The Abdus Salam International Centre for Theoretical Physics

2449-4

38th Conference of the Middle European Cooperation in Statistical Physics - MECO38

25 - 27 March 2013

Statistical Properties of Entanglement in Large Quantum Systems

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STATISTICAL PROPERTIES OF ENTANGLEMENT IN LARGE QUANTUM SYSTEMS

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Joint work with

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Camerino, Pisa, Roma, ICTP, Waseda Tokyo

MECO 38 - ICTP Trieste, 26 March 2013

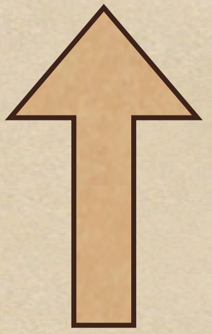
Outline

- Entanglement and random matrices
- Multipartite entanglement and frustration

Entanglement is not ONE but rather THE characteristic trait of quantum mechanics

Schroedinger

focus on: qubit = spin 1/2

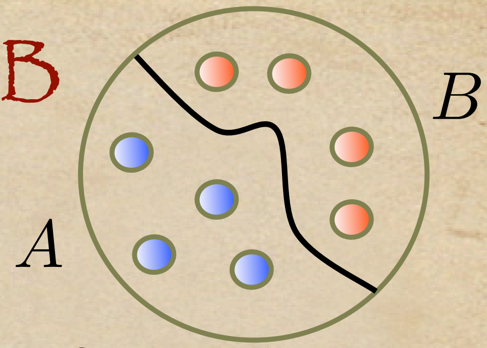


or

spin or two-level quantum system (dim = 2)

n spins $\Longrightarrow$ dim = 2^n = N (large)

Bipartite quantum system A + B



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = N \quad \dim \mathcal{H} = N^2$$

Pure state

$$|\psi\rangle \in \mathcal{H}$$

Reduced density matrix of A

$$\rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$$

Purity of subsystem A

$$\pi_A = \mathrm{tr}\rho_A^2$$

$$\pi_A = \pi_B \in [1/N, 1]$$

A measure of the amount of bipartite entanglement between A and B

State of the global system

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i}\, \underbrace{|u_i\rangle}_{A} \otimes \underbrace{|v_i\rangle}_{B}$$

Schmidt decomposition

$$\rho_A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

Partial density matrix

$$\pi_A = \sum_{i=1}^{N} \lambda_i^2$$

Purity

$\pi_A = 1$ Party A in a factorized state

$$\rho_A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Global system in a product state:
no entanglement

$$|\psi\rangle = |u_A\rangle \otimes |v_B\rangle$$

$\pi_A = 1/N$ Party A in a totally mixed state

$$\rho_A = \frac{1}{N} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Global system in a maximally bipartite entangled state

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |u_i\rangle \otimes |v_i\rangle$$

summarizing:

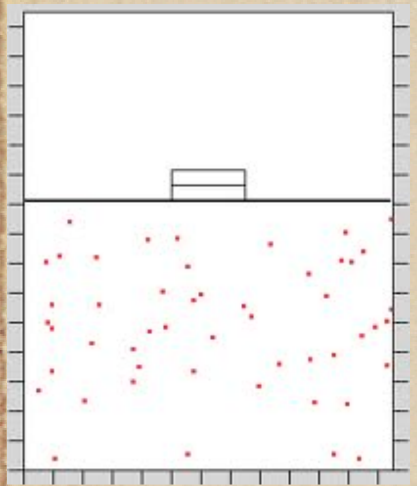
$$1/N \leq \pi \leq 1$$

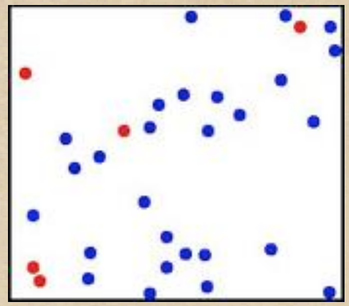
maximally entangled

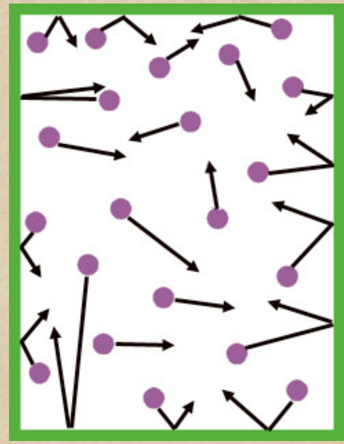
factorized

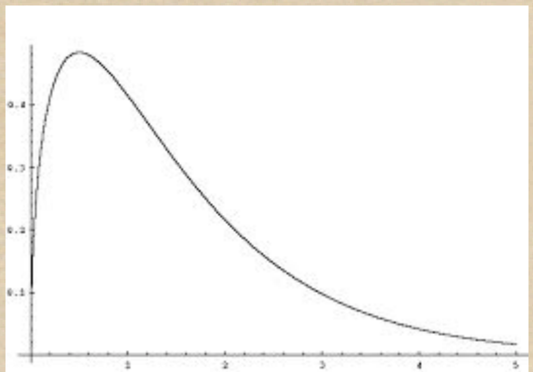
what happens in between?

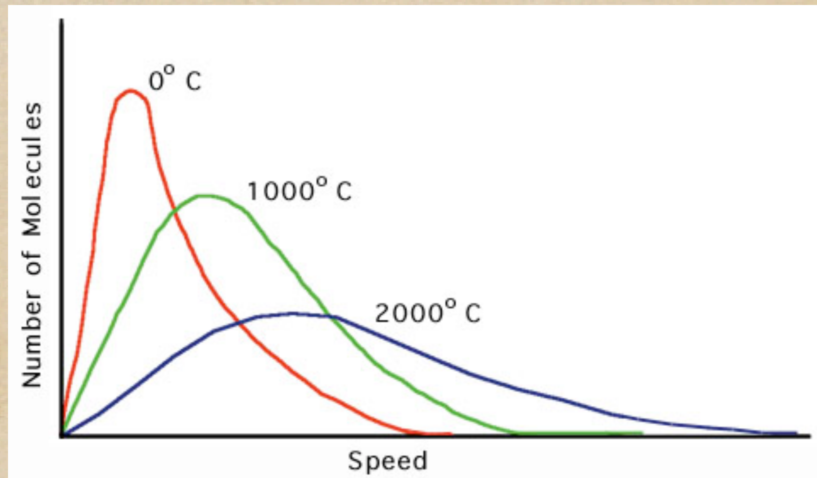
ideal gas









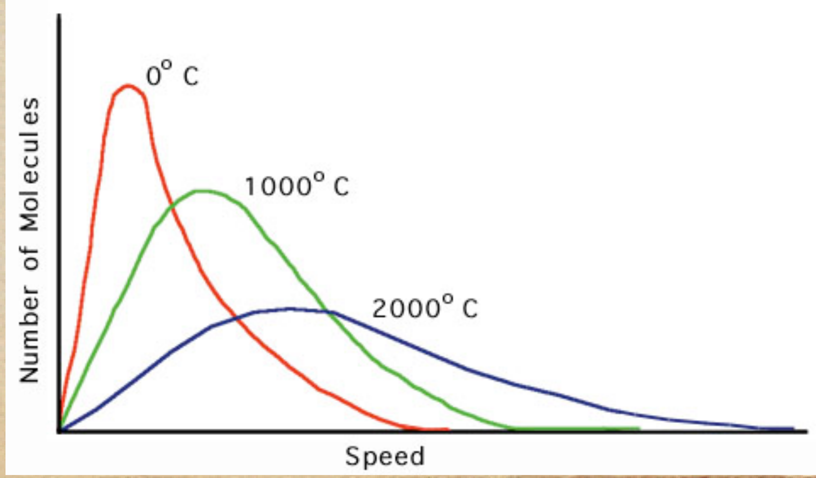


distribution ... as a function of Temp

can we ask same question for entanglement?

strategy

- fix purity (i.e. entanglement)
- pick a generic state of given purity
- find distribution of eigenvalues of reduced density matrix for different purities (i.e. entanglement)



Random states (all states equally probable)

What is a random state $|\psi\rangle$?

