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Chaos and transport in disordered classical nonlinear chains

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Discrete nonlinear Schrödinger equation with disorder

$$i \frac{d\psi_n}{dt} = \omega_n \psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^* \psi_n^2$$

Bosonic atoms in disordered optical lattices

Bose-Hubbard Hamiltonian:

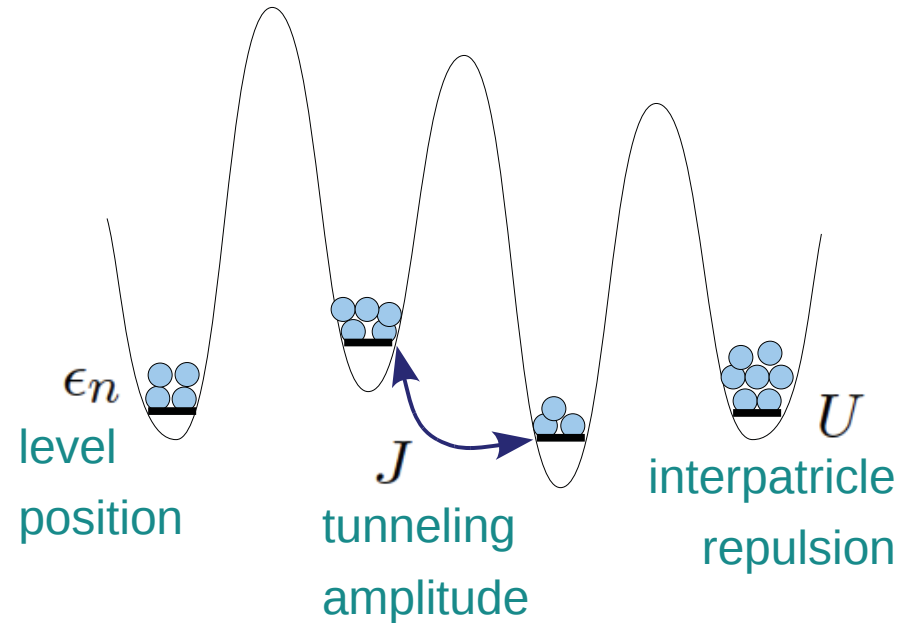
$$\hat{H} = \sum_n \epsilon_n \hat{b}_n^\dagger \hat{b}_n + U \sum_n \hat{b}_n^\dagger \hat{b}_n^\dagger \hat{b}_n \hat{b}_n - J \sum_n (\hat{b}_{n+1}^\dagger \hat{b}_n + \hat{b}_n^\dagger \hat{b}_{n+1})$$

bosonic operator

$$\hat{b}_n = \frac{\psi_n}{\sqrt{\hbar}} + \hat{\xi}_n$$

condensate wave function

quantum fluctuation



Billy *et al.*, *Nature* **453**, 891 (2008)

Roati *et al.*, *Nature* **453**, 895 (2008)

$$\frac{d\hat{b}_n}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{b}_n] \quad \longrightarrow \quad i\hbar \frac{d\psi_n}{dt} = \epsilon_n \psi_n - J(\psi_{n-1} + \psi_{n+1}) + \frac{U}{\hbar} |\psi_n|^2 \psi_n$$

The classical limit of a bosonic field: $N_n \equiv \langle \hat{b}_n^\dagger \hat{b}_n \rangle \rightarrow \infty$, $\hbar \rightarrow 0$, $N_n \hbar \rightarrow |\psi_n|^2$

Light in evanescently coupled 1D waveguides

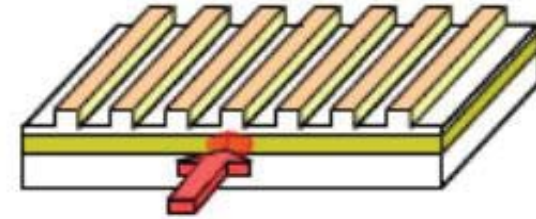
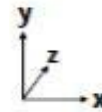
Wave equation for the electric field $E(\mathbf{r})e^{-i\omega t}$:

$$\epsilon(\mathbf{r}) \frac{\omega^2}{c^2} E + \nabla^2 E = \frac{\omega^2}{c^2} \chi^{(3)} |E|^2 E$$



inhomogeneous
dielectric
structure

Kerr
nonlinearity



Lahini *et al.*, *PRL* **100**, 013906 (2008)



transverse eigenmodes



$$E(\mathbf{r}) = \sum_n \psi_n(z) u_{\perp}(x - x_n, y) e^{ikz}$$



$k \gg \partial/\partial x, \partial/\partial y$ paraxial approximation

discrete
nonlinear
Schrödinger
equation

$$2ik \frac{\partial \psi_n}{\partial z} = q_{\perp, n}^2 \psi_n + \dots$$

Discrete nonlinear Schrödinger equation with disorder

$$i \frac{d\psi_n}{dt} = \underbrace{\omega_n \psi_n}_{\text{harmonic oscillators}} - \underbrace{\Omega(\psi_{n+1} + \psi_{n-1})}_{\text{nearest-neighbor coupling}} + \underbrace{g\psi_n^* \psi_n^2}_{\text{anharmonicity}}$$

Coordinate and momentum:
$$\psi_n = \frac{p_n}{\sqrt{2m\omega_n}} + \sqrt{\frac{m\omega_n}{2}} i q_n$$

Classical equations of motion:
$$\frac{dp_n}{dt} = -\frac{\partial H(p, q)}{\partial q_n}, \quad \frac{dq_n}{dt} = \frac{\partial H(p, q)}{\partial p_n}$$

Dimensionality:
$$\psi_n \sim \sqrt{\text{action}}$$

Anderson localization and nonlinearity

$$i \frac{d\psi_n}{dt} = \omega_n \psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^* \psi_n^2$$

disorder coupling nonlinearity

Anderson localization

$$i \frac{d\psi_n}{dt} = \omega_n \psi_n - \Omega(\psi_{n+1} + \psi_{n-1})$$

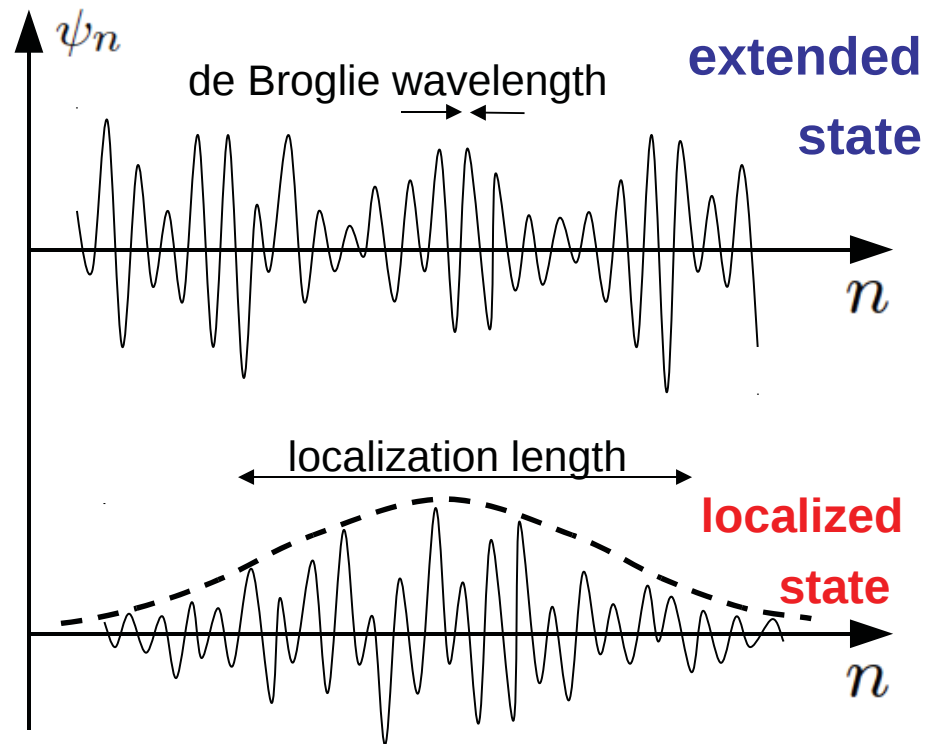
disorder coupling

Anderson localization

$$i \frac{d\psi_n}{dt} = \omega_n \psi_n - \Omega(\psi_{n+1} + \psi_{n-1})$$

disorder coupling

In one dimension
all eigenstates
are localized



Weak disorder: plane waves
↓
mean free path
↙
localization length (= 2 × m.f.p.)

