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Slow Dynamics on Quenched Complex Networks

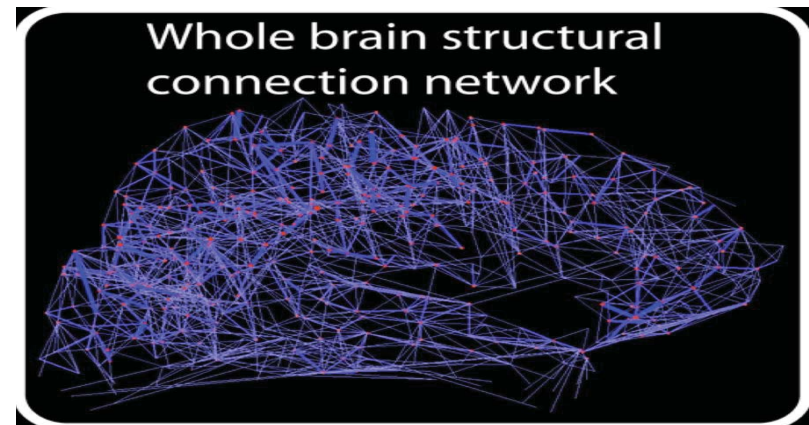
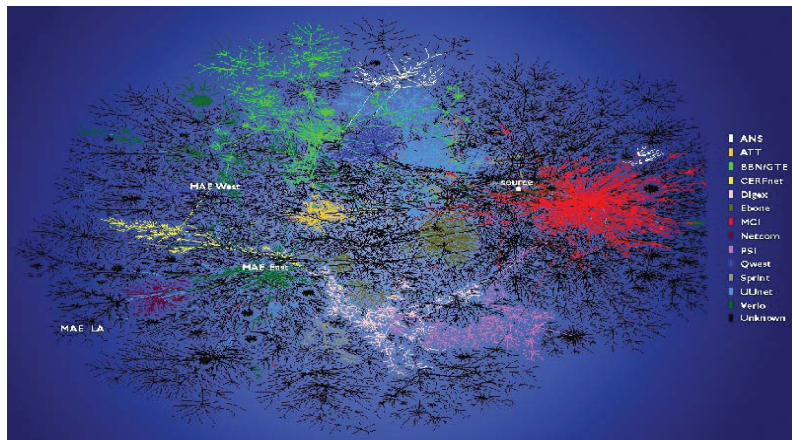
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Slow dynamics on quenched complex networks

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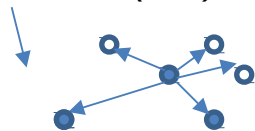
Statphys research → dynamical processes defined on complex networks

Expectation: small world topology → mean-field behavior → fast dynamics

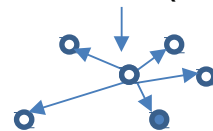


Prototype: **Contact Process (CP)** or **Susceptible-Infected-Susceptible (SIS)** two-state models:

Infect: $\lambda / (1+\lambda)$

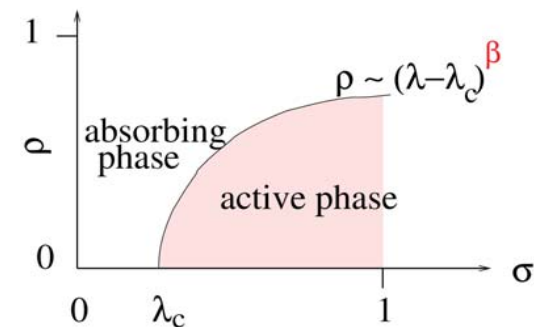


Heal: $1 / (1+\lambda)$



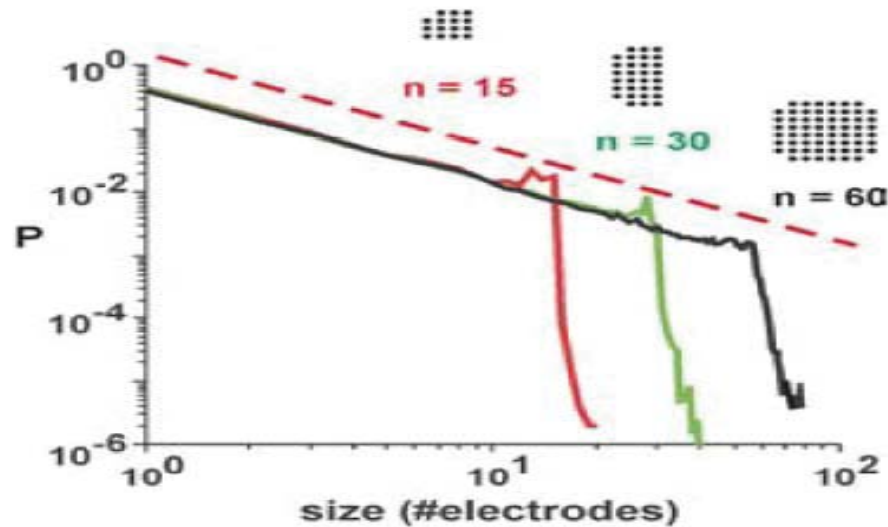
Order parameter : density of active (●) sites