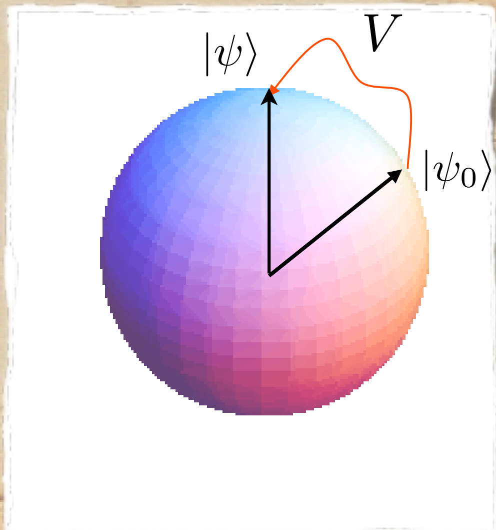
Uniform measure on unit sphere

States distributed according to the Haar measure of the unitary group $\mathcal{U}(\mathcal{H})$

$$|\psi\rangle = V|\psi_0\rangle \qquad V \in \mathcal{U}(\mathcal{H}) = \mathcal{U}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

Induced measure on the reduced density matrices of A

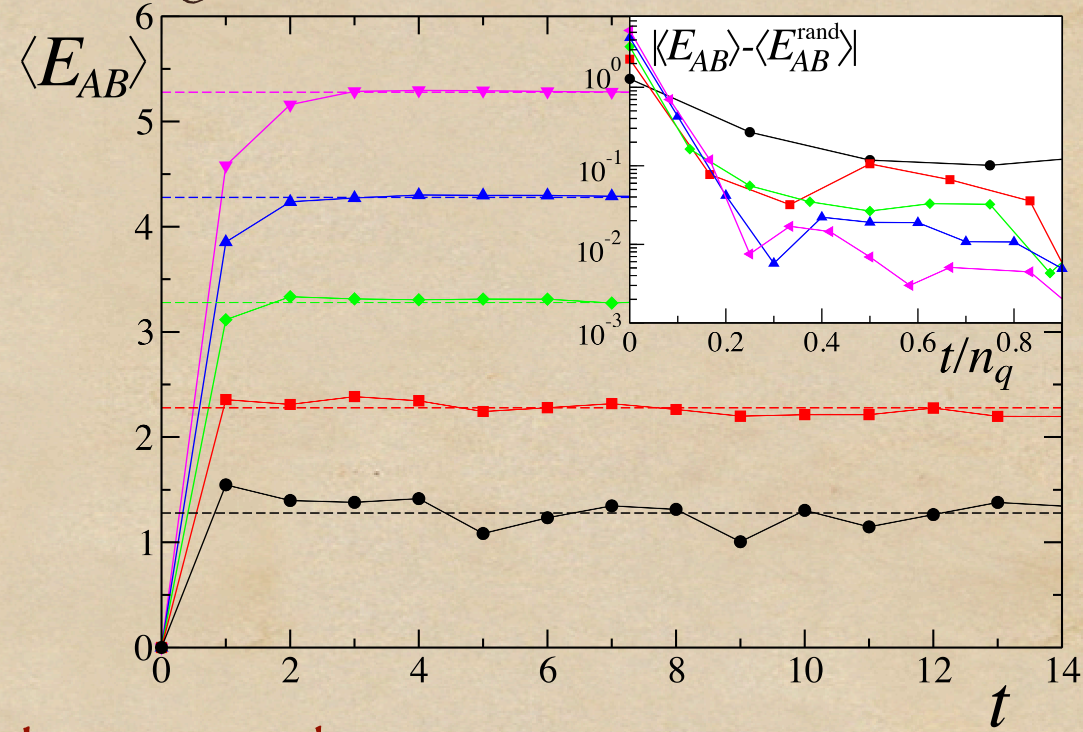
$$d\mu(\rho_A) \qquad \rho_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$$



Intermezzo: generation of typical states

Typical states can be efficiently generated by (generic) chaotic dynamics

Quantum sawtooth map



Scott and Caves, JPhA 36, 9553 (2003)

Mejia-Monasterio, Benenti, Carlo, Casati, PRA 71, 062324 (2005)

Rossini, Benenti, PRL 100, 060501 (2008)

Typical entanglement

How entangled is a typical state?

E Lubkin, JMP 19 1028 (1978)

$$\langle \pi_A \rangle = \langle \mathrm{tr} \rho_A^2 \rangle = 2/N$$

This is a lot! Close to the minimum, 1/N

$$\pi_A \in [1/N, 1]$$

> With Lucretius,[1] I find disorder in an initial state of the universe repugnant. Furthermore, a pure state remains pure. Entropy is somehow to be developed without fundamental entropy.

Variance?

$$\langle\langle \pi_A^2 \rangle\rangle = \langle \pi_A^2 \rangle - \langle \pi_A \rangle^2 = \frac{2}{N^4}$$

can be calculated directly.

D N Page, PRL 71 1291 (1993)

Sensitive calculation: both terms of order $1/N^2$

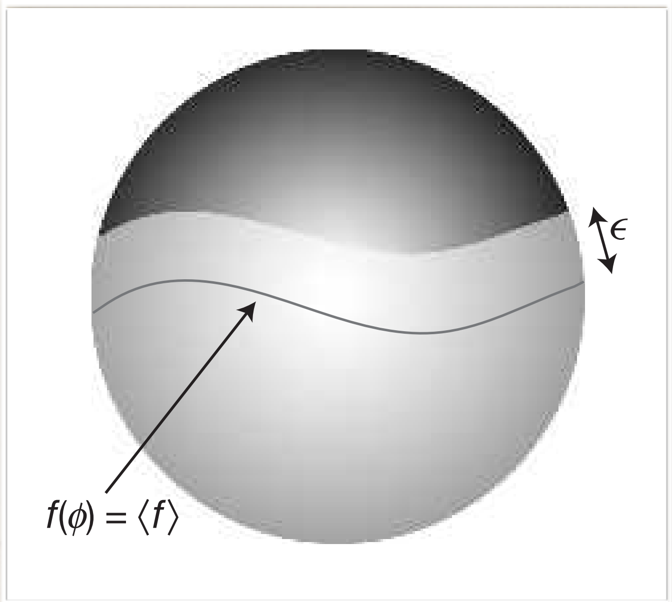
Higher moments

more and more complicated

O Giraud, JPA 40 2793 (2007)

An interesting mathematical phenomenon: concentration of measure

- The uniform measure on the N-dimensional sphere concentrates very strongly about any equator as N gets large (any polar cap has volume exponentially small in N).
- Levy's Lemma
  Any slowly varying function on the sphere takes values close to the average except for an exponentially small set.



