The Abdus Salam
International Centre
for Theoretical Physics

2449-7

38th Conference of the Middle European Cooperation in Statistical Physics - MECO38

25 - 27 March 2013

Interdisciplinary Applications of RMT

Zdzislaw BURDA

Institute of Physics, Jagiellonian University
Krakow
Poland

Strada Costiera, 11 - 34151 - Trieste - Italy • Tel. +39 0402240111 • Fax. +39 040224163 • sci_info@ictp.it • www.ictp.it
ICTP is governed by UNESCO, IAEA, and Italy, and it is a UNESCO Category 1 Institute

Interdisciplinary Applications of RMT

Zdzislaw Burda

Institute of Physics
Jagiellonian University
Krakow, Poland

MECO 38, ICTP Trieste, March 2013

RMT = Random Matrix Theory

Outline

- Examples of random matrices
- History and milestones
- Applications:
    - Planar combinatorics
    - Statistical + quantum fluctuations
    - Multivariate analysis
    - Maths: FRV, EVS, number theory;
- Examples
- Summary

Examples of random matrices

- Wigner matrices
$$A = (a_{ij})_{i=1\ldots N, j=1\ldots N}$$
  - a_{ij} are i.i.d. real/complex random variables
  - Symmetric/Hermitian $H = A + A^\dagger$

- Invariant ensembles: $H \to UHU^{-1}$
$$Z = \int DH \exp -N \mathrm{tr} V(H)$$

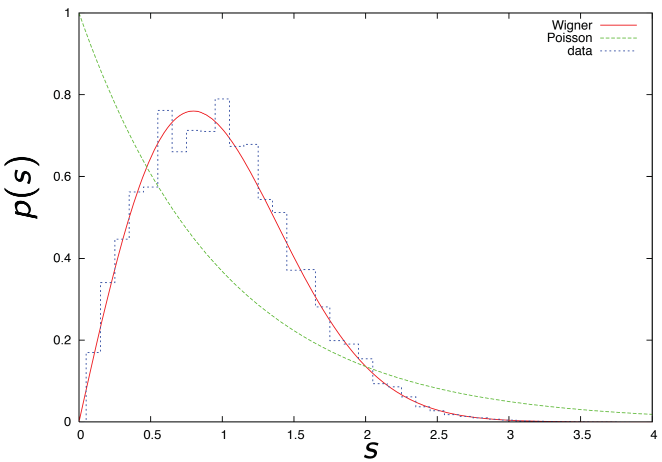
- Many other types: eg. random Toeplitz matrices, Haar truncated matrices, . . . , problem-oriented, home-made random matrices;

- Random matrix is a counterpart of random variable !
- Typically one is interested in the limit $N \to \infty$

Foundations of Random Matrix Theory

- Wigner: Hamiltonian for a large nucleus
- Hamiltonian as a large random matrix
- New type of physics: quantum + statistical fluctuations

- Level separation: $s = \frac{\Delta E}{\langle \Delta E \rangle}$
- Wigner surmise $p(s) = \frac{\pi s}{2}\ e^{-\pi s^2/4}$



Coulomb/Dyson gas

- Dyson: $P(H)DH = e^{-N\mathrm{tr}V(H)}DH$, e.g. $V(H) = H^2/2$
- Invariance: $H \to UHU^{-1}$
- $P(H)DH \longrightarrow p(\lambda_1, \ldots, \lambda_N)D\lambda$
- Analogous to $e^{-(x^2+y^2)}dxdy = re^{-r^2}drd\phi$
- Eigenvalue distribution

$$p(\lambda_1, \ldots, \lambda_N)D\lambda \sim \prod_{i<j} |\lambda_i - \lambda_j|^2 e^{-N\sum_k V(\lambda_k)} D\lambda = e^{-NE(\lambda_1,\ldots,\lambda_N)} D\lambda$$

$$E(\lambda_1, \ldots, \lambda_N) = \sum_k V(\lambda_k) - \frac{1}{N} \sum_{i,j,i\neq j} \ln |\lambda_i - \lambda_j|$$

- Mehta $p(\lambda_1, \ldots, \lambda_N) = \det_{ij} K(\lambda_i, \lambda_j)$
- Brezin,Itzykson,Parisi,Zuber perturbation + 't Hooft 1/N expansion

Applications of RMT

- 2d quantum gravity and strings
- counting: triangulations, pseudoknots, meanders, RNA, ...
- many-body quantum systems
- quantum chaos, quantum transport, entanglement
- QCD - Dirac operator
- complex (glassy) systems
- multivariate analysis
- financial risk management
- information theory
- wireless telecommunication
- free probability, number theory, evs, combinatorics ...

