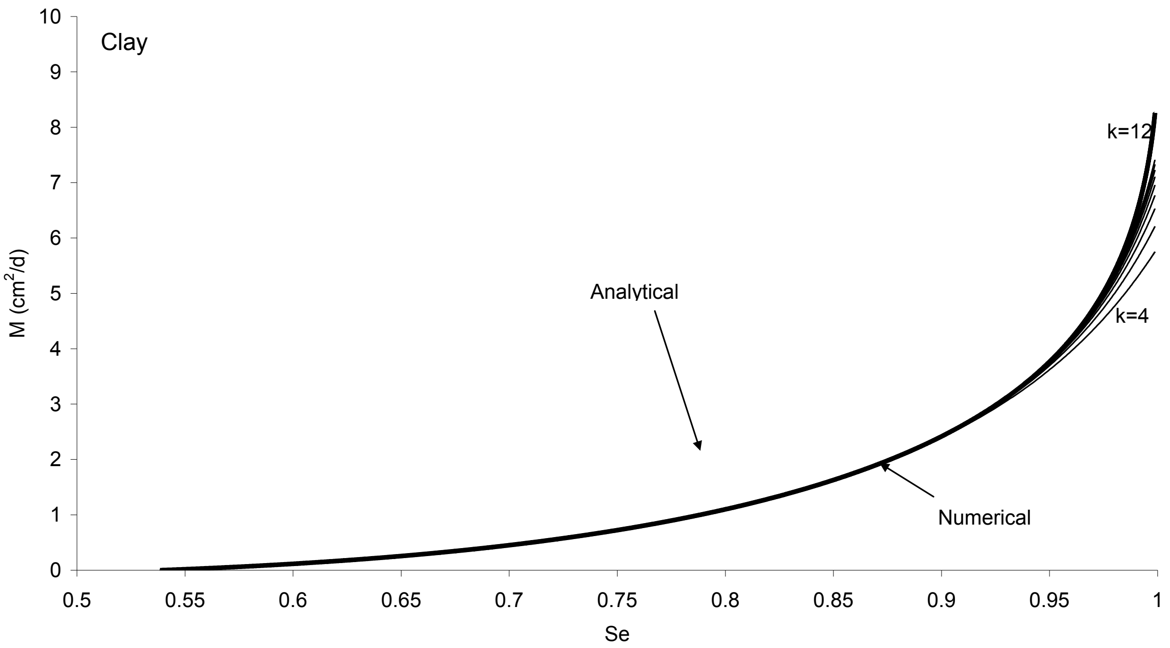
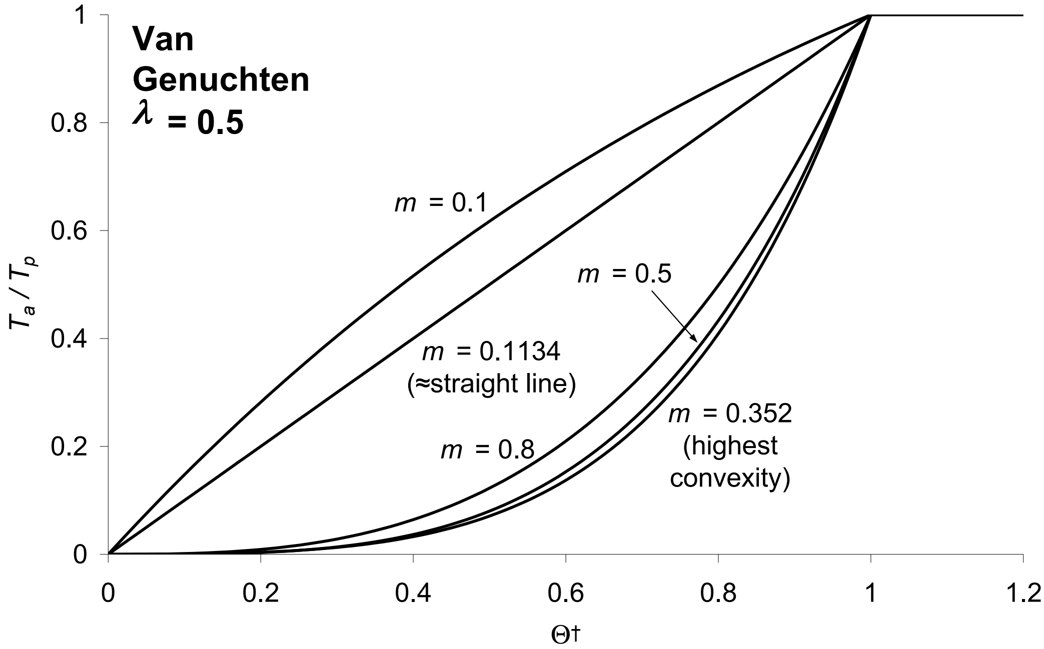
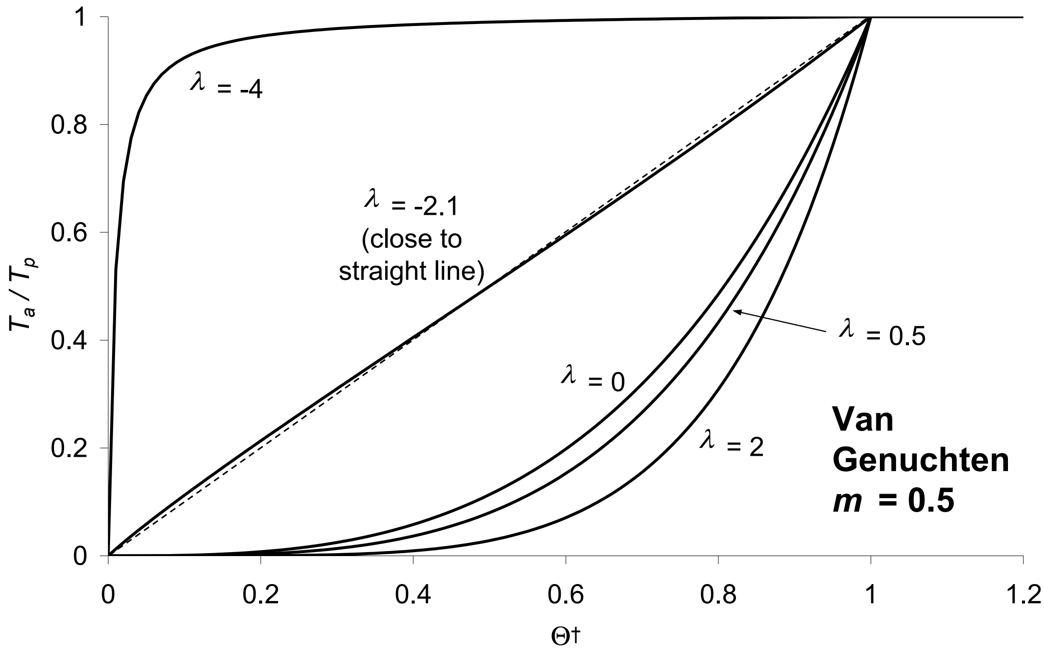


Figure 4. Comparison between analytical (results for **clay soil** according to equation 22 using different values of 'k' - from 4 to 12) and numerical solution showing the matric flux potential (M, $cm^2.d^{-1}$) as function of effective degree of saturation (Se), where α = 0.0195 cm^{-1}; m = 0.098286745; n = 1.109; Θs = 0.59 $m^3.m^{-3}$; Θr = 0.01 $m^3.m^{-3}$; λ = -5.901 and Ks = 4.53 $cm.d^{-1}$.



(a)



(b)

Figure 5. Relative transpiration T_a / T_p as a function of relative soil water content $\Theta^{\dagger}$ [$\Theta^{\dagger} = (\theta-\theta_w) / (\theta_l-\theta_w)$] for Van Genuchten type of soils (a) at different values of m with $\lambda = 0.5$ and (b) at different values of λ with $m = 0.5$.