Strong disorder:
localization on one site
↓
weak perturbative tails
on the neighboring sites

Anderson localization and nonlinearity

$$i \frac{d\psi_n}{dt} = \underbrace{\omega_n \psi_n}_{\text{disorder}} - \underbrace{\Omega(\psi_{n+1} + \psi_{n-1})}_{\text{coupling}} + \underbrace{g \psi_n^* \psi_n^2}_{\text{nonlinearity}}$$

STRONG disorder + WEAK coupling



STRONG localization

WEAK

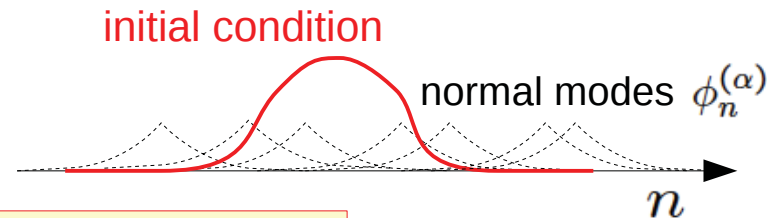
nonlinearity

What is the effect of a WEAK nonlinearity
on STRONG Anderson localization?

The problem of wave packet spreading

Linear system: Anderson localization

$$\psi_n(t) = \sum_{\alpha} c_{\alpha} \phi_n^{(\alpha)} e^{-i\omega_{\alpha} t}, \quad c_{\alpha} = \sum_n \phi_n^{(\alpha)} \psi_n(0)$$



The wave packet remains exponentially localized forever

Nonlinear system: interaction between the normal modes

$$i \frac{dc_{\alpha}}{dt} = \sum_{\beta\gamma\delta} V_{\alpha\beta\gamma\delta} e^{i(\omega_{\alpha} + \omega_{\beta} - i\omega_{\gamma} - i\omega_{\delta})t} c_{\beta}^* c_{\gamma} c_{\delta}$$

→ small correction?
 → chaotic behavior?

Numerical integration: $\langle \Delta x^2 \rangle \propto t^p$, $p \sim 0.3 - 0.4$ subdiffusion

Shepelyansky (1993); Molina (1998); Kopidakis *et al.* (2008); Pikovsky & Shepelyansky (2008); Skokos *et al.* (2009); Skokos & Flach (2010); Laptjeva *et al.* (2010); Bodyfelt *et al.* (2011)

Indications for slowing down: Mulansky *et al.* (2011); Michaely & Fishman (2012)

KAM theorem, perturbation theory: $p \rightarrow 0$

Bourgain & Wang (2008); Wang & Zhang (2009); Fishman *et al.* (2009); Johansson *et al.* (2010)

Experiment: subdiffusion (non-universal exponent) Lucioni *et al.* (2011)

Given

1. Strong localization
 2. Weak nonlinearity
 3. (Almost) arbitrary initial condition with extensive norm and energy
- } → worst conditions for transport

Question: will the system equilibrate at long distances, and how?

Answer: yes, by normal nonlinear diffusion:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

Mechanism: CHAOS

- concentrated on rare local chaotic spots
(as seen by Oganesyan, Pal, Huse, 2009)
- which produce stochastic force
- and redistribute the energy between oscillators

