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Nonperturbative Renormalization Group for the Kardar-Parisi-Zhang Equation

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Nonperturbative renormalization group for the Kardar-Parisi-Zhang equation

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Presentation outline

1. (Very short) recall of the Kardar-Parisi-Zhang equation
   - Interface growth
   - The KPZ equation
   - Multi-dimensional interface
2. Non Perturbative Renormalisation Group for KPZ
   - NPRG for nonequilibrium systems
   - Construction of an approximation scheme
3. Results
   - Phase diagram and critical exponents
   - Scaling in one dimension
   - Comparison with exact scaling functions

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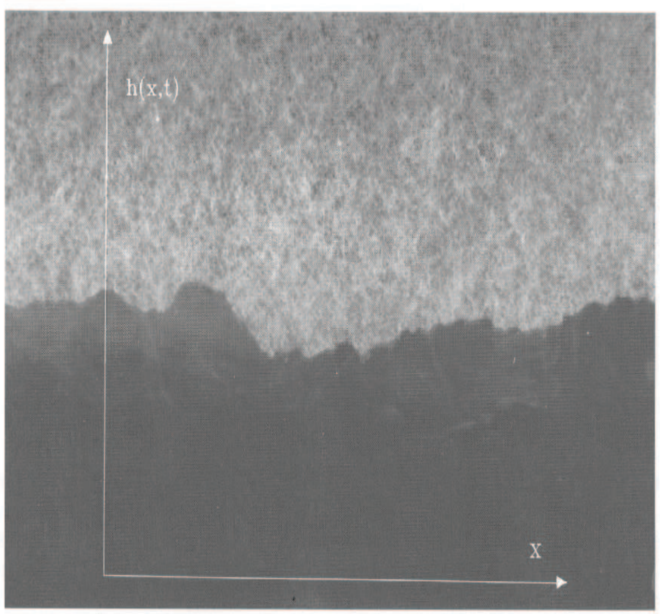
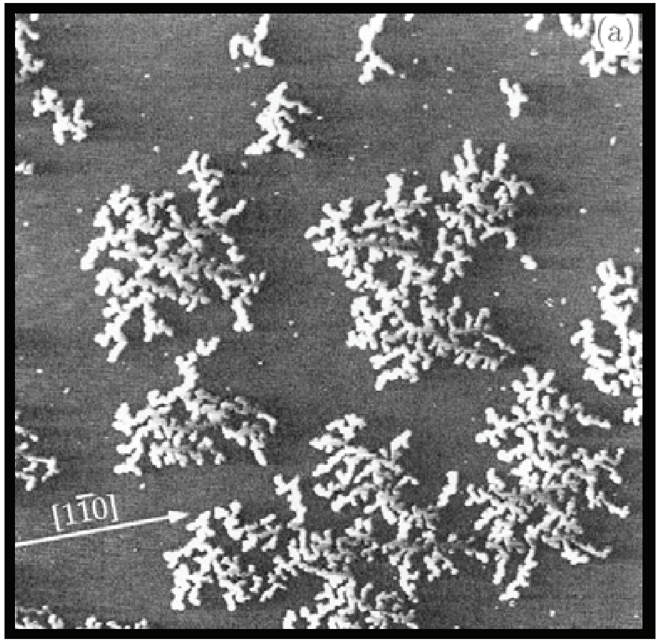
Some examples of moving interfaces



Paper combustion (1D)



Molecular Beam Epitaxy (2D)

Kinetic roughening

Generic scaling in all dimensions

Stationary correlation function (in the co-moving frame) :

$$C(t,\vec{x}) = \langle h(t,\vec{x})h(0,0)\rangle$$

Has the following behavior :

$$C(t,\vec{x}) \sim t^{2\chi/z} \qquad t \ll |\vec{x}|^z$$

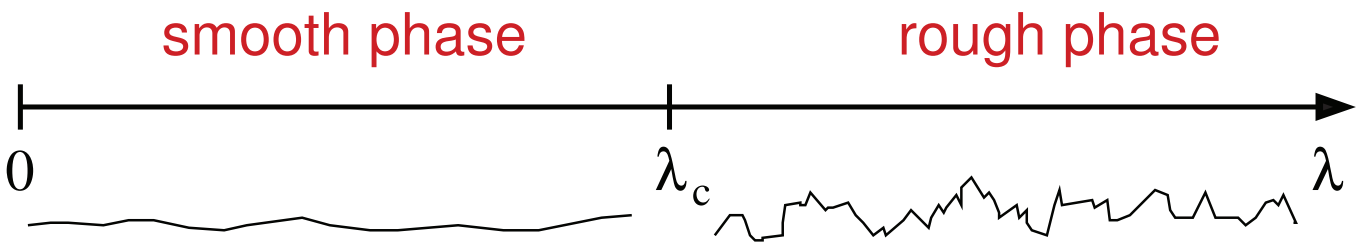
$$C(t,\vec{x}) \sim |\vec{x}|^{2\chi} \qquad t \gg |\vec{x}|^z$$

Scaling form : $C(t,\vec{x}) \sim |\vec{x}|^{2\chi} F(t/|\vec{x}|^z)$

- universal roughness χ and dynamical z exponents
- universal scaling function F

Kinetic roughening

Nonequilibrium continuous phase transition in $d > 2$



Edwards-Wilkinson	rough phase of KPZ
$(\lambda_{eff,IR} = 0)$	$(\lambda_{eff,IR} \neq 0)$
$\chi = (2 - d)/d$	$\chi > 0$
$z = 2$	$z + \chi = 2$

A continuous model : the KPZ equation

One of the simplest nonlinear Langevin equations

$$\frac{\partial h(t,\vec{x})}{\partial t} = \nu\,\nabla^2 h(t,\vec{x}) + \frac{\lambda}{2}\left(\nabla h(t,\vec{x})\right)^2 + \eta(t,\vec{x})$$

Kardar, Parisi and Zhang, **PRL** (1986)

- η : Gaussian uncorrelated white noise
  $\langle \eta(t,\vec{x})\eta(t',\vec{x}')\rangle = 2\,D\,\delta^d(\vec{x}-\vec{x}')\,\delta(t-t')$
- $\nu\nabla^2 h$: smoothening surface tension
- $\lambda(\nabla h)^2$: nonlinear growth along the local normal

Equivalence with other stochastic equations

- Burgers' equation for randomly stirred fluids ($\vec{u} \propto \nabla h$)
- Directed Polymer in Random Media ($Z = e^{(\lambda/2\nu)h}$)

The KPZ universality class

Large universality class

- discrete models ballistic deposition, Eden clusters, RSOS models, polynuclear growth. . .
- experiments paper combustion, bacterial colony growth, turbulent liquid crystal. . .