Regular, Euclidean lattice: **DP** critical point : $\lambda_c > 0$ between inactive and active phases

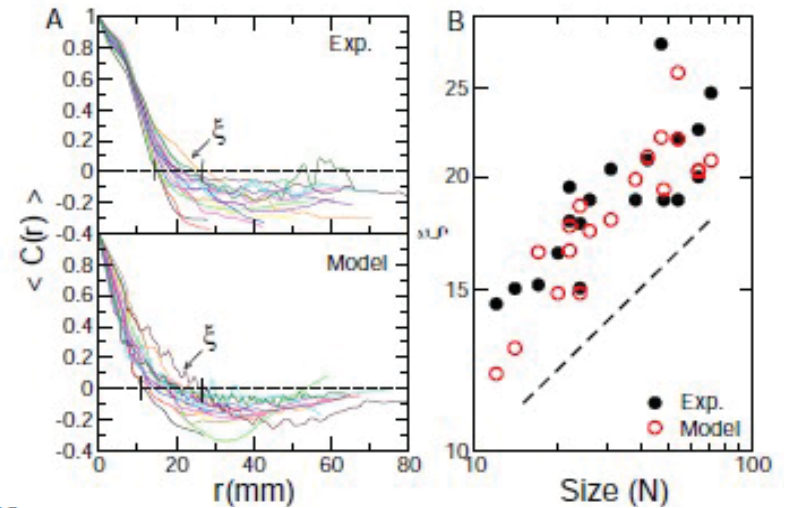


Observed power-law (slow) dynamics in networks

- Brain : Size distribution of neural avalanches
G. Werner : *Biosystems*, 90 (2007) 496,

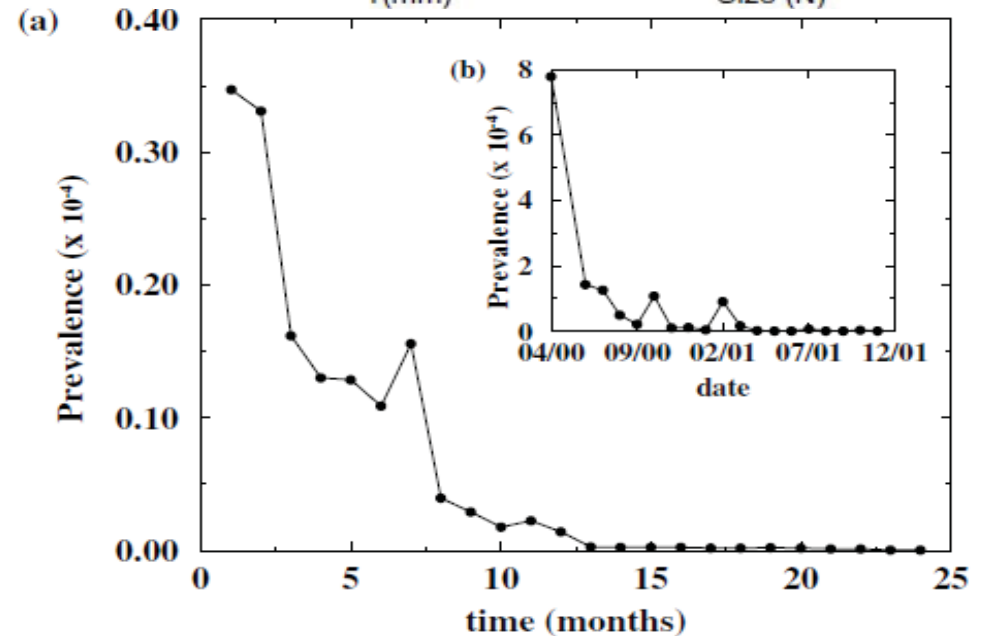


- Correlation length (ξ) diverges
Tagliazucchi & Chialvo (2012) :
Brain complexity born out of criticality.



Internet: worm recovery time is slow:

Can we expect slow dynamics in small-world networks ?



Scaling in nonequilibrium system

Scaling and universality classes appear in complex system due to : $\xi \rightarrow \infty$
i.e: near **critical points**, due to currents ...