Applications of RMT (1)

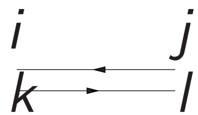
- 2d quantum gravity and strings
- counting: triangulations, pseudoknots, meanders, RNA, ...

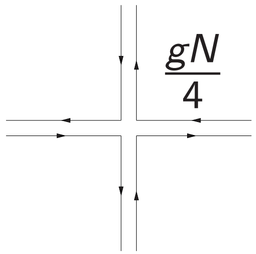
Random matrices and planar combinatorics

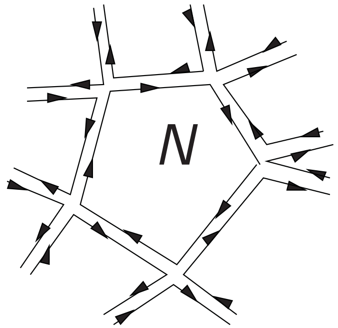
- Planar diagrams

$$F(N,g) = -\log \int DHe^{-N\mathrm{tr}V(H)} \,, \quad \text{where } V(H) = \frac{1}{2}H^2 + \frac{g}{4}H^4$$

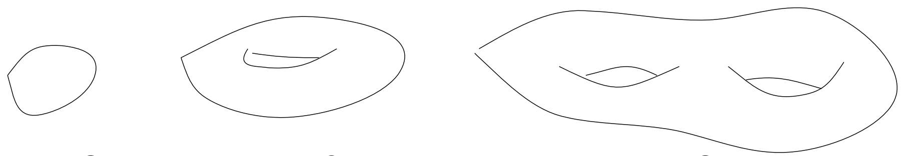
- Feynman rules $\langle H_{ik} H^*_{jl} \rangle = \frac{1}{N}\delta_{ij}\delta_{kl}$







- 't Hooft's expansion $N^{-L} N^V N^P = N^{V+P-L} = N^{2(1-h)}$



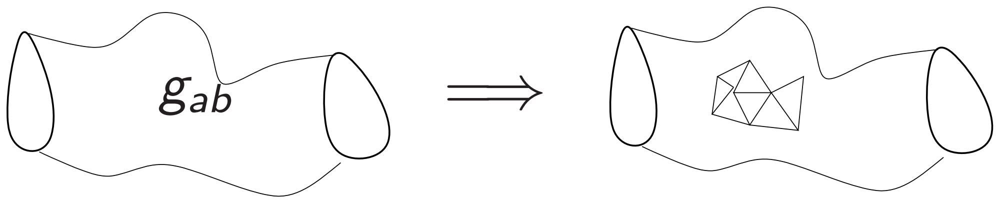
N^2 $\quad$ N^0 $\quad$ N^{-2} $\quad \ldots$

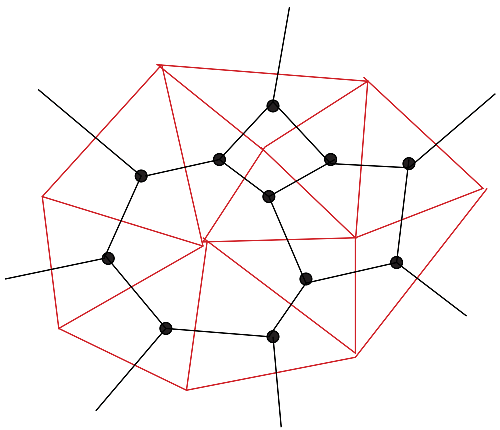
- Perturbative expansion g^V

2d quantum gravity and strings

$$Z = \int D[g_{ab}] e^{-S[g_{ab}]}$$

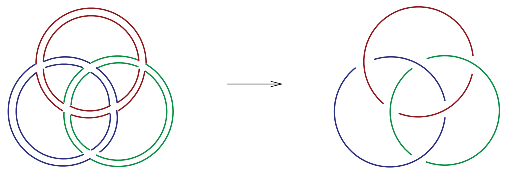
$$\int D[g_{ab}] \longrightarrow \sum_T$$