Very studied theoretically :

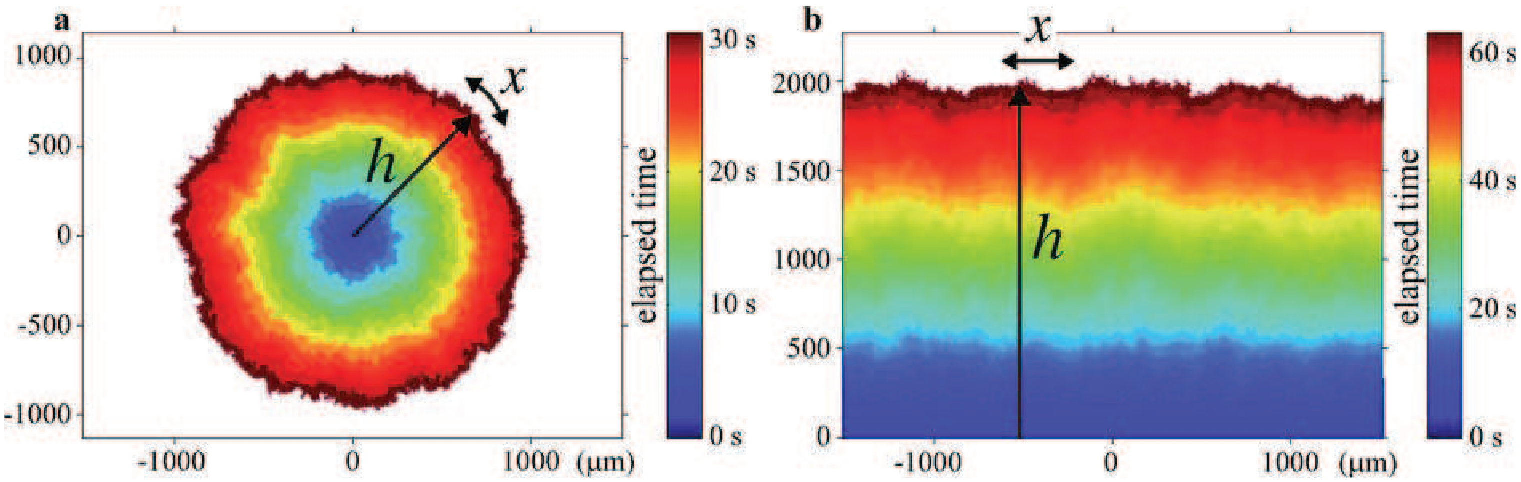
- *before 2000* : numerics, RG, approximate methods
  Frey and Täuber, **PRE** (1994),Wiese, **J. Stat. Phys.** (1998)
- *last decade* : exact results in $d = 1$ for $t \to \infty$
  Johansson, **Commun. Math. Phys.** (2000), Prähofer and Spohn, **PRL** (2000), Prähofer and Spohn, **J. Stat. Phys.** (2004), Sasamoto, **J. Phys. A** (2005).
- *since 2010* : solution at finite t in $d = 1$
  Sasamoto and Spohn, **PRL** (2010), **J. Stat. Phys.** (2010), Amir, Corwin and Quastel, **Commun. Pure Appl. Math.** (2011), Dotsenko, **Europhys. Lett.** (2010), Calabrese, Le Doussal and Rosso, **Europhys. Lett.** (2010), Calabrese and Le Doussal, **PRL** (2011), Imamura and Sasamoto, **J. Phys. A** (2011), **PRL** (2012).

One-dimensional interface

High-precision experiments

Convection of turbulent nematic liquid crystals

high-precision measurements of growing clusters with circular and flat geometries



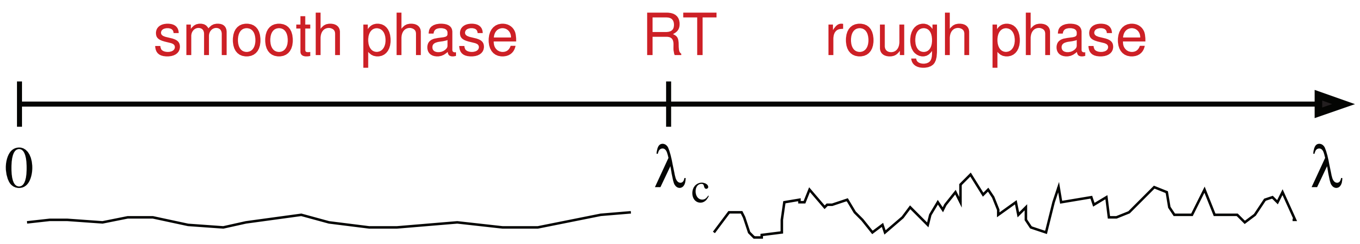
Takeuchi and Sano, **PRL** (2010), **J. Stat. Phys.** (2012), Takeuchi, Sano, Sasamoto, Spohn, **Sci. Rep.** (2011)

$\rightarrow$ Compares favourably with exact results.