Basic models are classified by universal scaling behavior in Euclidean, regular system

- **Why don't we see universality classes in models defined on networks ?**
- **Power laws are frequent in nature \leftrightarrow Tuning to critical point ?**

I'll show a possible way to understand these

Rare Region theory for **quench disordered CP**

- Fixed (quenched) disorder/impurity **changes the local birth rate** $\Rightarrow \lambda_c > \lambda_c^0$

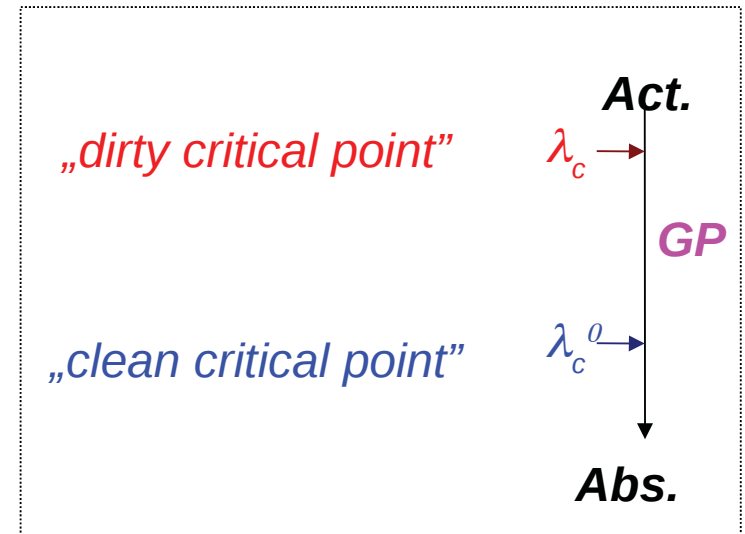
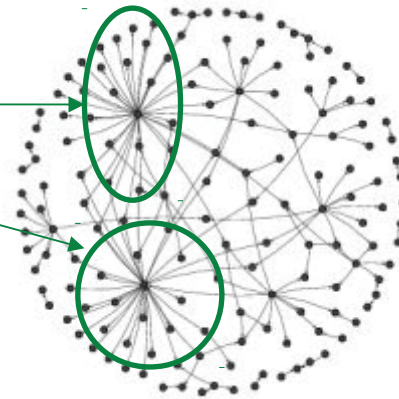
- **Locally active**, but arbitrarily large

Rare Regions

in the inactive phase due to the **inhomogeneities**

- Probability of RR of size L_R :

$$w(L_R) \sim \exp(-c L_R)$$



contribute to the density: $\rho(t) \sim \int dL_R L_R w(L_R) \exp[-t/\tau(L_R)]$

- For $\lambda < \lambda_c^0$: conventional (exponentially fast) decay

- At λ_c^0 the characteristic time scales as: $\tau(L_R) \sim L_R^z \Rightarrow$ saddle point analysis:

$$\ln \rho(t) \sim t^{d/(d+z)}$$

stretched exponential

- For $\lambda_c^0 < \lambda < \lambda_c$:

$$\tau(L_R) \sim \exp(b L_R):$$

Griffiths Phase

$$\rho(t) \sim t^{-c/b}$$

continuously changing exponents

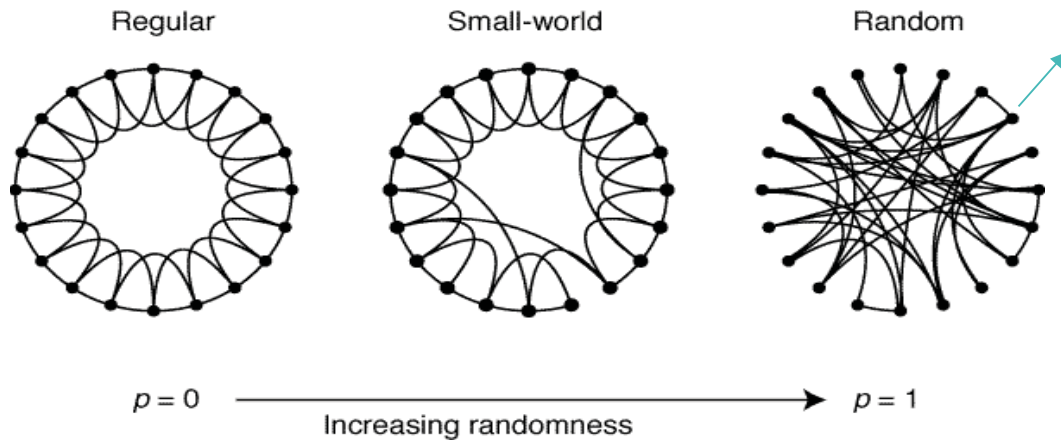
- At λ_c

$$\rho(t) \sim \ln(t)^{-\alpha} \quad \text{Infinite randomness fixed point scaling}$$

- In case of correlated RR-s with dimension $> d$: smeared transition

Basic network models

From regular to random networks:



Erdős-Rényi ($p = 1$)

Degree (k) distribution in $N \rightarrow \infty$ node limit:

$$P(k) = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

Topological dimension: $N(r) \sim r^d$

Above perc. thresh.: $d = \infty$

Below percolation $d = 0$

Scale free networks:



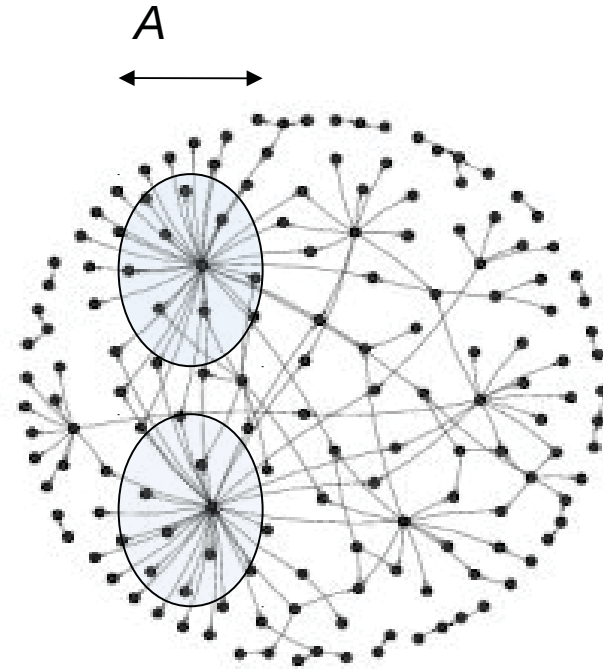
Degree distribution:

$$P(k) = k^{-\gamma} \quad (2 < \gamma < 3)$$

Topological dimension: $d = \infty$

Example: **Barabási-Albert**
lin. preferential attachment

Rare active regions in the absorbing phase: $\tau(A) \sim e^A$
→ slow dynamics (Griffiths Phase) ?



M. A. Munoz, R. Juhász, C. Castellano and G. Ódor, PRL 105, 128701 (2010)

1. Inherent disorder in couplings
2. Disorder induced by topology

Optimal fluctuation theory + simulations: YES

- In Erdős-Rényi networks below the percolation threshold
- In generalized small-world networks for **finite topological dimension**

CP + Topological disorder results

Generalized Small World networks: $P(l) \sim \beta l^{-2}$
 (link length probability)

Top. dim: $N(r) \sim r^d$ $d(\beta)$ finite:

$\lambda_c(\beta)$ decreases monotonically from
 $\lambda_c(0) = 3.29785$ (1d CP) to:

$\lim_{\beta \rightarrow \infty} \lambda_c(\beta) = 1$ towards mean-field CP value

$\lambda < \lambda_c(\beta)$: inactive, there can be
 locally ordered, rare regions due to more
 than average, active, incoming links

Griffiths phase: λ -dep. continuously changing
 dynamical power laws:

for example: $\rho(t) \sim t^{-\alpha(\lambda)}$

Logarithmic corrections!

Ultra-slow ("activated") scaling: $\rho \propto \ln(t)^{-\alpha}$ at λ_c

As $\beta \rightarrow 1$ Griffiths phase shrinks/disappears

Same results for: cubic, regular random nets
 higher dimensions?