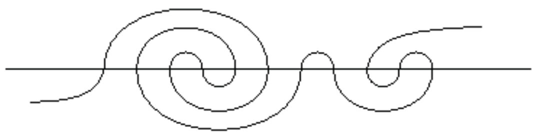
$$\phi^3 \Longleftrightarrow \Delta$$

Counting



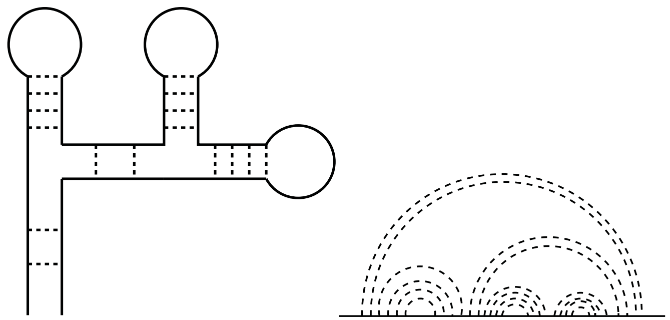
planar triangulations,
BIPZ, Tutte

tangles,
P. Zinn-Justin, Zuber



meanders,
Di Francesco, Golinelli, Guitter

loops, colorings,
Saclay group



RNA folding
Orland, Zee

Applications of RMT (2)

- many-body quantum systems
- quantum chaos, quantum transport, entanglement
- QCD - Dirac operator
- complex (glassy) systems

Quantities of interest

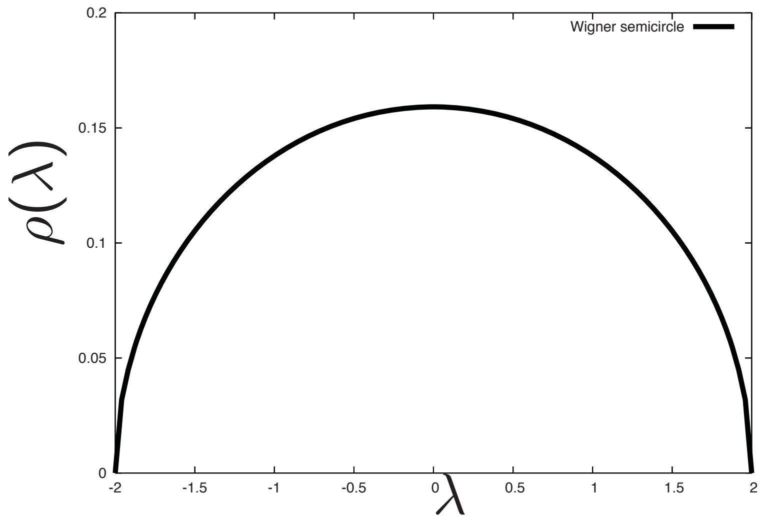
1. Joint eigenvalue distribution: $\rho(\lambda_1, \ldots, \lambda_N)$
2. Eigenvalue density:
   $\rho(\lambda) = \langle \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) \rangle$
3. Two-point correlation function:
   $\rho_2(\lambda, \lambda') = \left\langle \frac{1}{N^2} \sum_{ij} \delta(\lambda - \lambda_i)\delta(\lambda' - \lambda_j) \right\rangle$
4. Microscopic correlation function: $d\xi = \rho(\lambda) d\lambda$,
   $K(r) = \varrho_2(\xi, \xi'), \ \ r = \xi' - \xi$
5. Level spacing distribution: $P(s), \ \ s = \Delta\lambda / \langle \Delta\lambda \rangle$:
   $\cdots \leq \lambda_{a-1} < \lambda_a < \lambda_{a+1} \leq \ldots$

Gaussian unitary ensemble (GUE)