Multi-dimensional interface

RG treatements

Roughening Transition (RT) for $d > 2$



β-function for $g = \lambda^2 D / \nu^3$

- 2-loop in KPZ version

  Frey and Täuber, **PRE** (1994)

- all-order in DPRM version

  Wiese, **J. Stat. Phys.** (1998)

RT fixed point

critical exponents :

$\chi = 0$ and $z = 2$

superuniversal (in all d)

Rough phase fixed point

perturbative RG fails *to all order* $\longrightarrow$ strong coupling !

Multi-dimensional interface

Other approaches

Numerical approaches

- discrete models Tang *et al.* , (1992) Pagnani and Parisi , **PRE** (2013) E. Marinari *et al.* , (2012), Kelling and Ódor, **PRE** (2011) T. Halpin-Healy, **PRL** (2012)
- direct integrations Miranda and Reis, (2008)
- real space NRG Castellano *et al.*, (1998-99)

d	χ
2	0.384
3	0.304
4	0.256
$d_c = \infty$	0

Analytical approaches

- perturbative FRG $d_c \simeq 2.5$
  Le Doussal and Wiese, **PRE** (2005)
- Mode-Coupling Theory $d_c = 4$
  Frey, Täuber and Hwa, **PRE** (1996), Colaiori and Moore, **PRL** (2001)
- Self-Consistent Expansion $d_c = \infty$
  Schwartz and Edwards, (1992), Schwartz and Katzav, (2008)

} exponents, 1d scaling function

For $d > 1$, an analytical nonperturbative approach is needed to control the rough phase

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Field theory for the KPZ equation

Field theory for a Langevin equation

build generating functional to compute noise averages $\langle \mathcal{O} \rangle_\eta$

Janssen, **Z. Phys. B** (1976), de Dominicis, **J. Phys. (Paris)** (1976)

- introduce response field $\tilde{h}$ to enforce equation of motion
- integrate over the Gaussian noise distribution

Langevin dynamics : $\Longleftrightarrow$ generating functional :

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta \qquad \mathcal{Z} = \int \mathcal{D}\varphi \mathcal{D}[i\tilde{\varphi}]\, e^{-\mathcal{S}[\varphi,\tilde{\varphi}] + \int (j\varphi + \tilde{j}\tilde{\varphi})}$$

KPZ action

$$\mathcal{S}_{KPZ}[\varphi,\tilde{\varphi}] = \int d^d\vec{x}\, dt \left\{ \tilde{\varphi} \left[\partial_t \varphi - \nu \nabla^2 \varphi - \frac{\lambda}{2} (\nabla\varphi)^2 \right] - D\,\tilde{\varphi}^2 \right\}$$

Nonperturbative Renormalization Group for nonequilibrium systems

Berges, Tetradis and Wetterich, **Phys. Rep.** (2002), Canet, Delamotte, Chaté, **J. Phys. A** (2011),Berges, Mesterházy, (2012), Kloss, Kopietz, **PRB** (2011)

Wilson's philosophy : Scale dependent generating functional

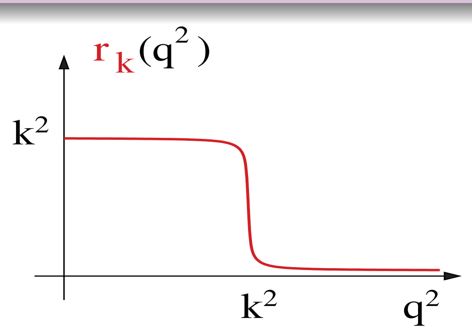
$$\mathcal{Z}_\kappa[j,\tilde{j}] = \int \mathcal{D}\varphi \mathcal{D}\tilde{\varphi}\; e^{-\mathcal{S}[\varphi,\tilde{\varphi}] - \Delta\mathcal{S}_\kappa + \int (j\varphi + \tilde{j}\tilde{\varphi})}$$

$$\Delta\mathcal{S}_\kappa[\psi,\tilde{\psi}] = \frac{1}{2}\int_{\vec{x},t} \left(\psi\;\tilde{\psi}\right) \mathcal{R}_\kappa(\nabla^2,\partial_t) \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$$

Special constraints for KPZ

from Ito's discretization, causality and symmetries

$$\mathcal{R}_\kappa(\omega,\vec{q}) = r_\kappa\left(\frac{q^2}{\kappa^2}\right) \begin{pmatrix} 0 & \nu_\kappa q^2 \\ \nu_\kappa q^2 & -2D_\kappa \end{pmatrix}$$



Nonperturbative Renormalization Group for nonequilibrium systems

Legendre transform

$$\Gamma_\kappa[\varphi,\tilde\varphi] + \log \mathcal{Z}_\kappa[j,\tilde j] = \int j\varphi + \tilde j\tilde\varphi - \Delta\mathcal{S}_\kappa[\varphi,\tilde\varphi]$$

Exact flow equation for Γ_κ

$$\partial_\kappa\Gamma_\kappa = \frac{1}{2}\,\mathrm{Tr}\int_{\omega,\vec q}\partial_\kappa\mathcal{R}_\kappa\cdot G_\kappa \quad \text{with} \quad G_\kappa = \left[\Gamma_\kappa^{(2)} + \mathcal{R}_\kappa\right]^{-1}$$

No exact solution, but useful to implement approximations not based in perturbation theory.

Construction of an approximation scheme

Quest of a scheme preserving symmetries

Our aim

calculate full *momentum and frequency dependent* two-point functions in the stationary regime.

$$\partial_\kappa[\Gamma_\kappa{}^{(2)}]_{ij} = \mathrm{Tr}\int \partial_\kappa \mathcal{R}_\kappa \cdot G_\kappa \cdot \left(-\frac{1}{2}\,[\Gamma_\kappa{}^{(4)}]_{ij} + [\Gamma_\kappa{}^{(3)}]_i \cdot G_\kappa \cdot [\Gamma_\kappa{}^{(3)}]_j\right) \cdot G_\kappa$$

$\Longrightarrow$An extremely efficient approximation scheme exist for correlation functions : close equations by expanding vertices in internal momentum and frequency.