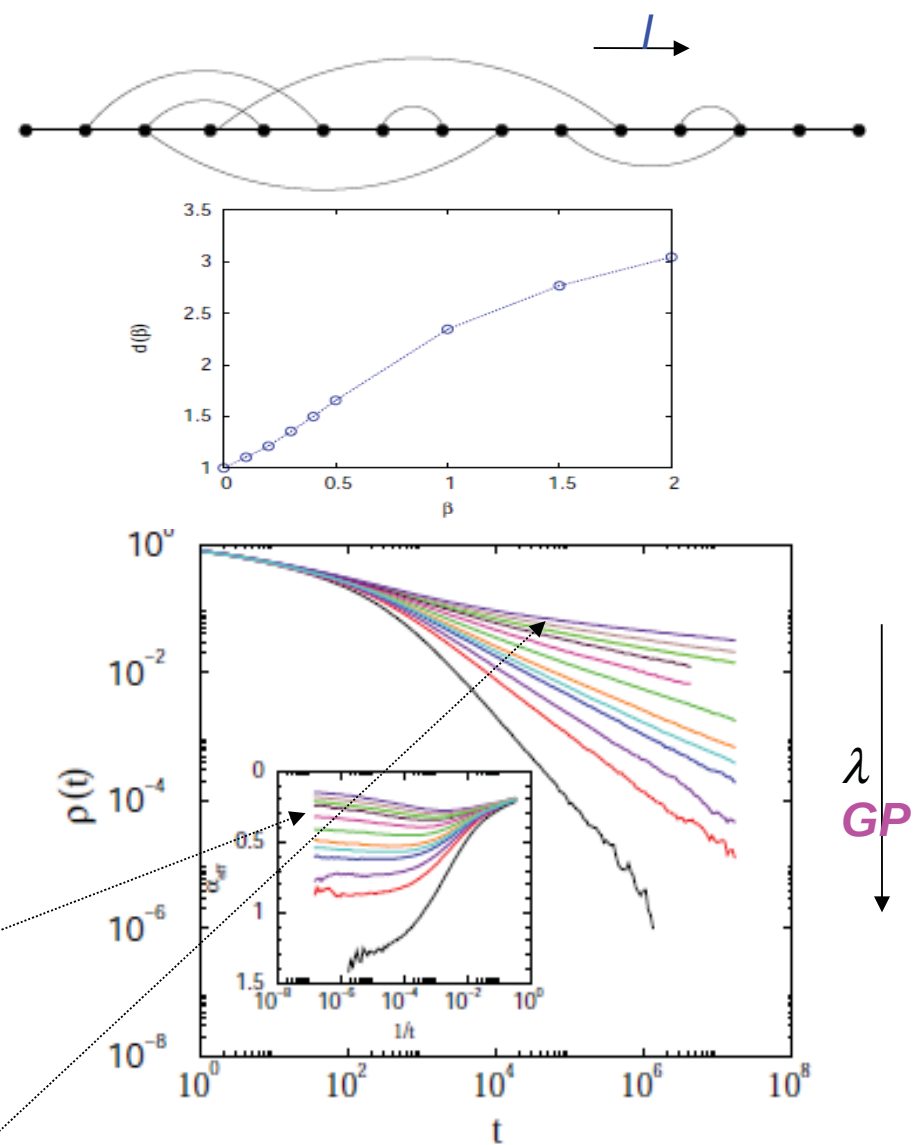


FIG. 3: Density decay in Benjamini-Berger networks with $s = 2$ and $\beta = 0.2$ for different values of λ (from top to bottom: 2.81, 2.795, 2.782, 2.77, 2.75, 2.73, 2.71, 2.70, 2.69, 2.67, 2.65, 2.6). Straight lines lie in the Griffiths phase. Inset: Corresponding effective exponents, illustrating the presence of corrections to scaling.

Contact Process on Barabási-Albert (BA) network

- Heterogeneous mean-field theory: conventional critical point, with linear density decay:

$$\rho(t) \sim [t \ln(t)]^{-1},$$

with logarithmic correction

- Extensive simulations confirm this:
- No Griffiths phase observed
- Steady state density vanishes at $\lambda_c \approx 1$ linearly, HMF: $\beta = 1$

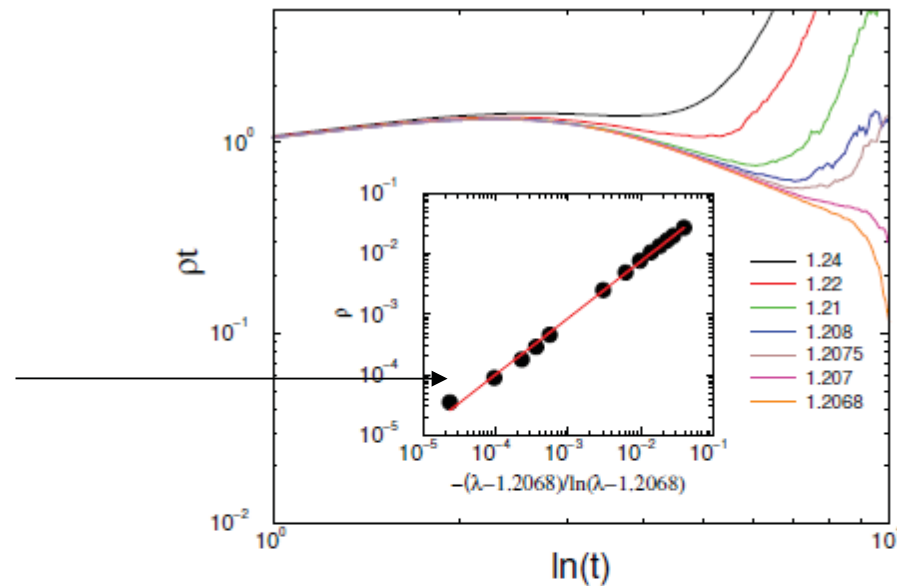


FIG. 1. Density decay ($t\rho(t)$) as a function of $\ln(t)$ for the CP on unweighted looped BA networks with $m = 3$ of size $N = 8 \times 10^7$. The different curves correspond to $\lambda = 1.2068, \dots, 1.24$ (bottom to top). Inset: Steady state density, showing agreement with HMF theory scaling. The full line shows a power-law fitting to the data points in the form $-0.36(5)x^{0.98(2)}$.

CP results on Barabási-Albert graphs

- Excluding loops slows down the spreading + Weights:
- *G. Ódor, R. Pastor-Storras PRE 86 (2012) 026117*
- *WBAT-I: $\omega_{ij} = \omega_0(k_i k_j)^{-\nu}$ hubs are suppressed*