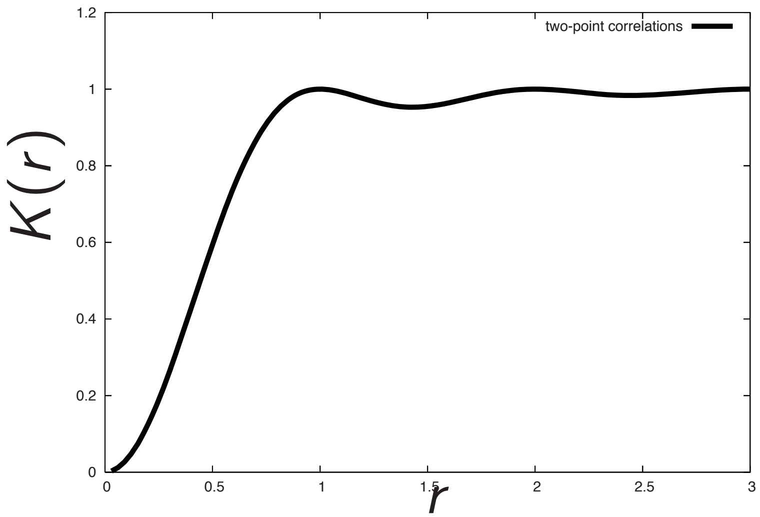
Wigner Law:

$\rho(\lambda) = \frac{1}{2\pi}\sqrt{1-\lambda^2/4}$

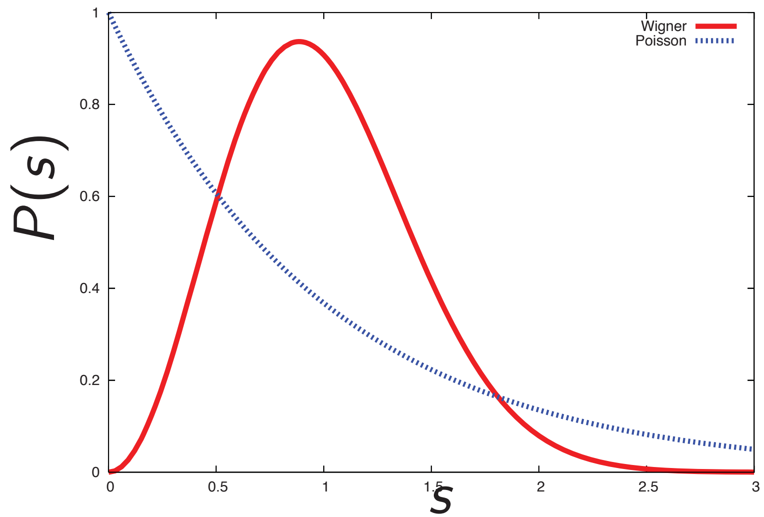
Universality !



$K(r) = 1 - \left(\frac{\sin \pi r}{\pi r}\right)^2$



$P(s) = \frac{32}{\pi^2} s^2 \exp(-\frac{4}{\pi} s^2)$

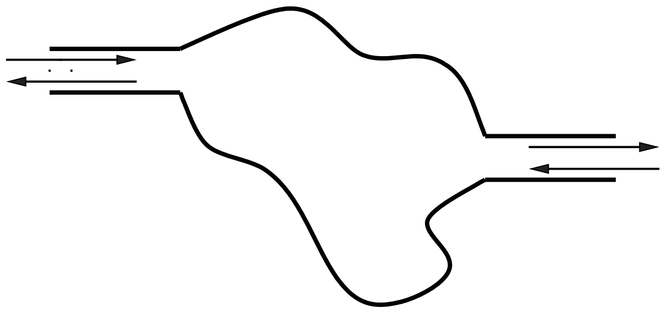


Quantum transport

- Beenakker quantum dots, wires, ...
- Scattering matrix

$$S = \begin{pmatrix} R & T' \\ T & R' \end{pmatrix}, \quad SS^\dagger = 1$$

- Conductance $G = \frac{2e^2}{\hbar} \mathrm{tr}\, TT^\dagger = \frac{2e^2}{\hbar} \sum_i \lambda_i$



Dirac operator in QCD

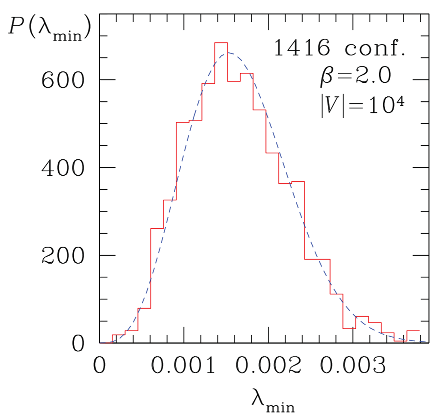
- Shuryak, Verbaarschot, Zahed
- Full QCD