Blaizot, Méndez-Galain and Wschebor, **Phys. Lett. B** (2006), Benitez *et al.*, **PRE** (2009)

Problem

Hard to make compatible with Ward identities amongst $\Gamma_\kappa^{(l,m)}$'s !

Symmetries

gauged shift symmetry

$$\varphi(t,\vec{x}) \to \varphi(t,\vec{x}) + c(t)$$

gauged Galilean symmetry

$$\varphi(t,\vec{x}) \to \vec{x}\cdot\partial_t\vec{v}(t) + \varphi(t,\vec{x}+\lambda\vec{v}(t))$$
$$\tilde{\varphi}(t,\vec{x}) \to \tilde{\varphi}(t,\vec{x}+\lambda\vec{v}(t))$$

time reversal symmetry in $d=1$

$$\varphi(t,\vec{x}) \to -\varphi(-t,\vec{x})$$
$$\tilde{\varphi}(t,\vec{x}) \to \tilde{\varphi}(-t,\vec{x}) + \frac{\nu}{D}\nabla^2\varphi(-t,\vec{x})$$

Ward identities

$$\Gamma_\kappa^{(1,1)}(\omega,\vec{p}=0) = i\omega$$

$$i\omega\frac{\partial}{\partial\vec{p}}\Gamma_\kappa^{(2,1)}(\omega,\vec{p}=\vec{0};\omega_1,\vec{p}_1) = \lambda\vec{p}_1\left[\Gamma_\kappa^{(1,1)}(\omega+\omega_1,\vec{p}_1) - \Gamma_\kappa^{(1,1)}(\omega_1,\vec{p}_1)\right]$$

$$2\Re\Gamma_\kappa^{(1,1)}(\omega,\vec{p}) = -\frac{\nu}{D}p^2\Gamma_\kappa^{(0,2)}(\omega,\vec{p})$$

Construction of an approximation scheme

Strategy

build a non-trivial Galilean invariant ansatz for Γ_κ

Analogy with fluid mechanics

introduce covariant time derivatives

$$\tilde{D}_t \equiv \partial_t - \lambda \nabla\varphi(t,\vec{x})\cdot\nabla \qquad D_t\varphi(t,\vec{x}) \equiv \partial_t\varphi(t,\vec{x}) - \frac{\lambda}{2}(\nabla\varphi(t,\vec{x}))^2$$

Second Order (SO) Ansatz for the effective action

$$\Gamma_\kappa[\psi,\tilde{\psi}] = \int_{t,\vec{x}} \left\{ \tilde{\psi} f_\kappa^\lambda D_t\psi - \frac{1}{2}\left[\nabla^2\psi f_\kappa^\nu \tilde{\psi} + \tilde{\psi} f_\kappa^\nu \nabla^2\psi\right] - \tilde{\psi} f_\kappa^D \tilde{\psi} \right\}$$

with f_κ^X three arbitrary functions $f_\kappa^X \equiv f_\kappa^X(-\tilde{D}_t^2, -\nabla^2)$

Canet, Chaté, Delamotte and Wschebor, **PRE** (2011)

Integration of the flow equations

It is convenient to use dimensionless renormalised quantities

- three running functions

$$\hat{f}^D_\kappa(\hat{\varpi}^2, \hat{p}^2) = f^D_\kappa(\omega^2, p^2)/D_\kappa \qquad \hat{p} = p/\kappa$$

$$\hat{f}^\nu_\kappa(\hat{\varpi}^2, \hat{p}^2) = f^\nu_\kappa(\omega^2, p^2)/\nu_\kappa \qquad \hat{\varpi} = \omega/(D_\kappa \kappa^2)$$

$$\hat{f}^\lambda_\kappa(\hat{\varpi}^2, \hat{p}^2) = f^\lambda_\kappa(\omega^2, p^2)$$

- two anomalous dimensions $\eta^D_\kappa = -\partial_s \ln D_\kappa$, $\eta^\nu_\kappa = -\partial_s \ln \nu_\kappa$
- one dimensionless coupling $\hat{g}_\kappa = \lambda^2 D_\kappa / {\nu_\kappa}^3 \kappa^{d-2}$

Numerical integration of the flow equations

at SO for $d = 1$ and performing the further approximation $\hat{f}^X_\kappa(\omega, \vec{p}) \to \hat{f}^X_\kappa(\vec{p})$ for $d > 1$.)

Results

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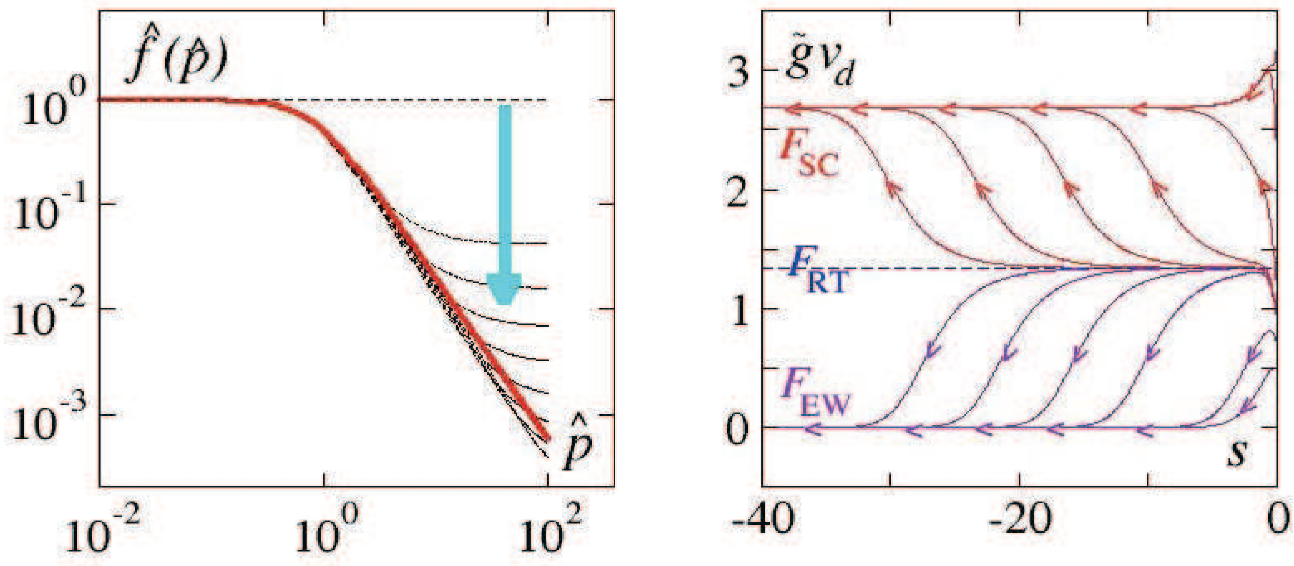
Integration of the flow equations