$$\omega_{ij} = \frac{|i-j|^x}{N}, \quad k_i \propto (N/i)^{1/2}$$

WBAT-II: disassortative

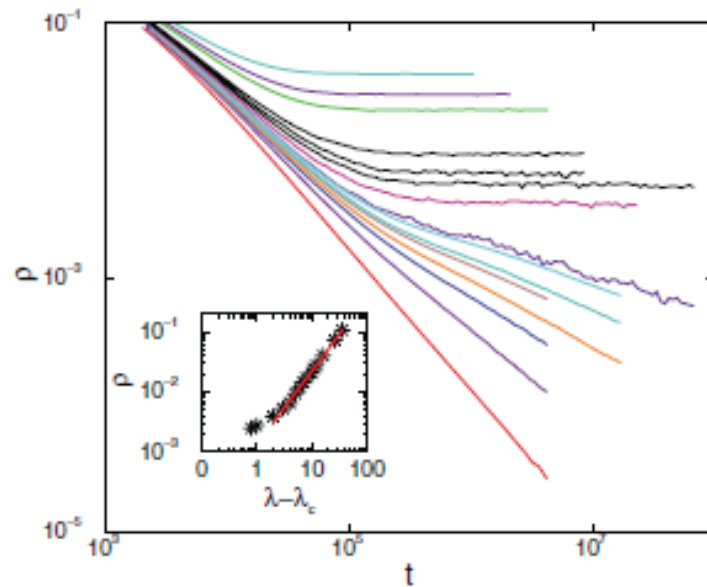


FIG. 3. (Color online) Density decay as a function of time for the CP on weighted BA trees generated with the WBAT-I model with exponent $\nu = 1.5$. Network size $N = 10^5$. Different curves correspond to $\lambda = 160, 156, 154, 149, 148, 147, 146, 145, 144.7, 144.2, 144, 143.5, 143, 142, 140$ (from top to bottom). Inset: Steady-state density.

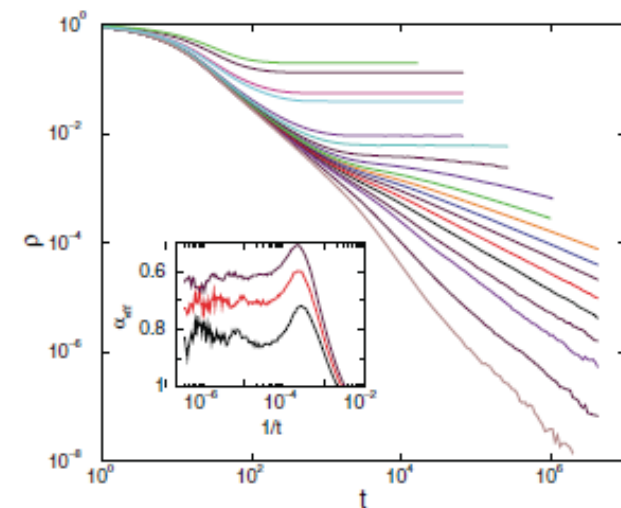


FIG. 7. (Color online) Density decay as a function of time $\rho(t)$ for the CP on weighted BA trees with a age-dependent weighting scheme (WBAT-II) with exponent $x = 2$. Network size $N = 10^5$. Different curves correspond to $\lambda = 6.75, 6.8, 6.85, 6.87, 6.9, 6.92, 6.94, 6.96, 6.98, 7, 7.04, 7.1, 7.2, 7.4, 8.5, 9, 12, 15$ (from top to bottom). Inset: Corresponding local slopes for $\lambda = 6.9, 6.92, 6.94$ (from bottom to top).

λ dependent density decay exponents: Griffiths Phases or Smeared phase transition ?

Do power-laws survive the thermodynamic limit ?

- Finite size analysis shows the disappearance of a power-law scaling:

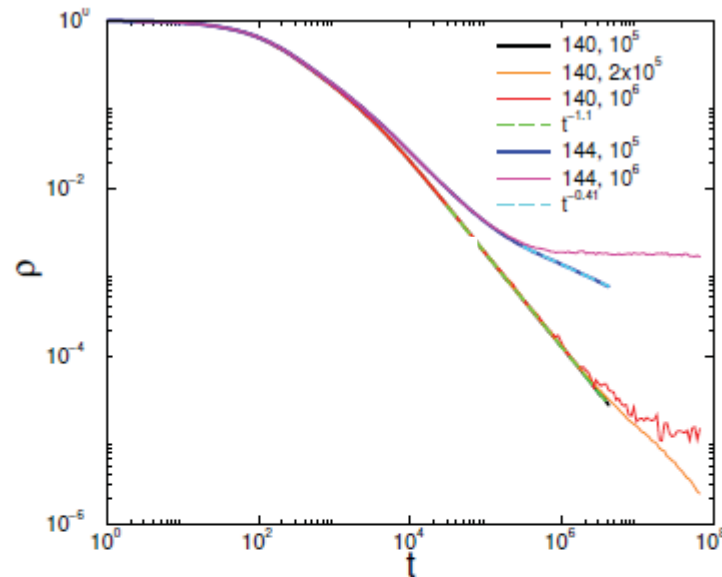


FIG. 5. Density decay as a function of time $\rho(t)$ for the CP on weighted BA trees with a multiplicative weighting scheme (WBAT-I) with exponent $\nu = 1.5$. Plots correspond to two sets of λ (upper branch: $\lambda = 144$, lower branch $\lambda = 140$) at different network sizes N . Dashed lines represent PL fittings. Inset: Initial time region of the same data, showing an stretched exponential behavior.