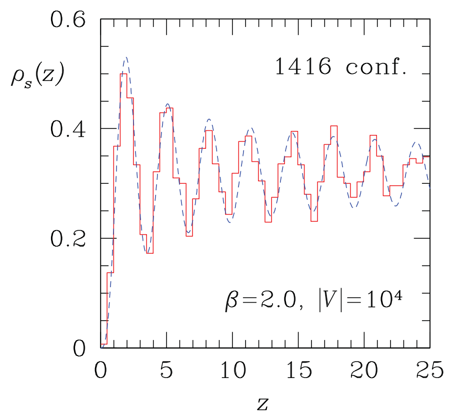
$$Z = \int DA \det^{N_f}(D+m) e^{-S_{YM}(A)}$$

$$D = \gamma_\mu \partial_\mu + ig\gamma_\mu A_\mu$$

- RMT approximation ($m = 0$)

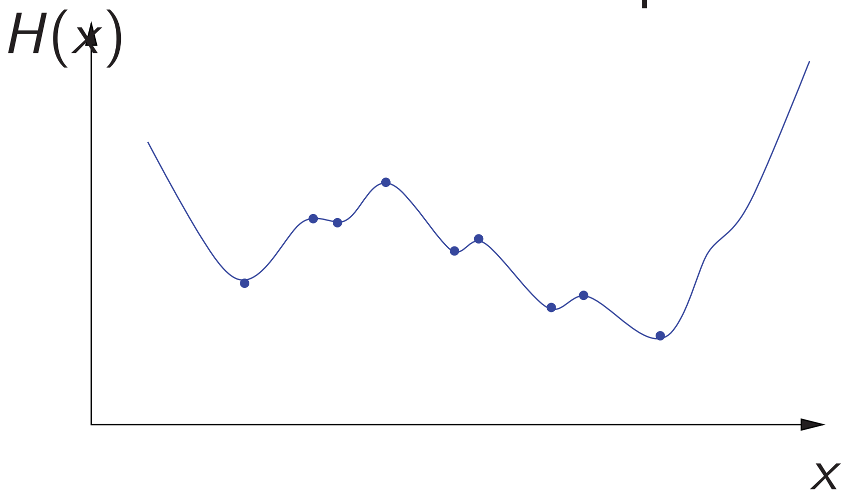
$$iD \longrightarrow \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}, \quad S_{YM} \longrightarrow N \mathrm{tr} HH^\dagger$$





Complex (glassy) systems

- Fyodorov
- Hamiltonian in random potential: $\langle H(\vec{x})H(\vec{y})\rangle \sim f\left(|\vec{y}-\vec{x}|\right)$



- Number of stationary points: $\partial_k H = 0$.
- Kac-Rice formula for density of stationary points/minima:

$$\rho(x) = \left\langle \left|\det \partial^2_{ij} H\right| \prod_{k=1}^{N} \delta(\partial_k H) \right\rangle$$

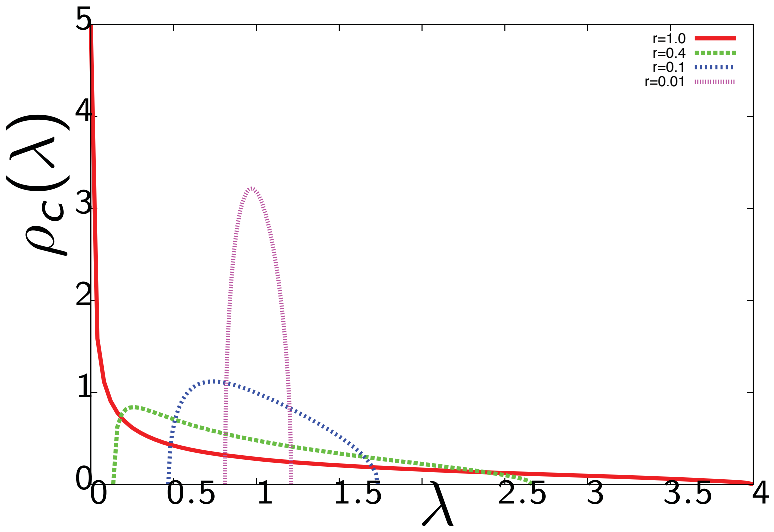
- Counting mapped onto calculations for Gaussian random matrices

Applications of RMT (3)

- multivariate analysis
- financial risk management
- information theory
- wireless telecommunication