see e.g.
Ledoux-Talagrand
Springer-Verlag book

Concentration of measure

Familiar example: central limit theorem

$$S_N = X_1 + \cdots + X_n$$

$$X_k = O(1)$$ independent random variables

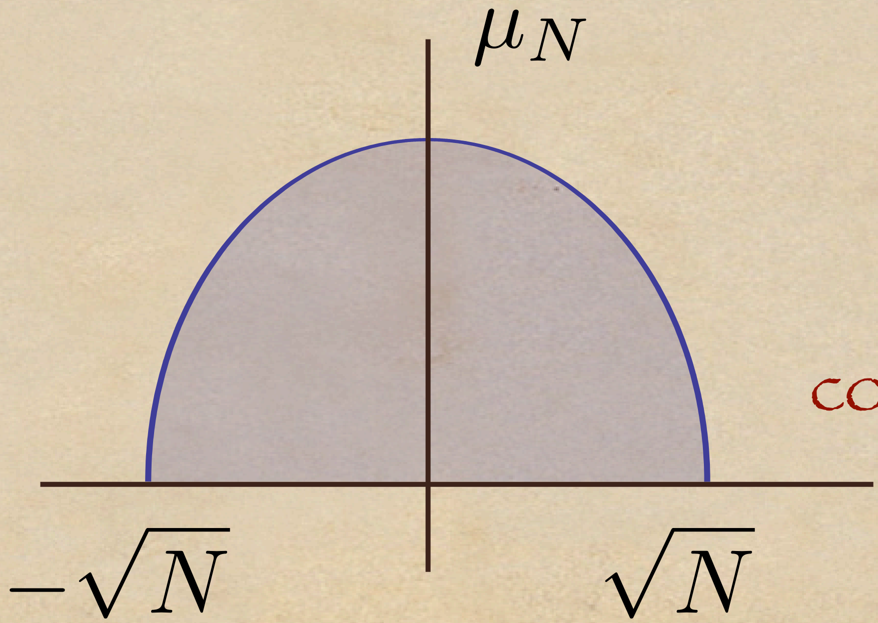
$$S_N = O(\sqrt{N})$$ sharp concentration

Concentration of measure

Noncommutative version:

Wigner's semicircle law

Spectral distribution of random matrices



sharp concentration

incidentally...

concentration of measure has been (beautifully) applied in the foundations of statistical mechanics:

entanglement yields typicality

(e.g. Boltzmann)

Popescu, Short, Winter arXiv:quant-ph/0511225

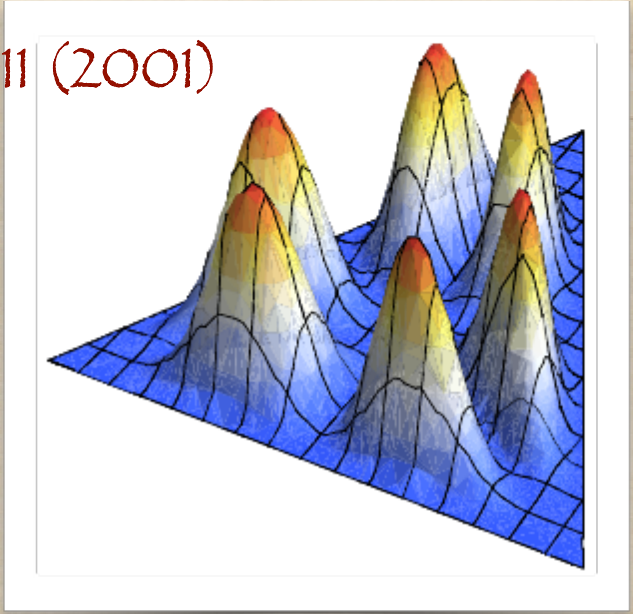
Joint distribution of Eigenvalues

$$\rho_A = U \Lambda U^\dagger \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$$

$$d\mu(\rho_A) = d\mu_H(U) \times d\nu(\Lambda)$$

$$d\nu(\Lambda) = p_N(\Lambda) d\lambda_1 \ldots d\lambda_N$$

Zyczkowski, Sommers, J Phys A 34, 7111 (2001)

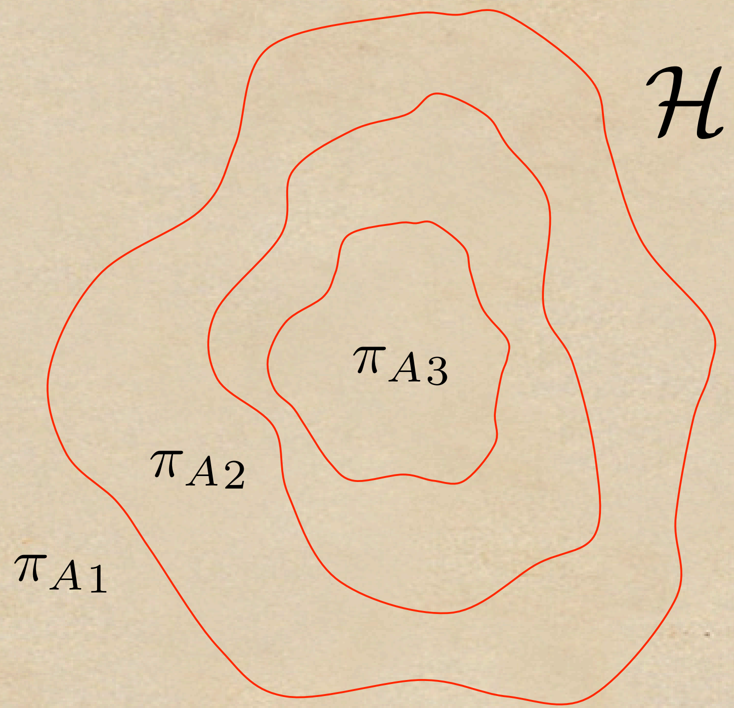


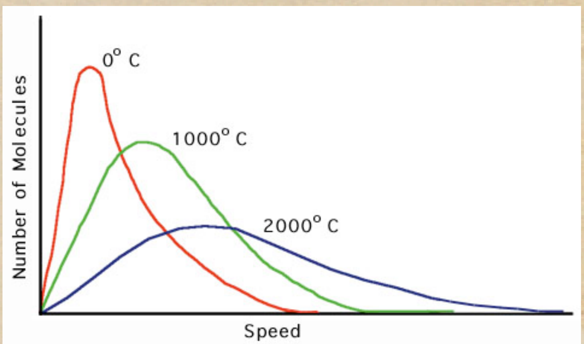
$$\boxed{p_N(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j)^2}$$

Eigenvalue repulsion!

Lloyd, Pagels, Ann Phys 188, 186 (1988)

Isoentangled (isopurity) manifolds



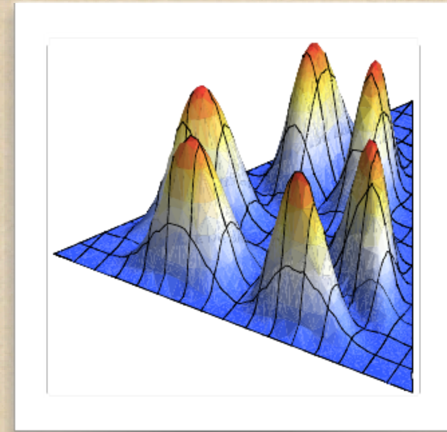


What is the typical distribution of the eigenvalues of ρ_A on each manifold?