Numerical integration of the flow equations

Renormalisation scale $s = \ln(\kappa/\Lambda)$ from $s = 0$ to $s \to -\infty$

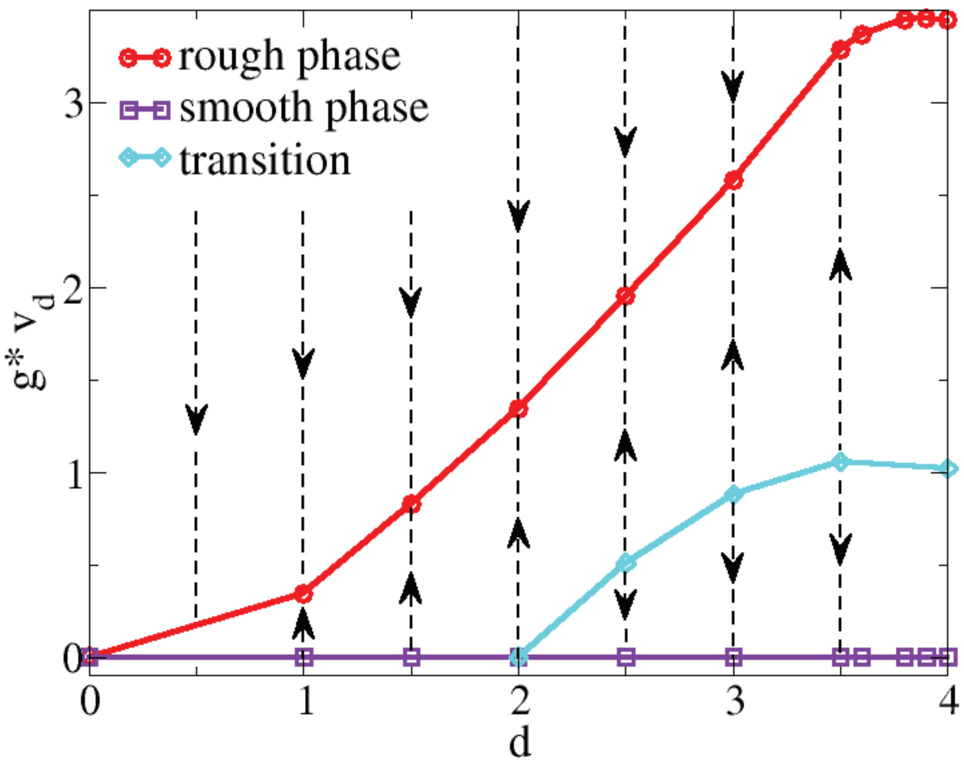


In dimensions $d > 2$, the flow always leads to one of the three fixed points EW, RT or KPZ.

Results in physical dimensions (NLO)

Phase diagram and critical exponents

Phase diagram



Critical exponents

at the fixed point

$\nu_\kappa \sim \kappa^{-\eta_*^\nu} = \kappa^{-\chi}$

and $z + \chi = 2$

d	χ NLO	χ num.
1	1/2	1/2
2	0.373	0.384
3	0.180	0.304

Canet, Chaté, Delamotte, Wschebor, **PRL** (2010), **PRE** (2011), Kloss, Canet, Wschebor, **PRE** (2012)

Results in one dimension (SO)

Generalized fluctuation-dissipation theorem in $d = 1$

- only one function left $\hat{f}^{\nu}_{\kappa} = \hat{f}^{D}_{\kappa} \equiv \hat{f}_{\kappa}$, $\hat{f}^{\lambda}_{\kappa} = 1$.
- Exponents

$$\nu_\kappa = D_\kappa, \qquad \eta^{\nu}_{\kappa} = \eta^{D}_{\kappa} \equiv \eta_\kappa \to \eta_* \stackrel{d=1}{=} 1/2$$

$$\chi = (2 - d + \eta^{D}_{\kappa} - \eta^{\nu}_{\kappa})/2 \stackrel{d=1}{=} 1/2,$$

$$z = 2 - \chi \stackrel{d=1}{=} 3/2$$

Dimensionless flow equations

$$\begin{aligned} \partial_s \hat{f}_\kappa(\hat{\varpi}, \hat{p}) &= \eta_\kappa \hat{f}_\kappa + \hat{p}\, \partial_{\hat{p}} \hat{f}_\kappa + (2 - \eta_\kappa)\hat{\varpi}\, \partial_{\hat{\varpi}} \hat{f}_\kappa + I_\kappa(\hat{\varpi}, \hat{p}) \\ \partial_s \hat{g}_\kappa &= \hat{g}_\kappa (2\eta_\kappa - 1) \end{aligned}$$

Decoupling of the nonlinear term

$I_\kappa \to 0$ in the regime $\hat{p}$ and/or $\hat{\varpi} \gg 1$

Canet, Delamotte, Chaté and Wschebor, **PRE** (2011)