Wishart matrices/population covariance

- Statistical system of N-degrees of freedom: X_i, $i = 1, \ldots, N$
- Covariance matrix: $C_{ij} = \langle X_i X_j \rangle \quad , \quad \langle X_i \rangle = 0 \quad .$
- Empirical (population) covariance matrix (Bai, Silverstein):
  - T measurements of X_i: x_{it}, $t = 1, \ldots, T$
  - $c_{ij} = \frac{1}{T} \sum_{t=1}^{T} x_{it} x_{jt} \quad , \quad$ aspect ratio $r = N/T$
- Example X_i i.i.d. $N(0,1)$:
  - $\rho_C(\lambda) = \delta(\lambda - 1)$
  - $\rho_c(\lambda) = \frac{1}{2\pi r \lambda} \sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)} \, , \quad \lambda_\pm = (1 \pm \sqrt{r})^2$

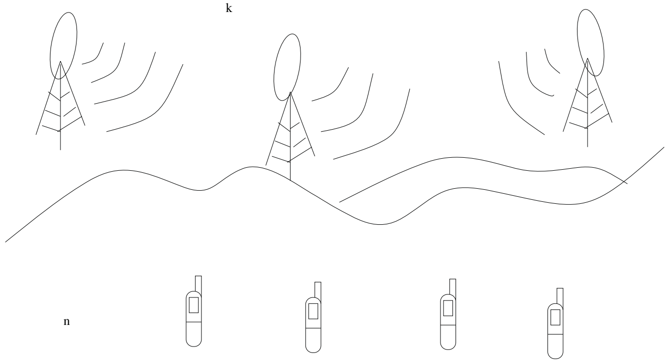


Portfolio selection (Markowitz)

- Revenue: $\langle X \rangle = \sum_{i=1}^{N} p_i \langle X_i \rangle$
- Risk: $\sigma^2(X) = \sum_{ij} p_i C_{ij} p_j = \sum_i \lambda_i v_i^2$
- Minimize risk for given expected revenue
- How to compute: C_{ij}?
- Historical data: c_{ij}
- $\rho_c(\lambda) \longrightarrow \rho_C(\lambda)$ (Burda, Jarosz, Jurkiewicz, Görlich)

Information theory and wireless telecommunication

- Shannon: $C = W \log_2(1 + SNR)$
- SNR = signal to noise ratio
- Telatar: $C = \min(N, K) W \log_2(1 + SNR)$
- K-senders, N-receivers



MIMO = multiple input multiple output

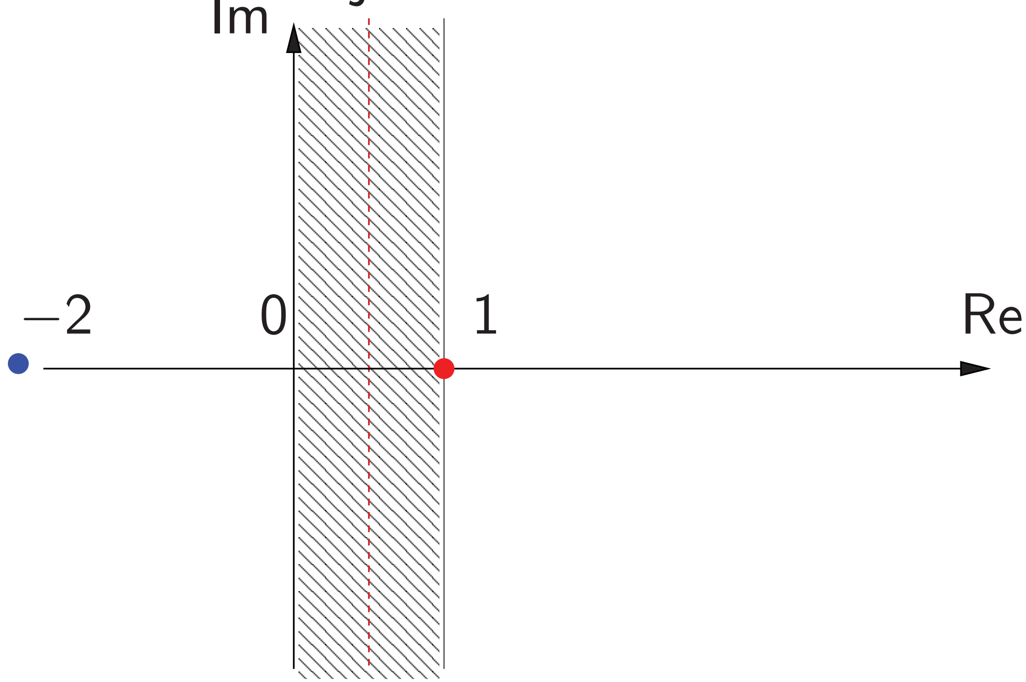
- $\vec{y} = H\vec{x} + \vec{n}, \qquad SNR = |x|/|n|$
- $C = \frac{1}{N}\left\langle \log \mathrm{tr}\left(1 + \mathrm{SNR} \cdot HH^{\dagger}\right)\right\rangle$ (Tulino, Verdú)