Sinolecka, Zyckowski, Kus, Acta Phys Pol B 33, 2081 (2002)

Typical entanglement spectrum

$$p_N(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j)^2$$



Find the most probable spectrum $\Lambda^* = (\lambda_1^*, \ldots, \lambda_N^*)$ on a given isopurity manifold:

$$p_N(\Lambda^*) = \max_{\Lambda} p_N(\Lambda)$$

with $\mathrm{tr}\Lambda^2 = \pi_A$

Constrained maximization problem

Typical entanglement spectrum

$$\max p_N(\Lambda) = \min V(\Lambda, \xi, \beta)$$

Unconstrained minimization

$\beta,\ \xi$ Lagrange multipliers (constraint)

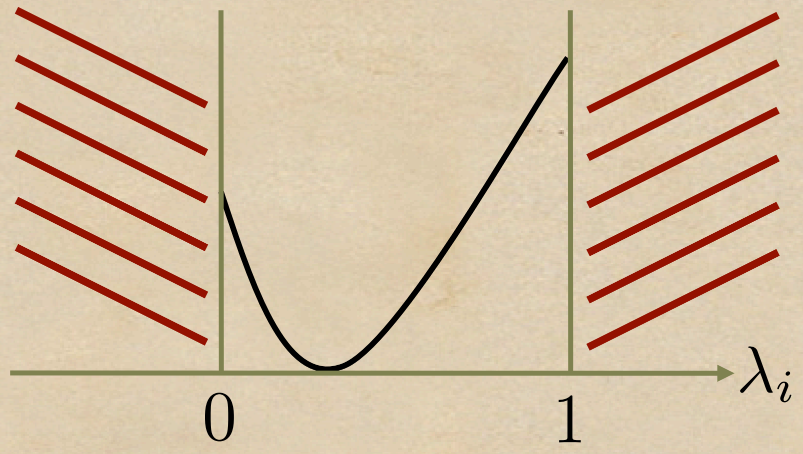
$$p_N(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j)^2$$

$$V(\Lambda, \xi, \beta) = -\frac{2}{N^2} \sum_{j<k} \ln |\lambda_j - \lambda_k| + \beta N \left(\sum_k \lambda_k^2 - \pi_A \right) + \xi \left(\sum_k \lambda_k - 1 \right)$$

Gas of eigenvalues on the interval

$$V(\Lambda) = \underbrace{\xi \sum_i \lambda_i + \beta N \sum_i \lambda_i^2}_{\text{harmonic}} \underbrace{- \frac{2}{N^2} \sum_{i<j} \ln |\lambda_i - \lambda_j|}_{\text{repulsive 2D Coulomb}}$$

Hard wall
from positivity



Hard wall
from unit trace

Dyson, J Math Phys 3, 157 (1962)

Saddle point equations

$$\frac{\partial V}{\partial \lambda_i} = \frac{\partial V}{\partial \xi} = \frac{\partial V}{\partial \beta} = 0$$

$$\rho(\lambda) = \frac{1}{N} \sum_k \delta(\lambda - N\lambda_k) \qquad \int \rho(\lambda)\, \mathrm{d}\lambda = 1$$

$$-\beta\lambda + P \int_0^\infty d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} - \frac{\xi}{2} = 0$$

$$\int_0^\infty \lambda \rho(\lambda)\, d\lambda = 1$$ Trace condition

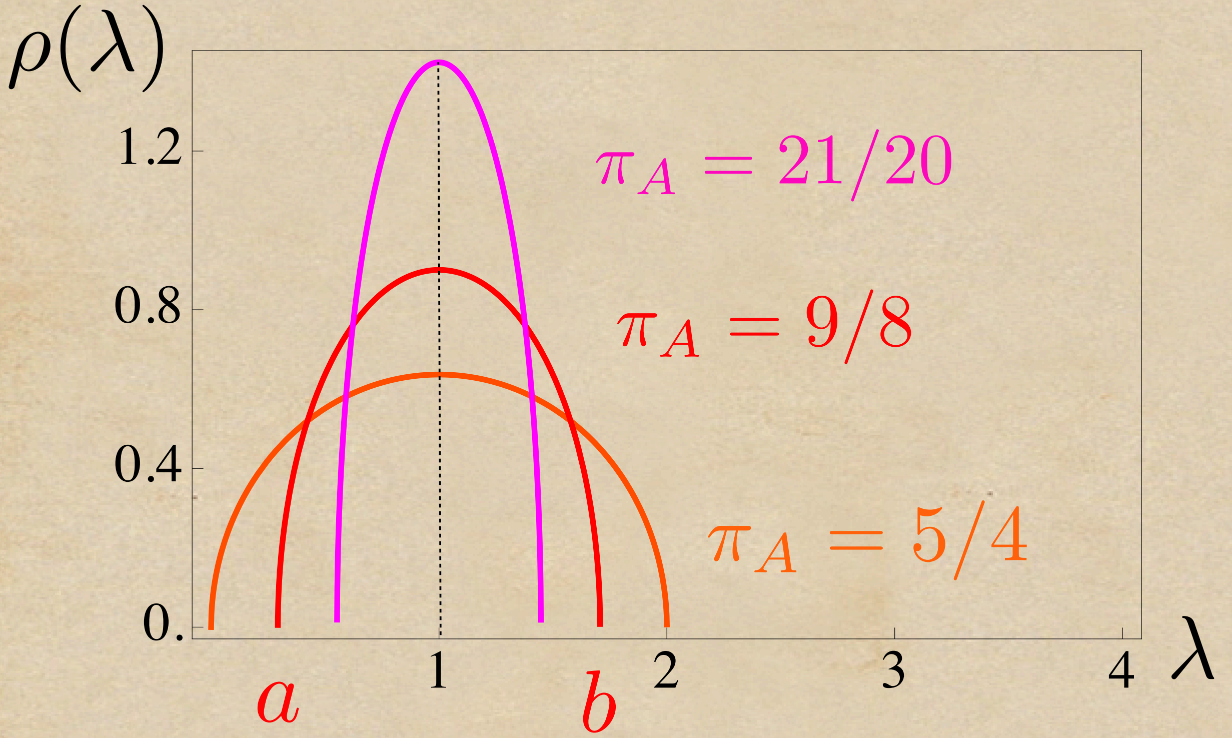
$$\int_0^\infty \lambda^2 \rho(\lambda)\, d\lambda = u$$ Fixed purity

Tricomi's equation

$$-\beta\lambda + P\int_0^\infty d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} - \frac{\xi}{2} = 0$$

solutions in compact support

$$1 < \pi_A < \frac{5}{4} \qquad \rho(\lambda) = \frac{\beta}{\pi}\sqrt{(\lambda - a)(b - \lambda)}$$



