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Semiclassical Theory of Quantum Quenches

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Semiclassical theory of quantum quenches

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Phys. Rev. Lett. **106**, 035701 (2011), arXiv:1011.3664

Phys. Rev. B **84**, 165117 (2011), arXiv:1106.5248

related work with

Benjamin Blaß (Saarbrücken)

EPL **93**, 30004 (2012), arXiv:1205.3303

Yu-Cheng Lin (Taipei)

Gergő Roósz (Budapest)

Phys. Rev. B **85**, 094417 (2012), arXiv:1201.3695

New J. Phys. **15**, 023036 (2013), arXiv:1210.7328

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Quantum quench dynamics

- Phenomena
sudden change of a parameter in the Hamiltonian
 - time: $t < 0 \rightarrow t > 0$
 - Hamiltonian: $\mathcal{H}_0 \rightarrow \mathcal{H}$
 - k-th eigenstate: $|\Psi_k^{(0)}\rangle \rightarrow |\Psi_k(t)\rangle$
 $|\Psi_k(t)\rangle = \exp(-it\mathcal{H})|\Psi_k^{(0)}\rangle$
- Studied quantities
 - observable \hat{A}
 $A(t) = \langle \Psi_k(t) | \hat{A} | \Psi_k(t) \rangle$
 - correlation function
 $C_{AB}(t_1, t_2) = \langle \Psi_k | \hat{A}(t_1) \hat{B}(t_2) | \Psi_k \rangle$
 $\hat{A}(t) = \exp(-it\mathcal{H})\hat{A}\exp(it\mathcal{H})$
- Experimental realizations
 - ultracold atomic gases in optical lattices
 - sudden change of parameters (c.f. through Feshbach resonance)
- weak couplings to dissipative degrees of freedom
- coherent time evolution
- Theoretical questions
 - nature of the stationary state
 - * non-integrable models - thermalization?
 - * integrable models - quasi-thermalization?
 - decay of correlations (space and time)
 - time-evolution of the entanglement entropy
- Questions studied in this talk
 - semiclassical theory
 - boundary and finite-size effects
 - inhomogeneous systems

Model and phase diagram

XY-chain in a transverse field

$$\mathcal{H} = -\frac{1}{2} \sum_l \left[\frac{1+\gamma}{2} \sigma_l^x \sigma_{l+1}^x + \frac{1-\gamma}{2} \sigma_l^y \sigma_{l+1}^y \right] - \frac{h}{2} \sum_l \sigma_l^z$$

Transverse Ising chain: $\gamma = 1$

$\sigma_l^{x,y,z}$: Pauli-matrices at site l

Free fermion representation

$$\mathcal{H} = \sum_p \varepsilon(p) \left(\eta_p^\dagger \eta_p - \frac{1}{2} \right)$$

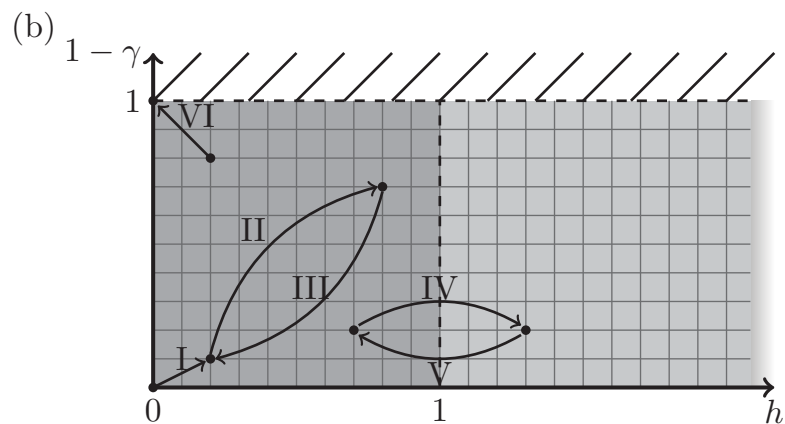
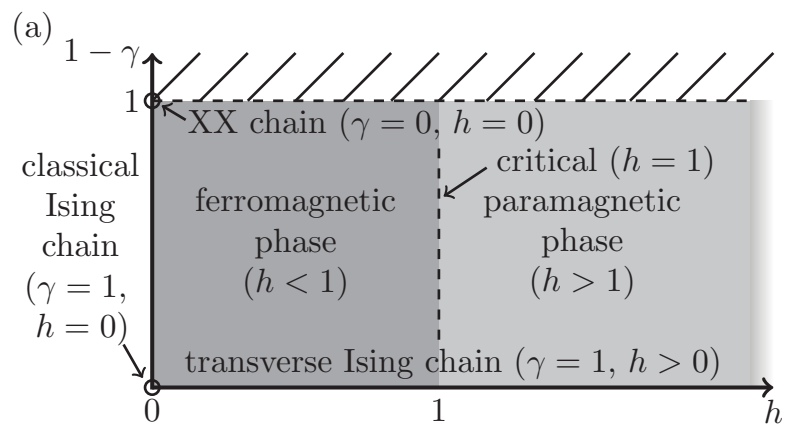
energy of modes: $\varepsilon(p) = \sqrt{\gamma^2 \sin^2 p + (h - \cos p)^2}$