Results in one dimension (SO)

General solution at the fixed point

in the regime $\hat{p}$ and/or $\hat{\varpi} \gg 1$

$$\hat{f}_*^D(\hat{\varpi},\hat{p}) = \frac{1}{\hat{p}^{\eta_D}}\hat{\zeta}_*^D\left(\frac{\hat{\varpi}}{\hat{p}^z}\right) \stackrel{d=1}{=} \frac{1}{\hat{p}^{1/2}}\hat{\zeta}\left(\frac{\hat{\varpi}}{\hat{p}^{3/2}}\right)$$

Generic scaling for the dimensionful correlation function

physical limit $\kappa \to 0$ at fixed ϖ, p

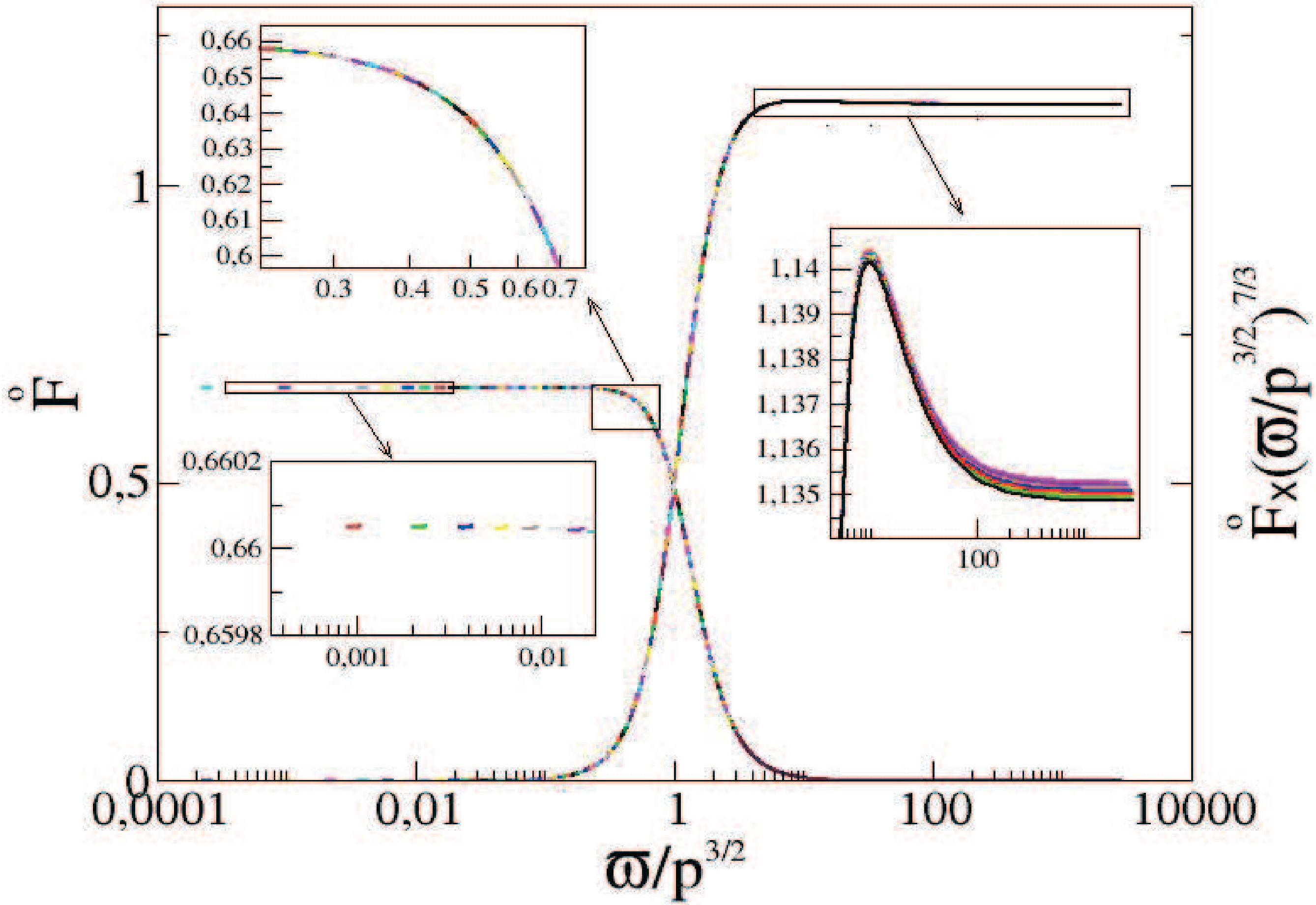
equivalent to $\hat{p} = p/\kappa$ and/or $\hat{\varpi} = \varpi/(\kappa^2 D_\kappa) \gg 1$

$$C(\varpi,p) = -\frac{\Gamma^{(0,2)}(\varpi,p)}{|\Gamma^{(1,1)}(\varpi,p)|^2} \stackrel{d=1}{=} \frac{2}{p^{7/2}} \frac{\hat{\zeta}\left(\frac{\hat{\varpi}}{\hat{p}^{3/2}}\right)}{\frac{\hat{\varpi}^2}{\hat{p}^3} + \hat{\zeta}^2\left(\frac{\hat{\varpi}}{\hat{p}^{3/2}}\right)}$$

$$\equiv \frac{2}{D_0 p^{7/2}}\mathring{F}\left(\frac{\varpi}{D_0 p^{3/2}}\right) \qquad D_\kappa = D_0\kappa^{-\eta_*^D}$$

Results in one dimension (SO)
Proof of generic scaling

Scaling function $\mathring{F}$ associated with the correlation function



Comparison with exact scaling functions

Definition of different scaling functions

Normalisations

scaling function g defined as $C(t,x) = \alpha\, t^{2/3}\, g(\beta\, x/t^{3/2})$
with arbitrary constants α and β