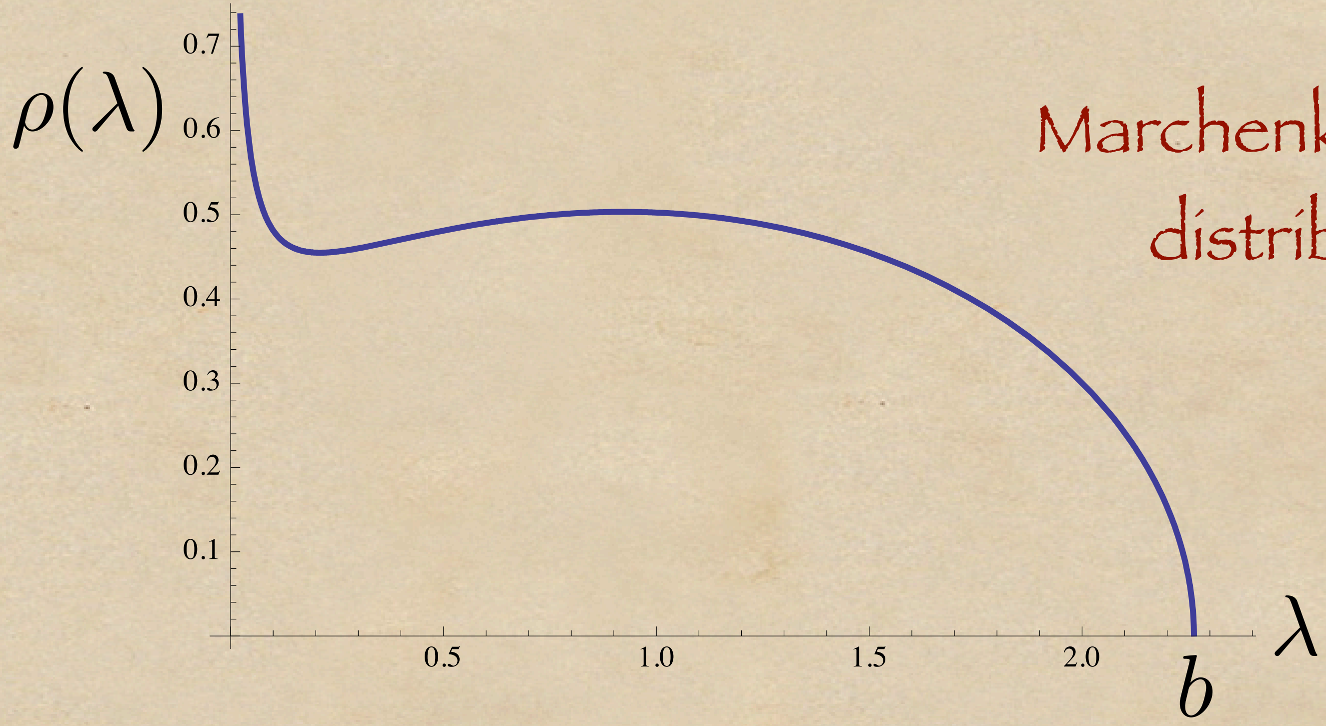
Wigner semicircle law

$(\pi_A \to N\pi_A)$

$$a = 1 - 2\sqrt{\pi_A - 1} \qquad b = 1 + 2\sqrt{\pi_A - 1}$$

$$\frac{5}{4} < \pi_A < 2$$

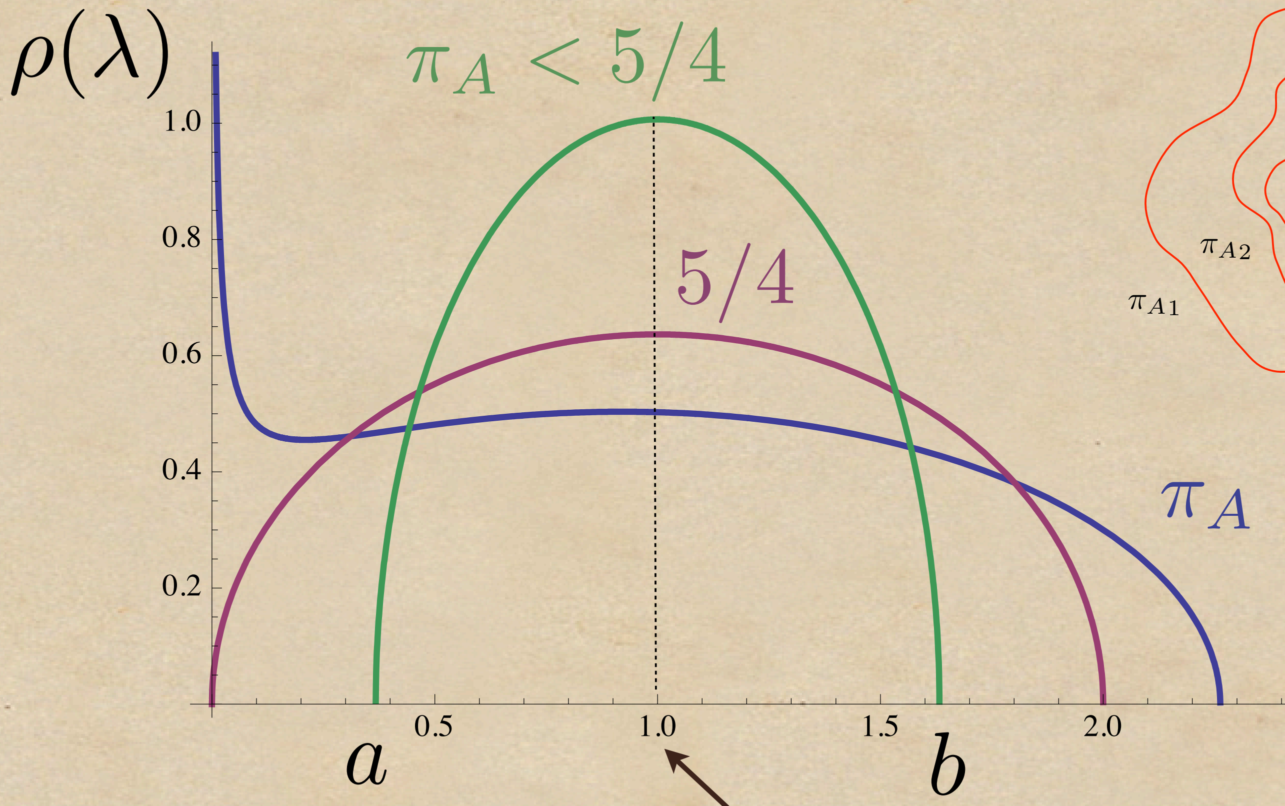
$$\rho(\lambda) = \frac{4}{\pi b^2}\sqrt{\frac{b-\lambda}{\lambda}}\left(b - 2 + \frac{2(4-b)}{b}\lambda\right)$$

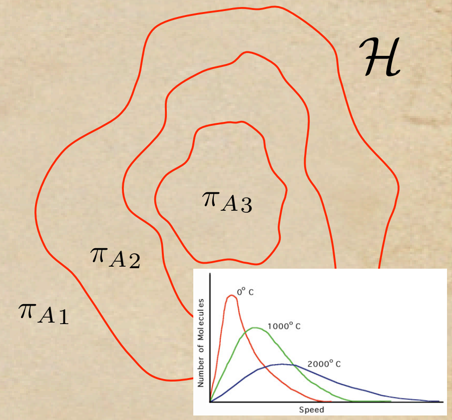


Marchenko-Pastur distribution

$$b = 6\left(1 - \sqrt{1 - 4\pi_A/9}\right)$$

summarizing: entanglement spectra (low purity)





max entangled states

$$\rho_A = \begin{pmatrix} 1/N & 0 & \dots & 0 \\ 0 & 1/N & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1/N \end{pmatrix}$$

