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The Bose-Einstein Condensation on Inhomogeneous Networks

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abstract

We present the new and unexpected results concerning the Bose–Einstein Condensation for the *Pure Hopping Model*, describing the thermodynamic behaviour of Bardeen–Cooper pairs in arrays of Josephson junctions located on non homogeneous networks. The amenable and the non amenable cases will be considered.

introduction

In the early paper: Burioni R., Cassi D., Rasetti M., Sodano P., Vezzani A. J. Phys. B **34** (2001), 4697–4710, it was shown the surprising fact that the critical density describing the condensation of Bardeen–Cooper Bosons for the pure hopping model can be finite also for low dimensional networks. Motivated by such a result, we investigated relevant spectral properties of the adjacency operator of non homogeneous networks. The graphs under investigation are obtained by adding density zero perturbations to periodic amenable networks, and homogeneous Cayley Trees. Apart from the natural mathematical interest, such spectral properties are relevant for the Bose Einstein Condensation for the pure hopping model describing Bardeen–Cooper pairs of Fermions (i.e. BCS Bosons) in arrays of Josephson junctions on non homogeneous networks. The resulting topological model is described by a one particle Hamiltonian which is, up to an additive constant, the opposite of the adjacency operator on the graph. In the condensation regime, the particles condensate on the perturbed graph, even in the configuration space due to non homogeneity. Roughly speaking, the system undergoes a sort of "dimension transition". We show for both amenable and non amenable situations, that it is enough to perturb in a negligible way the original graph in order to obtain a new network whose mathematical and physical properties dramatically change.

The present talk is based on the following papers:

–Fidaleo F., Guido D., Isola T. *Bose-Einstein condensation in inhomogeneous amenable graphs*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14** (2011), 149–197;

–Fidaleo F. *Harmonic analysis on perturbed Cayley trees*, J. Funct. Anal. **261** (2011), 604–634 (Erratum/addendum: J. Funct. Anal. **262** (2012), 4634–4637);

–Fidaleo F. *Harmonic analysis on perturbed Cayley trees II: Bose–Einstein condensation*, Infin. Dimens. Anal. Quantum Probab. Relat. Top., DOI: 10.1142/S0219025712500245.

To end we point out that, in recent experiments (Silvestrini P., Russo R., Corato V., Ruggiero B., Granata C., Rombetto S., Russo M., Cirillo M., Trombettoni A., Sodano P.: Phys. Lett. A **370** (2007), 499–503) it was found that the current is enhanced at low temperatures for non–homogeneous arrays of Josephson junctions in the Comb Graph, see Fig 2 below. Such a phenomenon might be indeed explained via the BEC of Bardeen–Cooper pairs.

the model

The framework is a sea of *Bardeen–Cooper pairs* in arrays of *Josephson junctions* on a network G: particles are located on vertices VG, and edges EG describe the presence of a Josephson junction. The distribution of the particles on the superconducting islands (vertices of the network) is governed by the Bose–Gibbs grand–canonical distribution.

The Hamiltonian of the system is the *Bose Hubbard Hamiltonian*

$$H_{BH} = m \sum_i n_i + \sum_{i,j} A_{i,j} \Big(V n_i n_j - J_0 a_i^\dagger a_j \Big) \,. \quad (1)$$

Here, $a_i^\dagger$ is the Bosonic creator, and $n_i = a_i^\dagger a_i$ the number operator on the site i. Finally, A is the adjacency operator whose matrix element $A_{i,j}$ in the place ij is the number of the edges connecting the site i with the site j. When m and V are negligible with respect to J_0, it might be expected that the hopping term dominates the physics of the system, at least for low temperatures. Thus, under this approximation, (1) becomes the quadratic *pure hopping* Hamiltonian given by

$$H_{PH} = -J \sum_{i,j} A_{i,j} a_i^\dagger a_j \,, \quad (2)$$

where the constant $J > 0$ is a mean field coupling constant which is in general different from the J_0 appearing in the more realistic Hamiltonian (1).

mathematical aspects and physical applications

Being the previous Hamiltonian a quasi–free (quadratic) one, it is enough to study the self-adjoint operator $-A$ on the one–particle space $\ell^2(VG)$. We put $J_0 = 1$ in (2), and normalizing such that the bottom of the spectrum of the energy is zero. The resulting Hamiltonian for the purely topological model under consideration is

$$H = \|A\|\mathbf{1} - A\,, \tag{3}$$

where A is the adjacency of the fixed graph G, acting on the Hilbert space $\ell^2(VG)$.