Applications of RMT (4)

- Free probability
- Number theory
- Extreme value statistics for repeling objects (Tracy-Widom) and related problems, combinatorics, (Johansson, Majumdar)

Zeros of Riemann Zeta function

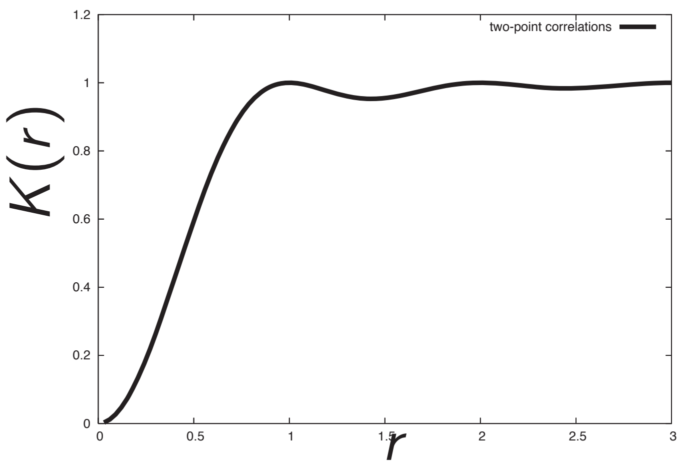
- Zeta function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$
- Riemann conjecture: nontrivial zeros $z_n = 1/2 + i\gamma_n$

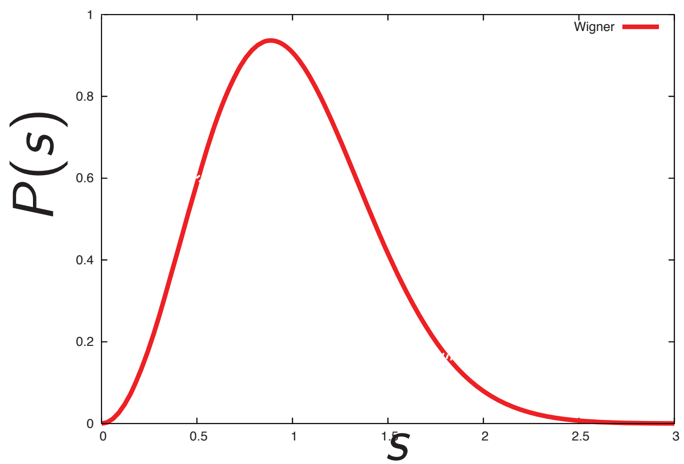


- Polya-Hilbert conjecture
- Montgomery-Dyson conjecture

Odlyzko-Montgomery law

- Number of zeros: $N(0 < \gamma_n < T) \sim T \ln T$
- Spacing: $s = \gamma_{n+1} \ln \gamma_{n+1} - \gamma_n \ln \gamma_n$
- Odlyzko paper: *The* 10^{20}*-th zero of the Riemann zeta function and* 175 *million of its neighbors*





Summary

- RMT - omnipresent in theoretical physics!
- Interphase of many branches of physics and mathematics!
- Sui generis branch of mathematics!
- Practical applications!
- Not discussed here: non-hermitean RMT
  - Dissipative systems
  - Time-lagged correlations $c_{ij} = \frac{1}{T} \sum_t x_{it} x_{jt-1}$
  - Chemical potential in QCD
  - Products of random matrices
- Reviews
  - G. Akemann, J. Baik, Ph. Di Francesco (eds): The Oxford Handbook of Random Matrix Theory, Oxford University Press 2011,
  - P. Vivo, Random Matrix Theory: a Matlab-based tutorial, 2013

Thank you!