Assumptions

$$i \frac{d\psi_n}{dt} = \underbrace{\omega_n \psi_n}_{\text{disorder}} - \underbrace{\Omega(\psi_{n+1} + \psi_{n-1})}_{\text{coupling}} + \underbrace{g\psi_n^* \psi_n^2}_{\text{nonlinearity}}$$

$$-\frac{W}{2} \leq \omega_n \leq \frac{W}{2} \qquad g > 0$$

1. **Strong localization:** $\frac{\Omega}{W} \equiv \tau \ll 1$ assumption about the Hamiltonian
2. **Weak nonlinearity:** $\frac{g|\psi_n|^2}{W} \sim \rho \ll 1$
 (nonlinear frequency shift \ll disorder)
 note the invariance under $\psi_n \rightarrow C\psi_n, \quad g \rightarrow C^{-2}g$
assumptions about the initial conditions
3. **High temperature:** $H < 0, \quad |H| \ll W \sum |\psi_n|^2$
 (no superfluid effects)

Thermalization and transport

Two conserved quantities: total energy H , total action $I = \sum |\psi_n|^2$

Local equilibration
in a finite time $\Rightarrow \mathcal{P}(\{\psi_n\}) \propto e^{-\beta(H-\mu I)}, \quad \beta \equiv 1/T$

Global equilibration: transport of the conserved quantities

Macroscopic action density: $\rho(x) = \frac{1}{L^*} \sum_{n=x-L^*/2}^{x+L^*/2} \frac{g|\psi_n|^2}{W} \approx \frac{gT}{|\mu|W}$
 \swarrow
length scale to be specified later

get rid of the energy density thanks to $|H| \ll WI$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

finite!

Diffusion coefficient

$$D(\rho) \sim \exp\left(-C \ln^2 \frac{1}{\tau^p \rho} \ln \frac{1}{\rho}\right) \quad \frac{1}{2} \leq p \leq 3$$

stronger than any
power law

$$\frac{1}{3} \left[1 + \ln\left(1 + \frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2} \leq C \leq 8 \left[1 + \ln\left(1 + \frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2}$$

double logarithm \sim constant

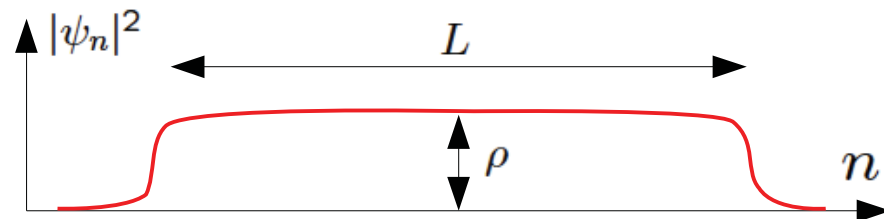
$$D^{-1} \text{ is self-averaging at distances } L^* \gg \exp\left(C \ln^2 \frac{1}{\tau^p \rho}\right)$$

Probability of chaos

Disorder realization $\{\omega_n\}$ }
 Initial condition $\{\psi_n\}$ } \rightarrow trajectory $\left\{ \begin{array}{l} \text{regular } \lambda = 0 \\ \text{chaotic } \lambda > 0 \end{array} \right.$

Lyapunov exponent $\lambda(\psi) = \lim_{t \rightarrow \infty} \lim_{\tilde{\psi}(0) \rightarrow \psi(0)} \frac{1}{t} \ln \frac{\|\tilde{\psi}(t) - \psi(t)\|}{\|\tilde{\psi}(0) - \psi(0)\|}$

Probability to be on a chaotic trajectory:



$$P_L(\tau, \rho) = \int_{-W/2}^{W/2} \prod_{n=1}^L \frac{d\omega_n}{W} \int \prod_{n=1}^L d^2\psi_n \delta(|\psi_n|^2 - \rho W/g) \Theta_{\text{chaotic}}(\{\omega_n\}, \{\psi_n\})$$