rescaled (!) $\pi_A \to 1$

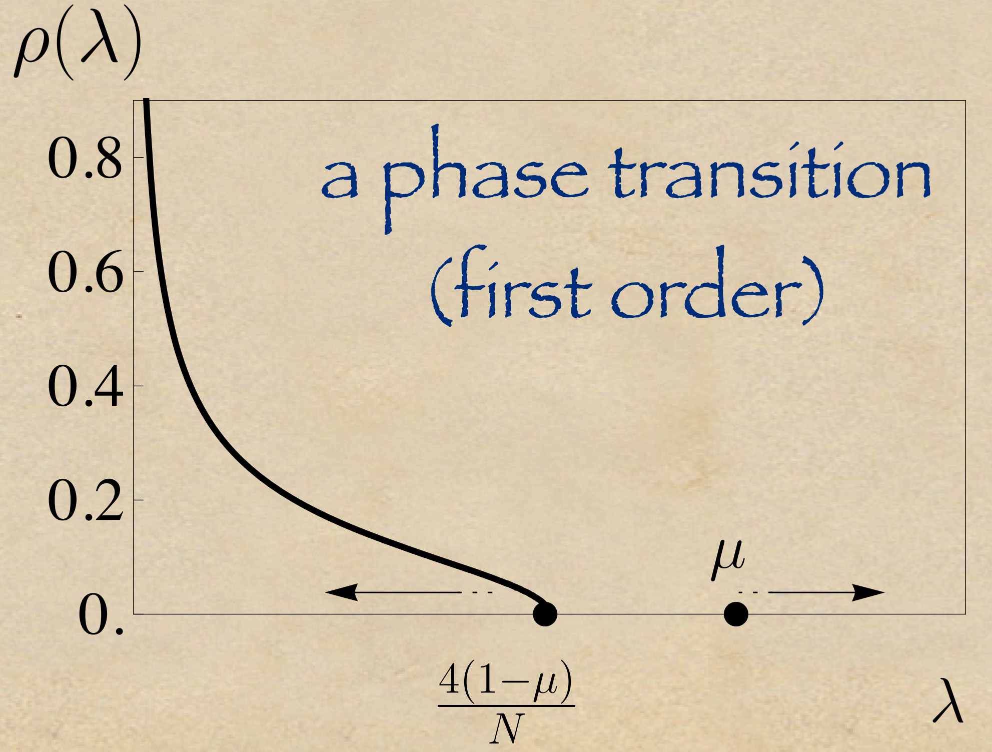
phase transition at $\pi_A = 5/4$

larger purity: evaporation of the largest eigenvalue

towards factorized state

$$\pi_A > 2/N$$

$$\rho_A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$



$$\pi_A = \mu^2 + 2/N$$

...so far purity

- but results can be extended to Renyi entropies of any order
- including "evaporation" of leading eigenvalue
- Nadal, Majumdar, Vergassola, PRL 104, 110501 (2010), J. Stat. Phys. 142 , 403 (2011)
- finally.............

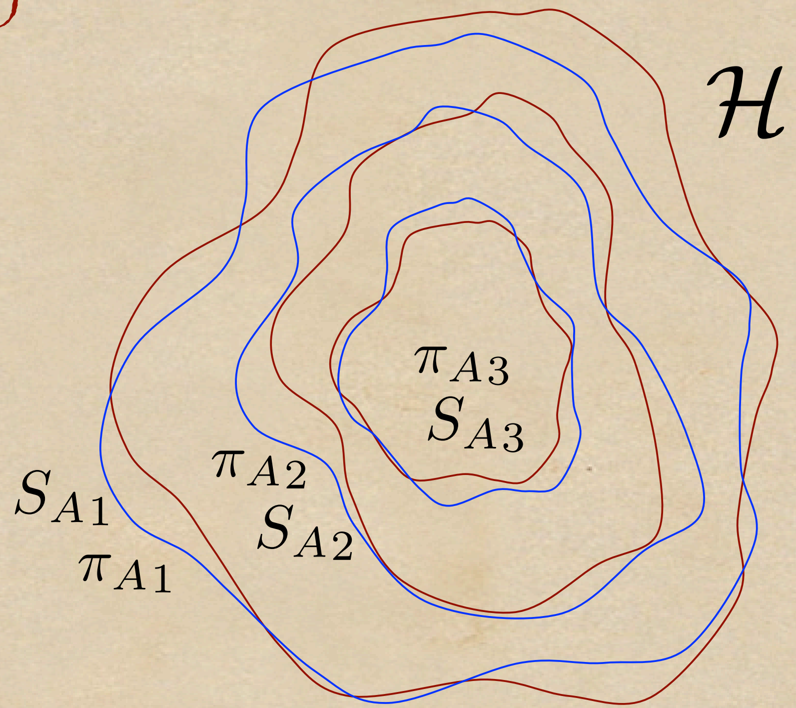
Isoentropic manifolds

$$\mathrm{tr}\rho_A^2 = \pi_A \longrightarrow -\mathrm{tr}\rho_A \ln \rho_A = S_A$$

purity

von Neumann

entropy



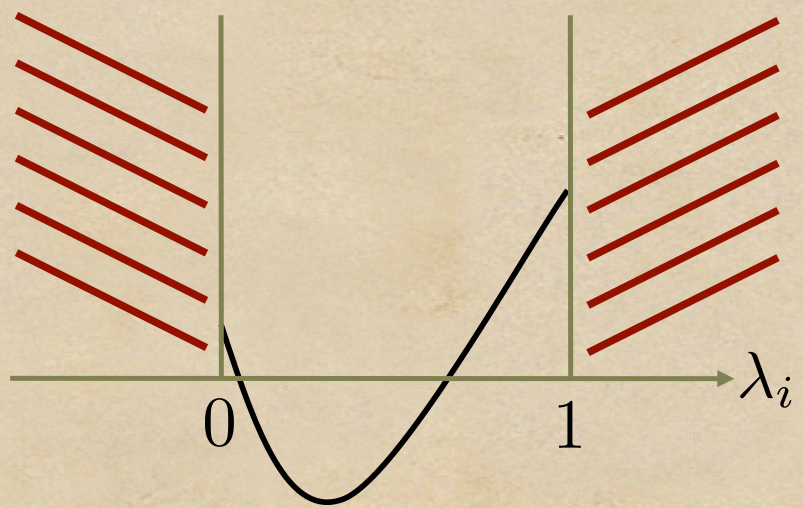
What is the typical distribution

of the eigenvalues of ρ_A on each manifold?

once again: Gas of eigenvalues on the interval

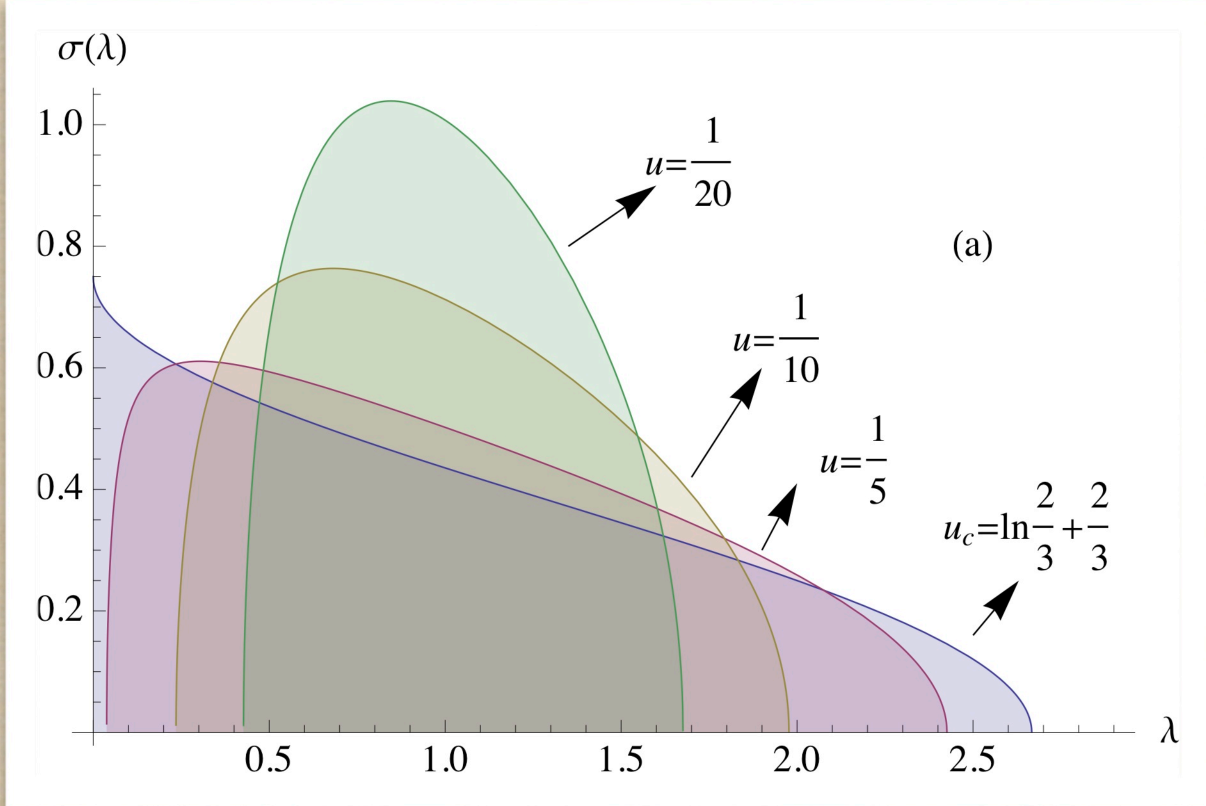
$$V(\Lambda) = \underbrace{\xi \sum_i \lambda_i + \beta N \sum_i \lambda_i \ln \lambda_i}_{\text{external potential}} - \underbrace{\frac{2}{N^2} \sum_{i<j} \ln |\lambda_i - \lambda_j|}_{\text{repulsive 2D Coulomb}}$$