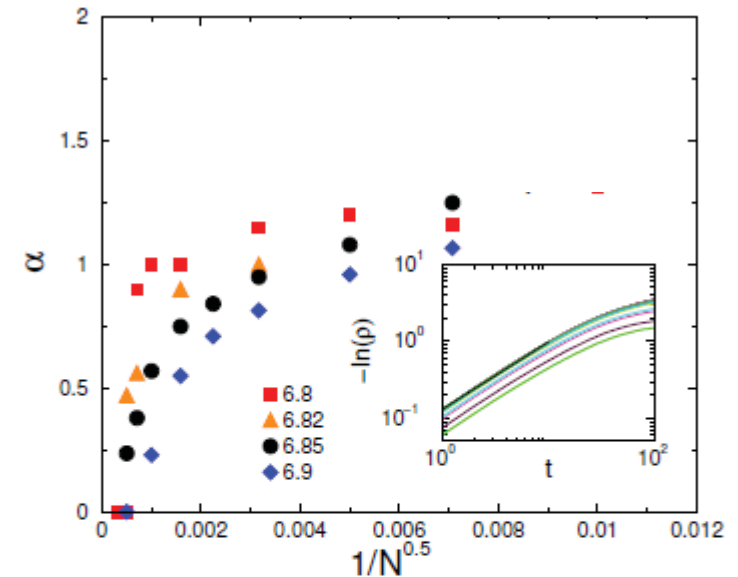


FIG. 8. Finite-size scaling analysis of the density decay exponent for $\lambda = 6.75$ (triangles), $\lambda = 6.8$ (boxes), $\lambda = 6.82$ (triangles), $\lambda = 6.85$ (bullets), $\lambda = 6, 9$ (rhombes) in the CP on weighted BA trees with a age-dependent weighting scheme (WBAT-II) with exponent $x = 2$. Top inset: $\rho(t)$ for $\lambda = 6.82$ ($N = 10^6$, $N = 4 \times 10^5$, $N = 10^5$ top to bottom). Bottom inset: Initial time density.

Power-law \rightarrow saturation explained by smeared phase transition:

- **High dimensional rare sub-spaces**

Percolation analysis of the weighted BA tree

We consider a network of a given size N , and delete all the edges with a weight smaller than a threshold ω_{th} .

For small values of ω_{th} , many edges remain in the system, and they form a connected network with a single cluster encompassing almost all the vertices in the network.

When increasing the value of ω_{th} , the network breaks down into smaller subnetworks of connected edges, joined by weights larger than ω_{th} .

The size of the largest ones (S_i) grows linearly with the network size N

↔ standard percolation transition.

These clusters, which can become arbitrarily large in the thermodynamic limit, play the role of correlated **RRs**, sustaining independently activity and smearing down the phase transition.

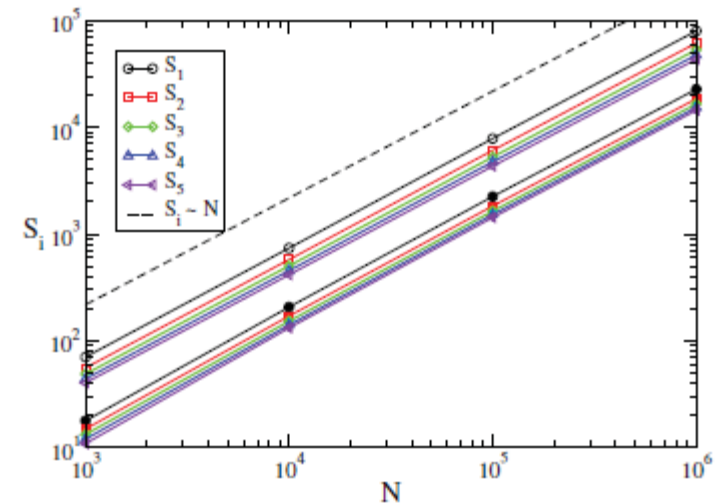


FIG. 6. Size S_i of the 5 largest clusters in a percolation analysis of the WBAT-I model with $\nu = 1.5$ for $\omega_{\text{th}} = 100\omega_{\text{min}}$ (hollow symbols) and $\omega_{\text{th}} = 1000\omega_{\text{min}}$ (full symbols), where ω_{min} is the minimum weight in the network. The size of all components grows linearly with network size N , and is therefore infinite in the thermodynamic limit.