or $e^{-g|\psi_n|^2/\rho W}$

strong weak
 localization nonlinearity

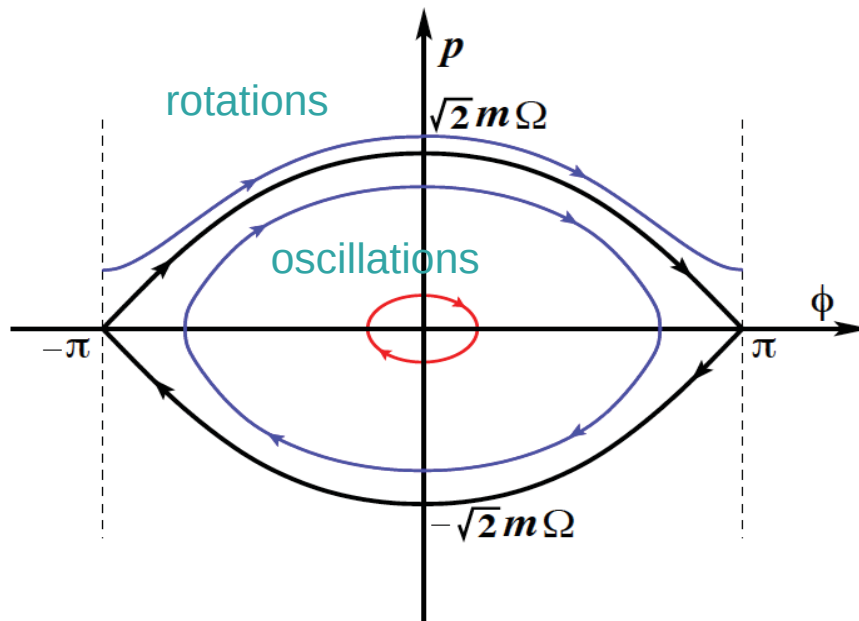
1 if chaotic
 0 if regular

$\tau \ll 1, \rho \ll 1 \quad \Rightarrow \quad \text{locality in space} \quad \Rightarrow \quad P_{L \gg 1}(\tau, \rho) = 1 - e^{-w(\tau, \rho)L}$

Basko, *Ann. Phys.* **326**, 1577 (2011); *Phys. Rev. E* **86**, 036202 (2012),
 Pikovsky & Fishman, *Phys. Rev. E* **83**, 025201 (2011)

Pendulum:
$$H(p, \phi) = \frac{p^2}{2m} - m\Omega^2 \cos \phi$$

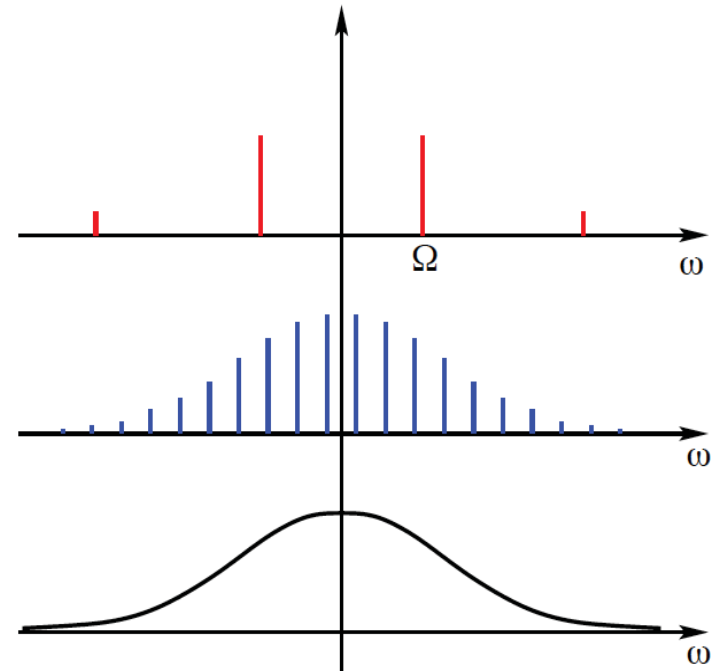
Phase space:



the period diverges at the separatrix



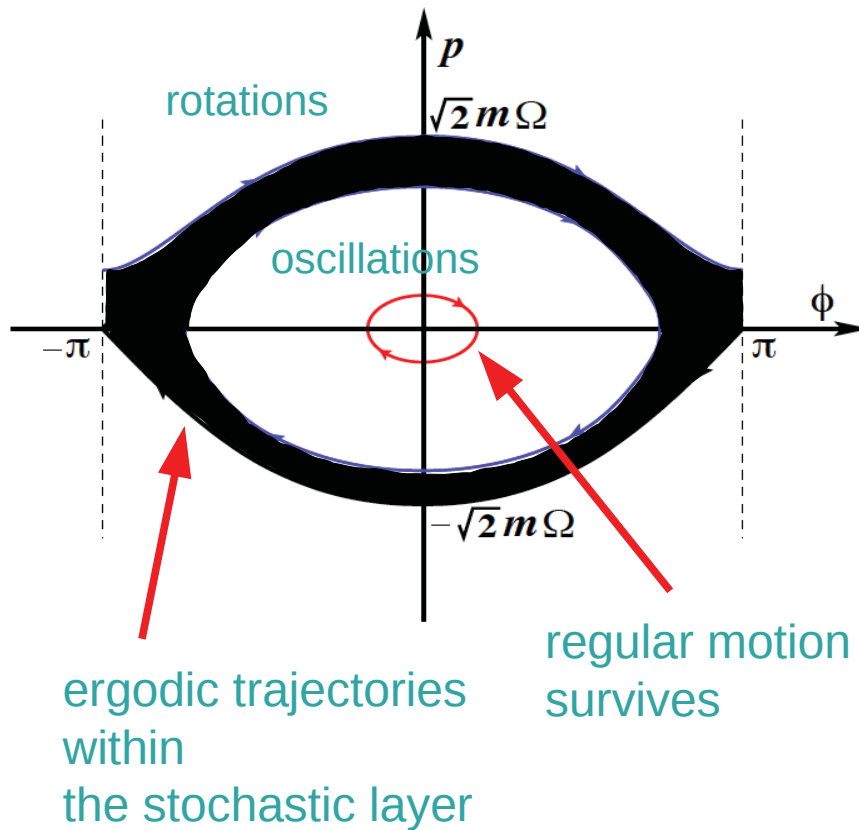
Spectrum
$$\int p(t) e^{i\omega t} dt$$



the separatrix motion has a continuous spectrum

Perturbed pendulum:

$$H(p, \phi, t) = \frac{p^2}{2m} - m\Omega^2 \cos \phi - V \cos(\phi - \omega t)$$



Stochastic layer area:

$$W_s \equiv \int_{\text{layer}} \frac{dp d\phi}{2\pi} \sim \frac{V}{\Omega} e^{-|\omega|/\Omega}$$

Melnikov-Arnold integral

$$|\omega| \gg \Omega$$

Continuous spectrum of the chaotic motion:

$$\left\langle e^{i\phi(t)} e^{-i\phi(t')} \right\rangle_{\omega} \sim \frac{1}{\Omega} e^{-|\omega|/\Omega}$$

review: B. Chirikov (1979)

Making a pendulum out of oscillators

two-oscillator
Hamiltonian:

$$H = \omega_1 |\psi_1|^2 + \frac{g}{2} |\psi_1|^4 + \omega_2 |\psi_2|^2 + \frac{g}{2} |\psi_2|^4 - \Omega (\psi_1^* \psi_2 + \psi_2^* \psi_1)$$

action-angle variables: $\psi_n = \sqrt{I_n} e^{-i\phi_n}$

$$H = \omega_1 I_1 + \frac{gI_1^2}{2} + \omega_2 I_2 + \frac{gI_2^2}{2} - 2\Omega \sqrt{I_1 I_2} \cos(\phi_1 - \phi_2)$$

canonical
transformation:

$$I = I_1 + I_2, \quad \phi = \frac{\phi_1 + \phi_2}{2}, \quad \tilde{I} = \frac{I_1 - I_2}{2}, \quad \tilde{\phi} = \phi_1 - \phi_2$$

$$H = \underbrace{H_0(I)}_{\text{constant}} + \underbrace{(\omega_1 - \omega_2)\tilde{I} + g\tilde{I}^2}_{\text{shift}} - \underbrace{2\Omega \sqrt{I^2/4 - \tilde{I}^2}}_{\text{almost constant}} \cos \tilde{\phi}$$

constant

shift

almost

constant $\Omega \ll W$

A third oscillator:

$$2\Omega \sqrt{I_2 I_3} \cos(\phi_2 - \phi_3)$$

perturbation of the pendulum

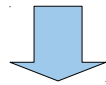
Three oscillators are sufficient
to generate chaos