The appearance of the BEC is connected with the asymptotics near zero, of the spectrum of the Hamiltonian. For vectors in the spectral subspace near zero (i.e. for small values of energies, and for $\mu \approx 0$), the Taylor expansion for the "Bose occupation function" relative to the *chemical potential* $\mu < 0$ leads to

$$\frac{1}{e^{\beta(H-\mu\mathbf{I})}-1} \approx [\beta(H-\mu\mathbf{I})]^{-1}$$
$$=\frac{1}{\beta}((\|A\|-\mu)\mathbf{I}-A)^{-1} \equiv \frac{1}{\beta}R_A(\|A\|-\mu)\,.$$

Then the mathematics of the BEC is reduced to the investigation of the spectral properties of the (more familiar object for mathematicians which is the) resolvent $R_A(\lambda)$, for $\lambda \approx \|A\|$.

The non homogeneous graphs we dealt with at the beginning were *density zero additive perturbations* of periodic lattices, Fig 1: the *comb graph* $\mathbb{Z} \dashv \mathbb{Z}$, see also Fig 2: the *star graph* (amenable situation). Very surprisingly, essentially the same situation also arises for some interesting non amenable examples such as negligible additive perturbations of homogeneous Cayley trees (Fig 3 and Fig 4).

hidden spectrum. It is the phenomenon that a part of the spectrum (for our purposes close to the bottom of the Hamiltonian, or equivalently to the norm of the Adjacency operator) does not contribute to the density of the eigenvalues (called in physics *integrated density of the states*) in the infinite volume limit. It is the combined effect of two facts: the perturbation is negligible but sufficiently big (in many models is indeed enough a finite additive perturbation) such that the density of eigenvalues does not change apart from a possible shift due to the increment of the norm of the perturbed Adjacency. Put $\delta := \|A^X\| - \|A^Y\| < 0$ (it has the meaning of a chemical potential, see below), we get for the integrated density of the states of $-A^Y$,

$$F_Y(x) = F_X(x+\delta)\,. \tag{4}$$

This always leads to the *hidden spectrum*, that is the part of the spectrum close to the bottom of the Hamiltonian which does not contribute to the density of the states. The immediate consequence is

$$\rho_c^Y(\beta) = \int \frac{\mathrm{d}F(x)}{e^{\beta(x-\delta))}-1} = \rho^X(\beta,\delta) < +\infty\,. \tag{5}$$

Namely, **in presence of the hidden spectrum the critical density of the model is always finite** independently on the geometrical dimension of the network.

transience/recurrence character. The (analytic) definition of *T/R character* of the adjacency is as follows. The Adjacency is said to be recurrent if

$$\lim_{\lambda\downarrow\|A\|}\langle R_A(\lambda)\delta_x,\delta_x\rangle = +\infty$$

for some, and then equivalently for all $x \in G$. Otherwise it is transient.* From a mathematical point of view, this means that $\delta_x \in \mathcal{D}_{R_A(\|A\|)^{1/2}}$, which turns out to be equivalent that for the quadratic form,

$$\langle (e^{\beta(H-\mu\mathbf{1})} - 1)^{-1}\delta_x, \delta_x \rangle < +\infty \,, \quad x \in G \,.$$

The last condition is the necessary and sufficient condition for the existence of locally normal states (i.e. those for which the local density is finite) describing BEC. Thus, **it is possible to have the BEC (for the pure hopping model) if and only if the Adjacency is transient**.

*When the Hamiltonian is (the opposite of) the discrete Laplacian, it is the generator of a random walk on the network and the T/R character has the standard probabilistic meaning.

The algebra, the states and the existence of the dynamics.

The two–point function candidates for locally normal states exhibiting BEC assumes the form

$$\omega_D(a^\dagger(u_1)a(u_2)) := \left\langle (e^{\beta H} - \mathbf{1})^{-1} u_1, u_2 \right\rangle + D\langle u_1, v\rangle\langle v, u_2\rangle\,, \quad u_1, u_2 \in \mathfrak{h}\,. \tag{6}$$

Typically, v is a Perron–Frobenius weight (i.e. non normalizable) and $\mathfrak{h} \subset \ell^2(G)$ is a dense subspace containing $\{\delta_x \mid x \in G\}$ (to have a reasonable local description) which is invariant for the dynamics. It is possible to prove in all the models under consideration, that the sequence made of the unique Perron–Frobenius eigenvectors for the finite volume Adjacency (i.e. the finite volume ground state wave function) all normalised at 1 on a fixed root, converges point wise to a Perron–Frobenius weight which can be considered as the ground state wave function, describing the density of possible condensate.[†] A delicate issue is to prove that (6) is meaningful for the dense subset

$$\mathfrak{h} := \text{span}\left\{ e^{\imath tH}\delta_j \mid t \in \mathbb{R}\,, j \in Y \right\}$$

in the transient case. This means that $\mathfrak{h}$ is invariant for the dynamics, and is in the domain of the bilinear form appearing in (6) and describing the density of the condensate. **In the transient situation, the states with two point function in (6) are meaningful and describe locally normal states exhibiting BEC and fulfilling the Kubo–Martin–Schwinger boundary condition w.r.t. the dynamics on $CCR(\mathfrak{h})$ generated by the pure hopping Hamiltonian.**

the wave function of the ground state. The PF (generalized) eigenvector is nothing but