Spectral Analysis of networks with quenched (disordered) topology

Master (rate) equation of SIS for occupancy prob. at site i :

$$\frac{d\rho_i(t)}{dt} = -\rho_i(t) + (1 - \rho_i(t)) \sum_j A_{ij} \lambda \rho_j(t) \omega_{ij} \quad (4)$$

For $t \rightarrow \infty$ the system evolves into a steady state, with the probabilities expressed as

$$\rho_i = \frac{\lambda \sum_j B_{ij} \rho_j}{1 + \lambda \sum_j B_{ij} \rho_j} \quad (5)$$

Weighted (real symmetric) Adjacency matrix: $B_{ij} = A_{ij} \omega_{ij}$,

Express ρ_i on orthonormal eigenvector ($\mathbf{f}_i(\Lambda)$) basis:

$$\rho_i = \sum_{\Lambda} c(\Lambda) f_i(\Lambda). \quad (6)$$

$$c(\Lambda) = \lambda \sum_{\Lambda'} \Lambda' c(\Lambda') \sum_{i=1}^N \frac{f_i(\Lambda) f_i(\Lambda')}{1 + \lambda \sum_{\tilde{\Lambda}} \tilde{\Lambda} c(\tilde{\Lambda}) f_i(\tilde{\Lambda})}. \quad (7)$$

$$\lambda_c = 1/\Lambda_1 \quad \Lambda_1(N) \propto N^{1/4}$$

Total infection density vanishes near λ_c as :

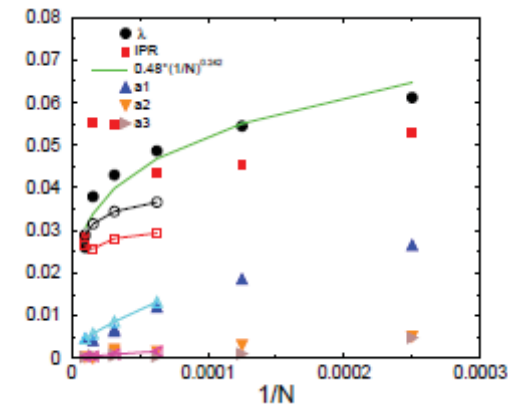
$$\rho(\lambda) \approx \alpha_1 \tau + \alpha_2 \tau^2 + \dots, \quad (8)$$

where $\tau = \lambda \Lambda_1 - 1 \ll 1$ with the coefficients

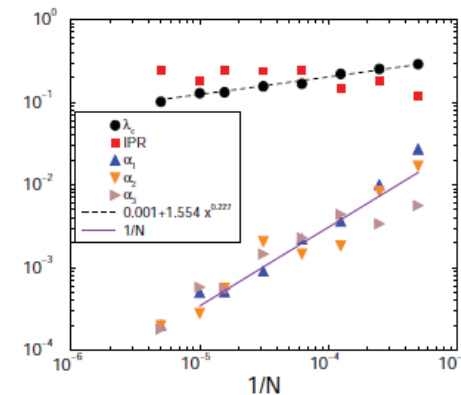
$$\alpha_j = \sum_{i=1}^N f_i(\Lambda_j) / [N \sum_{i=1}^N f_i^3(\Lambda_j)]. \quad (9)$$

To describe the localization of the components of $\mathbf{f}(\Lambda_1)$ [19] used the inverse participation ratio

$$IPR(\Lambda) \equiv \sum_{i=1}^N f_i^4(\Lambda), \quad (10)$$



Unweighted BAT



WBAT-II

Network	$1/\Lambda_1$	c	IPR	α_1	α_2	α_3
BA	0.017	0.32	0.014	4×10^{-4}	10^{-7}	10^{-8}
BAT	~ 0	0.24	0.055	10^{-4}	10^{-7}	10^{-9}
WBAT-I	~ 0	0.9(1)	~ 0	0.35(1)	5×10^{-8}	10^{-6}
WBAT-II	0.001	0.227	0.25	4×10^{-3}	3×10^{-5}	10^{-3}

Localization, strong rare-region effects in case of WBAT-II networks !

Summary

- *Quenched disorder in complex networks can cause slow dynamics :
Rare-regions \rightarrow (Griffiths) phases \rightarrow **no tuning or self-organization needed !***
- *In **finite dim.** (for CP) GP can occur **due to topological disorder***
- *In **infinite dim**, scale-free, BA network mean-field transition of CP
with logarithmic corrections (HMF+simulations)*
- *In **BA trees** non mean-field transition observed*
- *In **weighted BA trees** non-universal, slow, power-law dynamics
can occur for finite N , but in the $N \rightarrow \infty$ limit saturation is observed*
- *Smearred transition can describe this,
percolation analysis confirms the existence of arbitrarily large dimensional
sub-spaces with (correlated) large weights*
- *Acknowledgements to : HPC-Europa2, OTKA, FuturICT.hu*

[1] M. A. Munoz, R. Juhasz, C. Castellano, and G. Ódor, *Phys. Rev. Lett.* 105, 128701 (2010)

[2] G. Ódor, R. Juhasz, C. Castellano, M. A. Munoz, *AIP Conf. Proc.* 1332, Melville, NY (2011) 172

[3] R. Juhasz, G. Ódor, C. Castellano, M. A. Munoz, *Phys. Rev. E* 85, 066125 (2012)

[4] G. Ódor and Romualdo Pastor-Satorras, *Phys. Rev. E* 86, 026117 (2012)

[5] G. Ódor, *arXiv:1301.4407*