The price of making a pendulum

To find a separatrix:

$$\omega_1 + gI_1 = \omega_2 + gI_2$$

$$gI_1, gI_2 \ll W$$



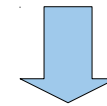
need $|\omega_1 - \omega_2| \ll W$

Look for a resonance

(guiding resonance)

To create the stochastic layer:

the pendulum frequency $\sim \sqrt{\Omega g I}$



$$\exp\left(-\frac{|\omega_2 - \omega_3|}{\sqrt{\Omega g I}}\right)$$

Look for another resonance

(layer resonance)

Making a pendulum out of more oscillators

$$-\Omega(\psi_1^*\psi_2 + \psi_2^*\psi_1) + \omega_2\psi_2^*\psi_2 - \Omega(\psi_2^*\psi_3 + \psi_3^*\psi_2)$$

 effective coupling 1 \leftrightarrow 3: $\frac{\Omega^2}{\omega_1 - \omega_2}(\psi_1^*\psi_3 + \psi_3^*\psi_1)$

works when $\omega_1 \approx \omega_3 \neq \omega_2$

Coupling + nonlinearity \rightarrow effective couplings of the form

$$\psi_1^*\psi_2^*\psi_3^*\psi_4\psi_5\psi_6 \rightarrow \cos(\phi_1 + \phi_2 + \phi_3 - \phi_4 - \phi_5 - \phi_6)$$

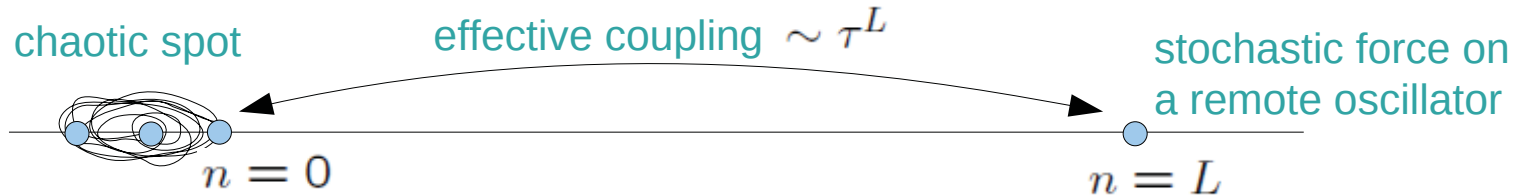
Guiding and layer resonances can be generated
in high orders of the perturbation theory

Competition: number of combinations \leftrightarrow power of the coupling constants

Chaos comes from rare resonant triples:

$$w \sim \min \left\{ \frac{\Omega}{W} \rho, \rho^2 \right\}$$

Arnold diffusion



Effective coupling $2V_{m_1 \dots m_N} \cos(m_1 \phi_1 + \dots + m_N \phi_N)$

action conservation: $m_1 + \dots + m_N = 0$

Change in actions due to the stochastic force after time t

$$\langle \delta I_n \delta I_{n'} \rangle \sim t V_{m_1 \dots m_N}^2 \frac{m_n m_{n'}}{\Omega} \exp\left(-\frac{|m_1 \omega_1 + \dots + m_N \omega_N|}{\Omega}\right)$$

probability = $f(\{I_n\}) W_s g |\vec{m}^g|^2 \delta(m_1^g (\omega_{n_1^g} + g I_{n_1^g}) + \dots + m_N^g (\omega_{n_N^g} + g I_{n_N^g})) \prod_n dI_n$

distribution function

stochastic layer area

guiding resonance

$$W_s \frac{\partial f}{\partial t} = \sum_{n, n'} \frac{\partial}{\partial I_n} W_s D_{nn'} \frac{\partial f}{\partial I_{n'}}$$

Constraints on the diffusion:

1. Total action is conserved
2. Total energy is conserved
3. The system stays on the guiding resonance

Conclusions

1. Strong Anderson localization + weak nonlinearity → weak chaos
2. Rare chaotic spots play the role of a bath
3. They produce transport by driving the Arnold diffusion