- *g dimensionless* $\longrightarrow$ fixes dependence in λ, ν, D
- *g universal* $\longrightarrow$ fixes dependence in D_0
- numerical normalisation fixed according to [*]

Introduction of 3 functions

$$f(y) = g''(y)/4$$
$$\tilde{f}(k) = 2\int_0^\infty dy\, \cos(ky) f(y)$$
$$\mathring{f}(\tau) = 2\int_0^\infty dk\, \cos(k\tau)\tilde{f}(k^{2/3})$$

Normalized functions

$$\mathring{F}(\tau) \to 2\sqrt{A}\,\mathring{f}(\sqrt{A}\,\tau)$$
$$A = 2\lambda^2 D/(\nu^3 {D_0}^2)$$

Canet, Delamotte, Chaté, Wschebor, **PRE** (2011)

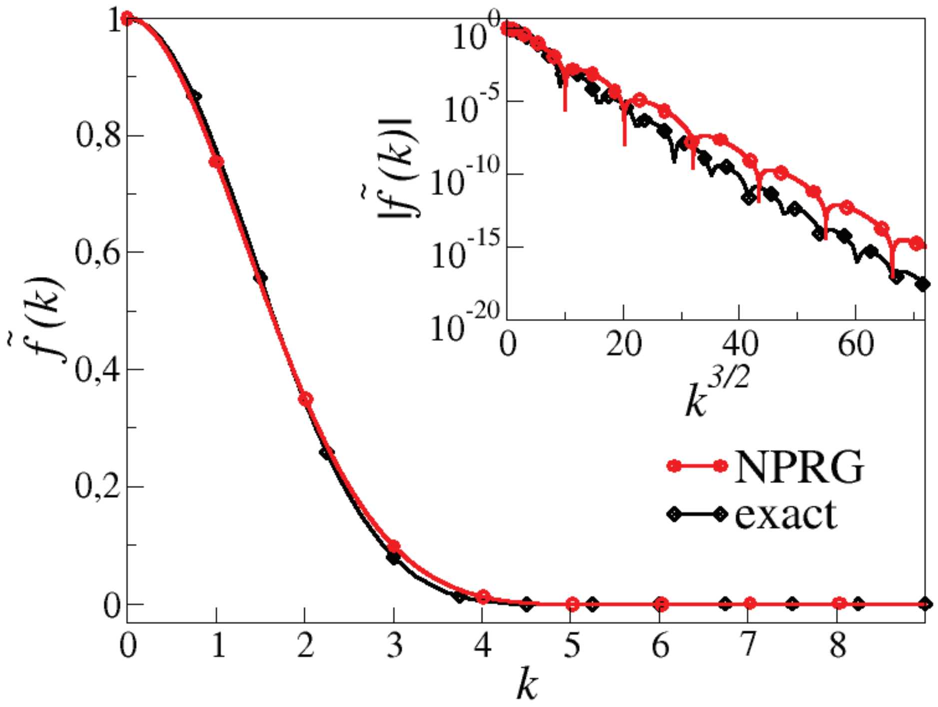
* Prähofer and Spohn, **J. Stat. Phys.** (2004)

Comparison with exact scaling functions

NPRG at SO scaling function in Fourier space

Scaling function $\tilde{f}(k) = \int_0^\infty \frac{d\tau}{\pi} \cos(\tau k^{3/2})\, \mathring{f}(\tau)$

Canet, Delamotte, Chaté and Wschebor, **PRE** (2011), Prähofer and Spohn, **J. Stat. Phys.** (2004)



Asymptotics

$\tilde{f} \sim \cos(a_0 k^{3/2}) e^{-b_0 k^{3/2}}$ for $k \to \infty$

	a_0	b_0
NPRG	0.28(5)	0.49(1)
exact	1/2	1/2

correct scale $k^{3/2} \neq k$ (contrary to MCT and SCE)

Comparison with exact scaling functions

Universal amplitude ratio

$$g_0 = c \int_0^\infty d\tau\ \tau^{2/3}\ \mathring{f}(\tau)$$

$c = 2\Gamma(1/3)/\pi^2$

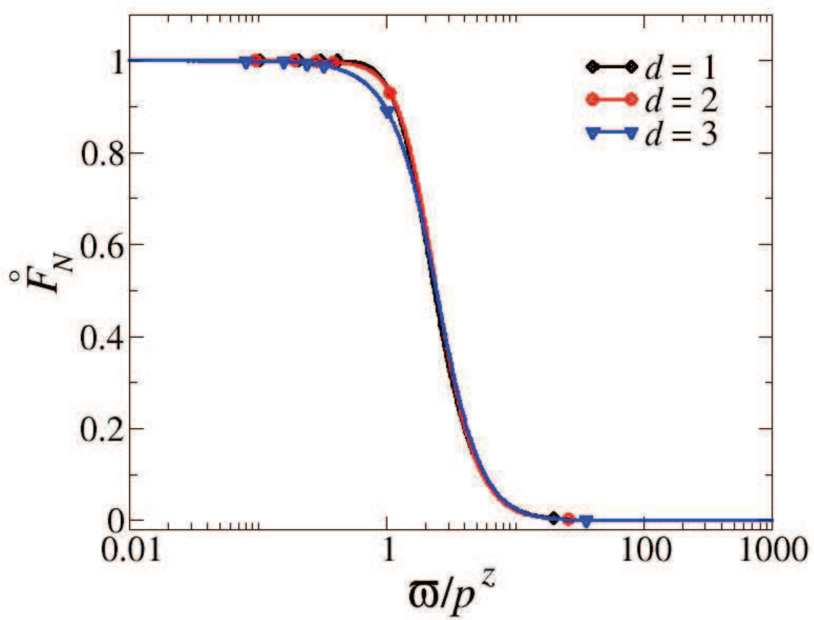
	g_0
exact	1.15039
NPRG	1.19(1)