[†]The Perron–Frobenius weight is in general no longer unique in transient case.

the (generalized) wave function of the physical ground state.[‡]. Then it describes the distribution of the condensate in the configuration space. It is possible to prove as a general fact, that it decays exponentially far away to the perturbation. At first glance, due to inhomogeneity, particles condensate also in configuration space, and the system undergoes a "dimension transition" which can be well explained as follows. The condensate distribution is well described by the Perron–Frobenius dimension d_{PF} (amenable cases). Consider the ball $\Lambda_n \uparrow G$ of radius n centered in any fixed root of the graph. Consider the Perron–Frobenius eigenvector v, previously described. The *geometrical dimension* d_G of G is defined to be a if $|\Lambda_n| \sim n^a$. The *Perron–Frobenius dimension* $d_{PF}(G)$ of G is defined to be b if $\left\| v\lceil_{\ell^2(\Lambda_n)} \right\| \sim n^{b/2}$. Looking at (6), the portion of condensate is described by the last addendum of the

[‡]Here "generalized" stands for non normalizable.

l.h.s.. In the finite region Λ, such a density of the condensate is roughly given by

$$C_D(\Lambda) \approx \frac{1}{|\Lambda|}\sum_{x\in\Lambda} D\langle\delta_x, v\rangle\langle v, \delta_x\rangle = D\frac{\|v\lceil_\Lambda\|^2}{|\Lambda|}\,. \tag{7}$$

If the graph is transient (condition under which it is possible to exhibit locally normal states describing BEC), we look at d_{PF}. If $d_{PF} < d_G$, $C_D(\Lambda) \to 0$ when $\Lambda \uparrow G$. it is then impossible to exhibit locally normal states exhibiting BEC whose mean particle density

$$\rho(\omega) := \limsup_{\Lambda\uparrow G} \rho_\Lambda(\omega)$$

is greater than ρ_c. In this situation **we are able to construct only locally normal states** ω **exhibiting BEC for which** $\rho(\omega) = \rho_c$. To summarize, such a very intriguing new, surprising and fascinating situation is summarized in the following table describing what happens in the amenable situation.

	ρ_c	R/T	d_G	d_{PF}	BEC	ρ–BEC
$\mathbb{Z}^d$, $d < 3$	∞	R	d	d	no	no
$\mathbb{Z}^d$, $d \geq 3$	$< \infty$	T	d	d	yes	yes
star graph	$< \infty$	R	1	0	no	no
$\mathbb{Z}^d \dashv \mathbb{Z}$, $d < 3$	$< \infty$	R	$d + 1$	d	no	no
$\mathbb{Z}^d \dashv \mathbb{Z}$, $d \geq 3$	$< \infty$	T	$d + 1$	d	yes	no
$\mathbb{N}$	∞	T	1	3	yes	no
$\mathbb{N} \dashv \mathbb{Z}$	$< \infty$	T	2	3	yes	yes
$\mathbb{N} \dashv \mathbb{Z}^2$	$< \infty$	T	3	3	yes	yes

Here, $A \dashv B$ is the comb–shaped graph whose base–point is A, ρ_c is the critical density, R/T denotes the transience/recurrence of the adjacency, BEC (ρ–BEC) denotes the existence of locally normal states exhibiting BEC (exhibiting BEC at mean densities $\rho > \rho_c$). The last examples involving $\mathbb{N}$ deserve of further explanation. The network $\mathbb{N}$ admits the BEC (i.e. there exist locally normal states exhibiting BEC) because it is transient. In addition, the combined effect that the critical density is infinite and $d_{PF} > d_G$ tells us that the states with BEC have infinite mean density. The case $\mathbb{N} \dashv \mathbb{Z}$ is also very interesting. In fact, it admits BEC because it is transient. But there is a gap in $(\rho_c, +\infty)$ for the possible mean density of locally normal states because $\rho_c < +\infty$ and $d_{PF} > d_G$: the states without condensate have mean density in the interval $(0, \rho_c]$ whereas the states exhibiting BEC has infinite mean density.

examples

We end with the description of the properties described above for some pivotal example of the graphs under consideration.

finite additive perturbations (see Fig 1):
finite critical density (provided the perturbation is sufficiently big to modify the norm of the adjacency), recurrent (as the PF eigenvector is normalizable), $d_{PF} = 0$.