Hard wall
from positivity



Hard wall
from unit trace

Entanglement spectra

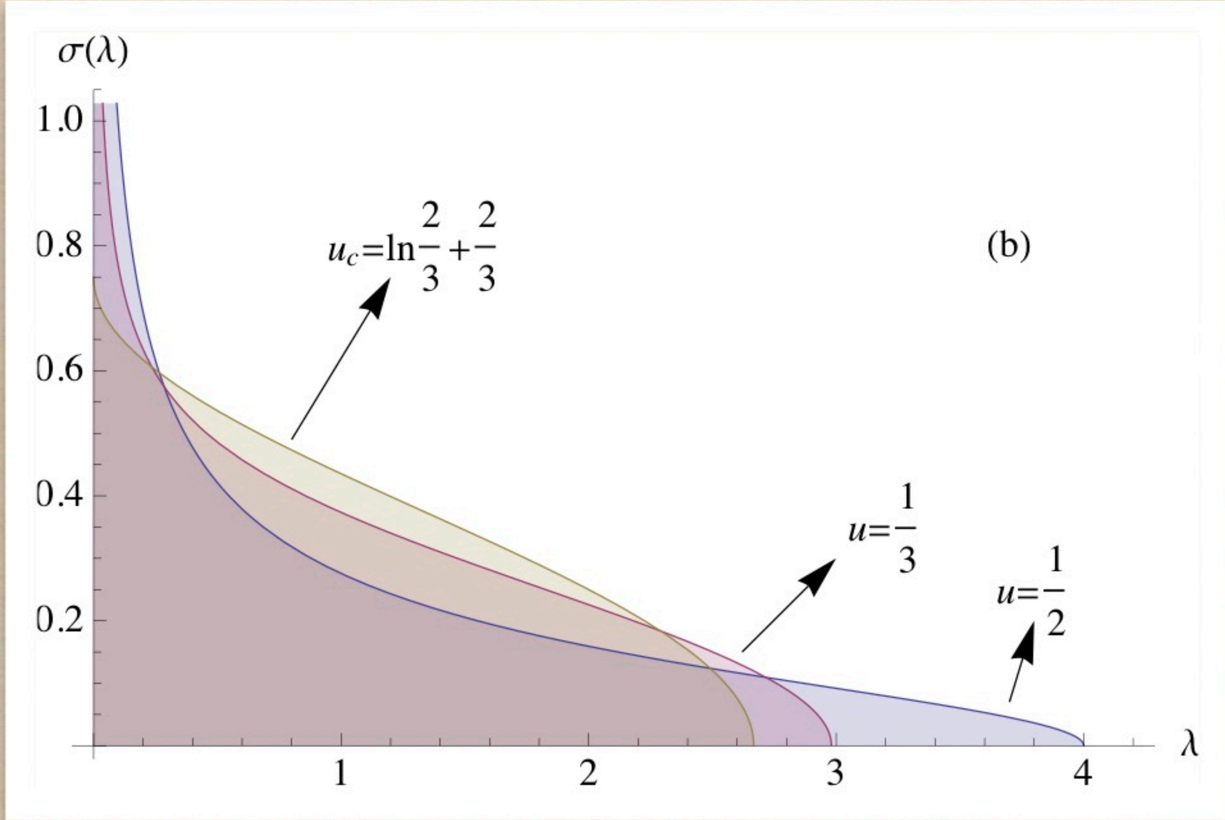


Deformed semicircle law

$$S_A = \ln N - u \qquad\qquad u < u_c \simeq 0.26$$

Facchi, Florio, Parisi, Pascazio, Yuasa, arXiv 1302.3383 (2013)

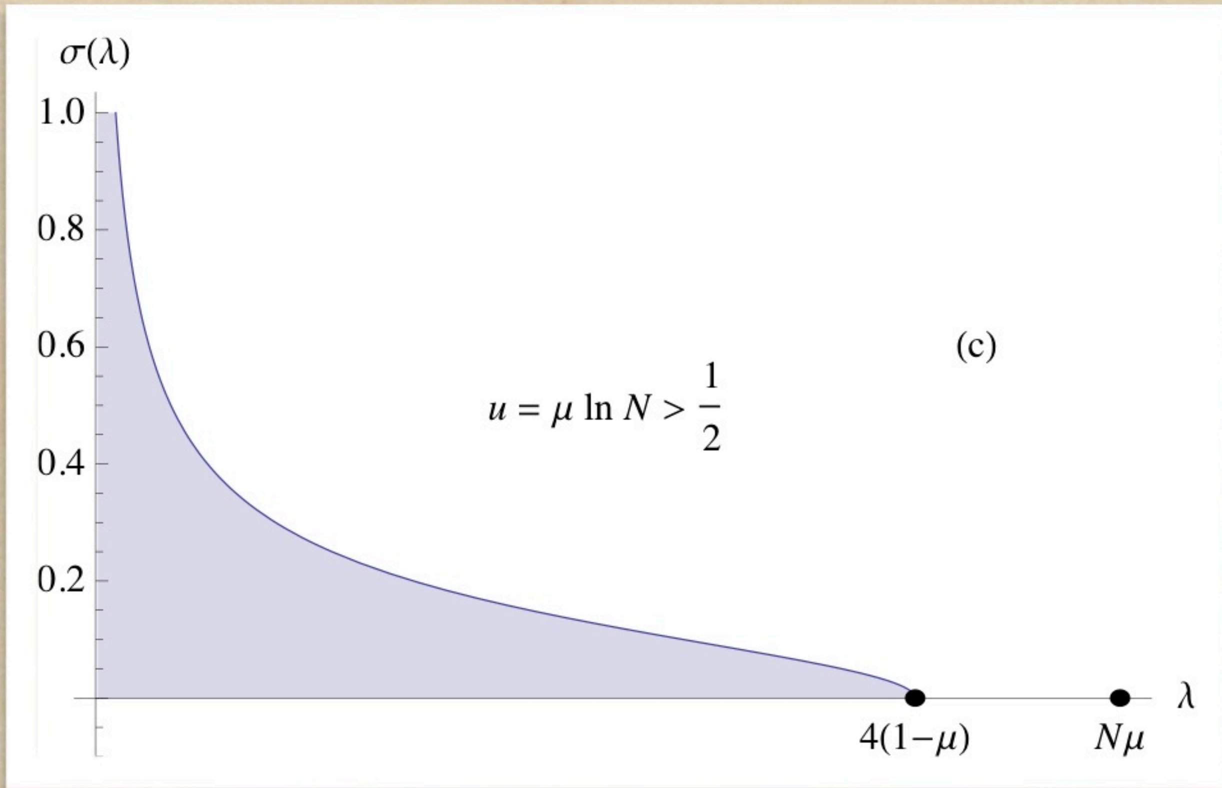
Entanglement spectra



Deformed M-P law

$$S_A = \ln N - u \qquad u > u_c \simeq 0.26$$

Entanglement spectra



Eigenvalue evaporation

$$S_A = \ln N - u \qquad\qquad u > 1/2$$

From Bipartite to Multipartite Entanglement

The purity π_A completely defines the BIPARTITE ENTANGLEMENT.
It depends on the bipartition.