Canet, Delamotte, Chaté and Wschebor, **PRE** (2011), Prähofer and Spohn, **J. Stat. Phys.** (2004)

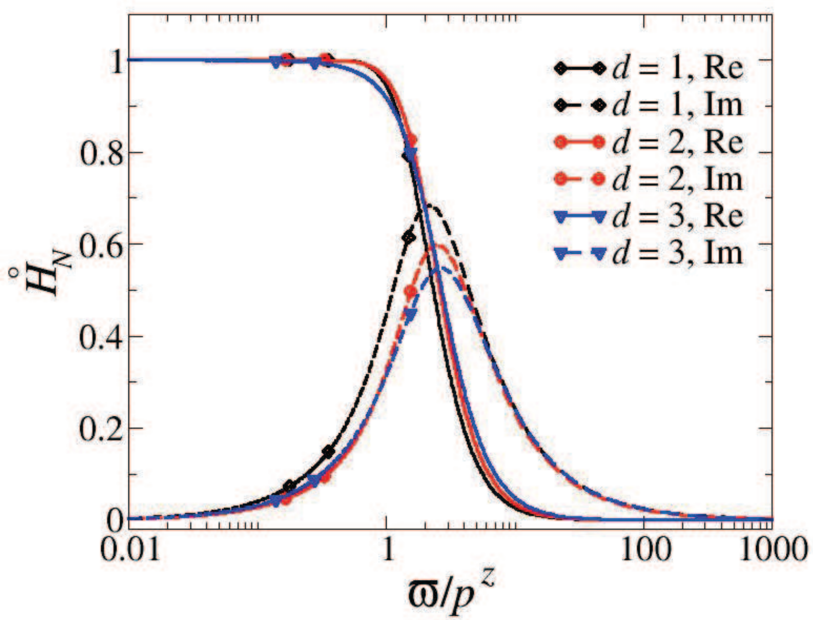
Correlation and response functions in $d \neq 1$

NPRG at NLO

Correlation function



Response function



Kloss, Canet and Wschebor, **PRE** (2012)

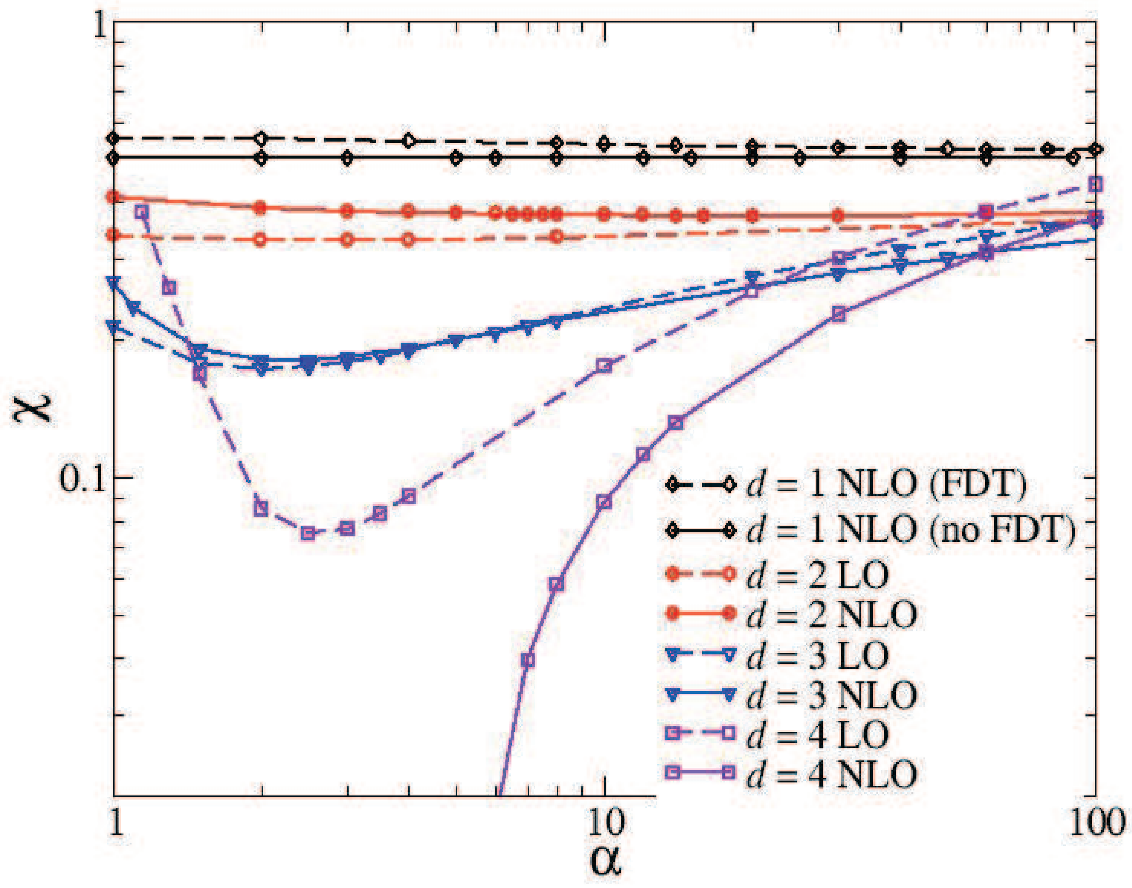
Summary and outlook

Summary

- NPRG framework for nonequilibrium systems
- approximation scheme for the KPZ equation
- scaling functions match very well exact results in $d = 1$
- first predictions of scaling functions for $d > 1$

Outlook

- height probability distribution
- upper critical dimension d_c
- Navier and Stokes equation


1
0.1
χ
1
10
100
α
d = 1 NLO (FDT)
d = 1 NLO (no FDT)
d = 2 LO
d = 2 NLO
d = 3 LO
d = 3 NLO
d = 4 LO
d = 4 NLO