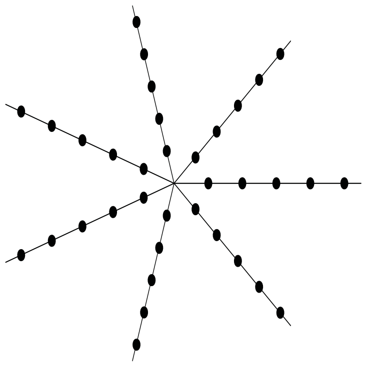

Fig 1: the star graph.

Comb graphs $G^d := \mathbb{Z}^d \dashv \mathbb{Z}$ (see fig):
finite critical density, recurrent if and only if $d \leq 2$, $d = d_{PF} < d_G = d + 1$.

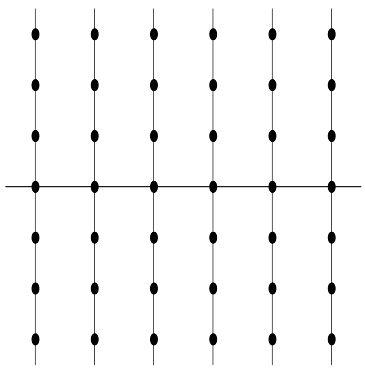

Fig 2: the comb graph $\mathbb{Z} \dashv \mathbb{Z}$.

the graph $\mathbb{N}$:
infinite critical density, transient, $3 = d_{PF} > 1$.

Comb graphs $H^2 := \mathbb{N} \dashv \mathbb{Z}^2$:
finite critical density, transient, $3 = d_{PF} = d_G$.

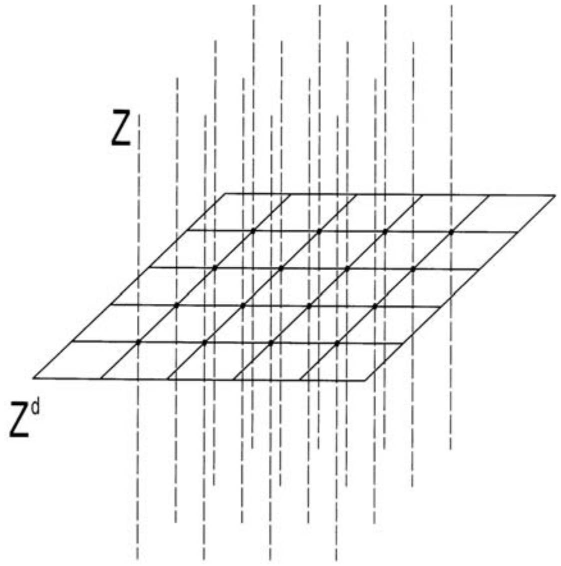

Fig 2 bis: the comb graph $\mathbb{N} \dashv \mathbb{Z}^2$.

The mathematical aspects of the BEC are extended to exponentially growing graphs such as the perturbed Cayley tree.

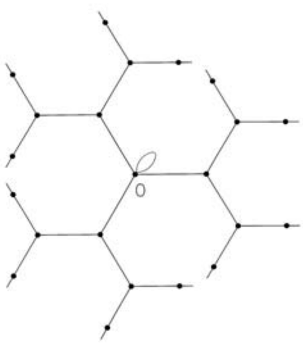

Fig 3: finite perturbation of the Cayley Tree of order 3.

It is recurrent and PF–"0" dimensional.

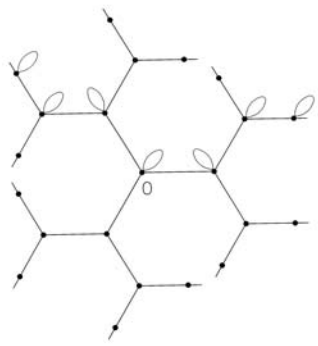

Fig 4: perturbation of the Cayley Tree of order 3 along $\mathbb{Z}$.

It is recurrent and PF–"1" dimensional.

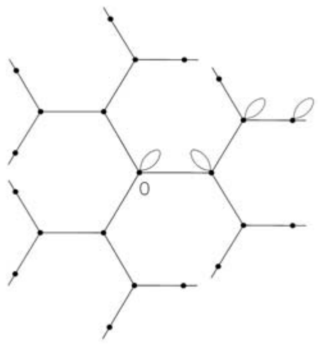

Fig 4: perturbation of the Cayley Tree of order 3 along $\mathbb{N}$.

It is transient and PF–"3" dimensional.

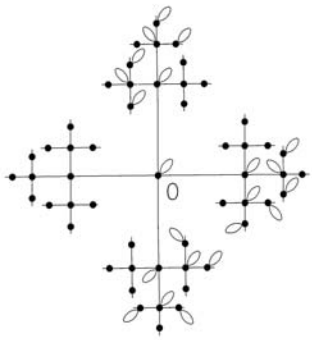

Fig 5: perturbation of the Cayley Tree of order 4 along a Cayley subtree of order 3.

It is transient and has the same PF–behavior as the basepoint.