What about MULTIPARTITE ENTANGLEMENT?
The numbers needed to characterize the system scale exponentially with its size.

Statistical methods

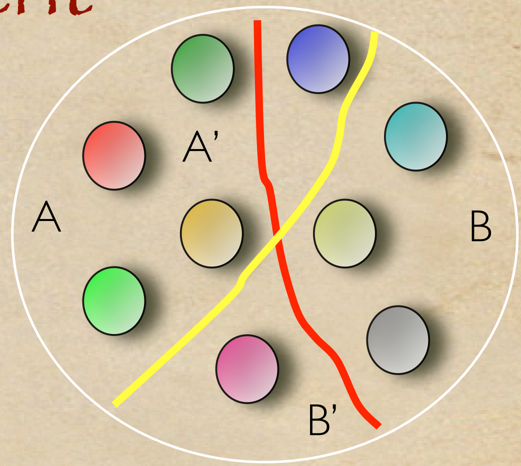
Seminal ideas from
Man'ko, Marmo, Sudarshan, Zaccaria JPA 2002
Parisi: complex systems

- The distribution of π_A characterizes the entanglement of the system.
- The average will be a measure of the amount of entanglement in the system, while the variance will measure how well such entanglement is distributed

Facchi, Florio, Pascazio, PRA 2006

Multipartite entanglement

Define the Potential of Multipartite Entanglement
as the average purity over balanced bipartitions



$$\pi_{\mathrm{ME}}^{(n)}(|\psi\rangle) = \mathbb{E}[\pi_A] = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \pi_A$$

$$\pi_A = \mathrm{Tr}(\rho_A^2)$$

Minimizers of the potential of multipartite entanglement:

Maximally Multipartite Entangled State (MMES)

$$\frac{1}{2^{n_A}} \leq \pi_{\mathrm{ME}}^{(n)}(|\psi\rangle) \leq 1$$

If the lower bound is saturated, the MMES is "perfect"

Facchi, Florio, Parisi, Pascazio PRA 2008; JPA 2009, JPA 2010

frustration

- Frustration in humans and animals arises from unfulfilled needs.
- Freud related frustration to goal attainment and identified inhibiting conditions that hinder the realization of a given objective.
- In the psychological literature one can find many diverse definitions, but roughly speaking, a situation is defined as frustrating when a physical, social, conceptual or environmental obstacle prevents the satisfaction of a desire.

frustration

- Interestingly, definitions of frustration have appeared even in the jurisdictional literature and appear to be related to an increased incidence of parties seeking to be excused from performance of their contractual obligations.
- There, ``Frustration occurs whenever the law recognises that without default of either party a contractual obligation has become incapable of being performed because the circumstances in which performance is called for would render it a thing radically different from that which was undertaken by the contract.... It was not this I promised to do. (Defined by Lord Radcliffe 1956 in Davis Contractors Ltd v Fareham Urban District Council [1956] AC 696 at 729 and adopted by the High Court of Australia in Codelfa Construction Pty. Ltd. v. State Rail Authority of NSW (1982) 149 CLR 337 at [1956] A.C. p729)

frustration

- In physics, this concept must be mathematized
- e.g., Mezard, Parisi and Virasoro 1987 Spin Glass Theory and Beyond (World Scientific, Singapore)

paradigmatic example

Three characters A, B anc C are not good friends and do not want to share a room.

Only two rooms available.

At least two of them will have to share a room ⟹ unfulfilled needs

⟹ frustration arises

$$H = S_A S_B + S_A S_C + S_B S_C \qquad (S_i = \pm 1)$$

each term = + or -

but their sum is larger than -3

(minimum "cost" being -1) ⟹ frustration

cost function

one or the other room

in general: cost function

$$H = -\frac{1}{2}\sum_{i \neq j} J_{ik} S_i S_k$$

Frustration

Is it possible in general to have $\pi_A = 1/2^{n_A}$? for all bipartitions and saturate the lower bound?

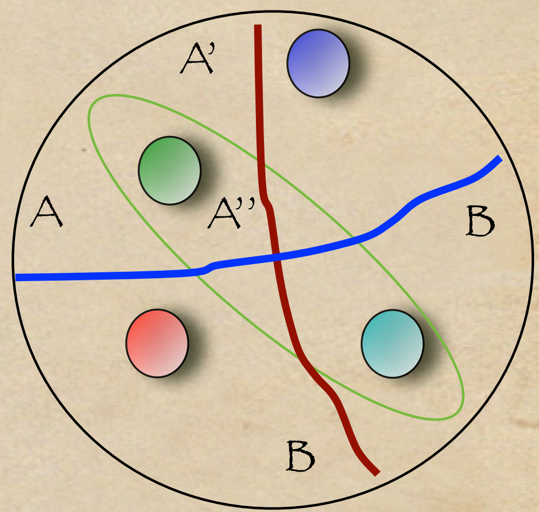
if NO $\rightarrow$ FRUSTRATION

$$n = 4$$

$$min\{\pi_A\} = \frac{1}{4}$$

$$min\{\frac{1}{2}(\pi_A + \pi_{A'})\} = \frac{1}{4}$$

$$min\{\pi^{(4)}_{ME}\} = \frac{1}{3}(\pi_A + \pi_{A'} + \pi_{A''}) = \frac{1}{3} \neq \frac{1}{4}$$



Summary for qubits

n	qubit perfect MMES
2,3	yes
4	no
5,6	yes
7	no (numerical evidence)
≥ 8	no

Frustration

(Scott 2004)

Partition function

Hamiltonian $H(z) = \pi_{\mathrm{ME}}(|\psi\rangle)$ $|\psi\rangle = \sum z_k |k\rangle$

Partition function $Z(\beta) = \int d\mu_C(z) e^{-\beta H(z)}$

inverse temperature β and μ_C uniform measure

Regimes

$$Z(\beta) = \int d\mu_C(z) e^{-\beta H(z)}$$

$\beta \to +\infty$ — MMES — minimizers of the Hamiltonian

$\beta \to 0$ — Typical states — $\mu = \langle \pi_A \rangle_0 = \dfrac{N_A + N_{\bar{A}}}{N+1} \sim \dfrac{2}{\sqrt{N}}$

$\beta \to -\infty$ — Separable states — maximizers of the Hamiltonian

Cumulants

First cumulant: Average of the Potential of Multipartite Entanglent on typical states

$$\langle H \rangle_0 = \frac{N_A + N_{\bar{A}}}{N+1} \sim \frac{2}{\sqrt{N}}$$

$$N_A = N_{\bar{A}} = \sqrt{N}$$

$$N \to +\infty$$

Second cumulant: Variance

if the bipartitions were indipendent:

$$\bar{\sigma}^2 = \langle H^2 - \langle H \rangle_0{}^2 \rangle_0 \sim \frac{3\sqrt{2}}{N^{4-\log_2 3}} = O(N^{-2.415})$$

$$\bar{\sigma}^2_{\text{ind}} \sim N^{-3}$$

Third cumulant: Skewness

$$\kappa_0^{(3)}[H] = \langle (H - \langle H \rangle_0)^3 \rangle_0 \sim 67.443\, N^{-4.1583}$$