Bogoliubov angle: $\tan \Theta_p = -\gamma \sin p / (h - \cos p)$

Quantum quench:

$$t < 0 \quad \rightarrow \quad t > 0$$

$$h_0, \gamma_0 \quad \rightarrow \quad h, \gamma$$



equilibrium phase diagram and 6 quench protocols

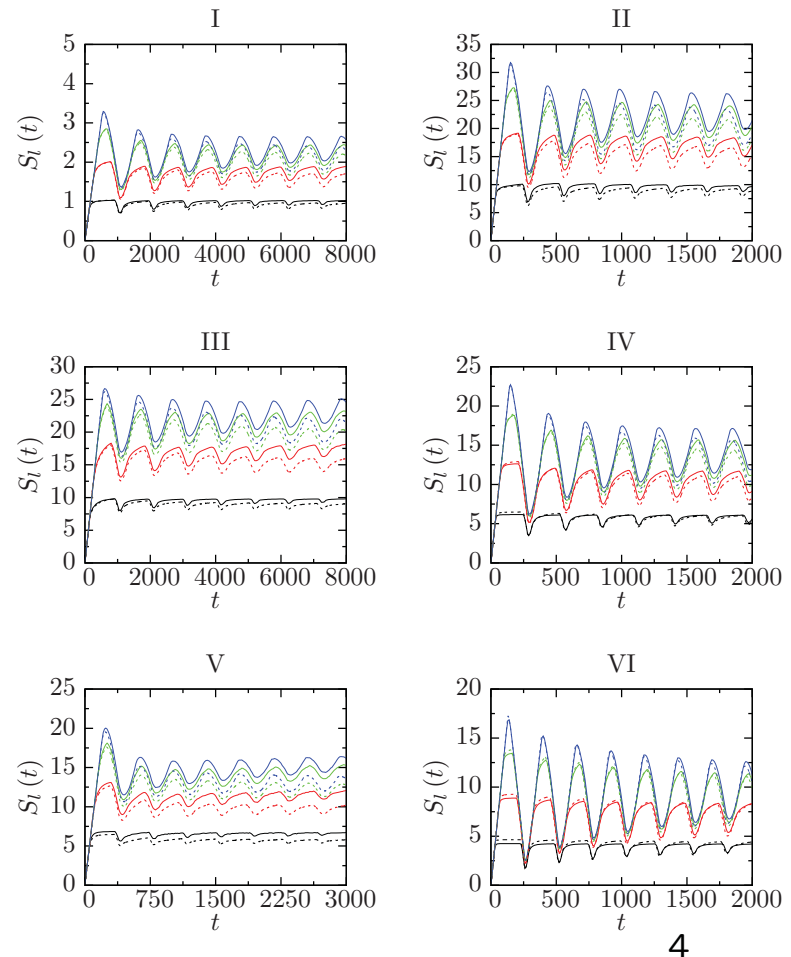
Time-evolution of the entanglement entropy

- Block of $i = 1, 2, \dots, \ell$ spins
- entanglement (von Neumann) entropy:

$$\mathcal{S}_\ell = \text{Tr}_{i \leq \ell} [\rho_\ell \log \rho_\ell]$$
- reduced density matrix:

$$\rho_\ell = \text{Tr}_{i > \ell} |\Psi_0^{(0)}\rangle \langle \Psi_0^{(0)}|$$
- its time evolution

$$\rho_\ell(t) = \exp(i\mathcal{H}t)\rho_\ell \exp(-i\mathcal{H}t)$$
- in the free-fermion basis reconstructed from the $2\ell \times 2\ell$ correlation matrix
- time-regimes
 - starting linear increase
 - constant plateau
 - linear decrease
 - (quasi)periodic repetition



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dynamical entropy for the 6 quench protocols
 $(L = 256, l = 32 : -, 64 : -, 96 : -, 128 : -)$.

Semiclassical theory - entropy

- at $t = 0$ QP pairs are created

$$\eta_{\pm p}^\dagger |\Psi_0^{(0)}\rangle$$

homogeneously in space, which are quantum entangled.

- for $t > 0$ they move ballistically with constant velocity $\pm v_p$:

$$v_p = \frac{\partial \varepsilon(p)}{\partial p} = \frac{\sin p [h - (1 - \gamma^2) \cos p]}{\varepsilon(p)}$$

- if at $t > 0$ one of the QP pairs is in the block and its partner is in the environment: there is a contribution s_p to the entanglement entropy. Otherwise the contribution is 0.

- \mathcal{S}_ℓ = sum of the contributions of the pairs

- $s_p = -(1 - f_p) \ln(1 - f_p) - f_p \ln f_p$
entropy of free fermions

- $f_p = f_p(h_0, \gamma_0; h, \gamma)$: the occupation probability of mode p

$$f_p = \langle \Psi_0^{(0)} | \eta_p^\dagger \eta_p | \Psi_0^{(0)} \rangle = \frac{1}{2} (1 - \cos \Delta_p)$$

- Δ_p : difference of the Bogoliubov angles:

$$\cos \Delta_p = \frac{(\cos p - h_0)(\cos p - h) + \gamma \gamma_0 \sin^2 p}{\varepsilon(p) \varepsilon^0(p)}$$

- In the $L \rightarrow \infty$, $\ell \gg 1$ limit:

$$\mathcal{S}_\ell(t) = t \frac{1}{2\pi} \int_0^\pi dp v_p s_p, \quad t < l/v_{max}$$

$$\mathcal{S}_\ell(t) = l \frac{1}{2\pi} \int_0^\pi dp s_p, \quad t \gg l/v_{max}$$

exact results (Fagotti & Calabrese)

- for finite L excellent agreement

Correlation function and local magnetization

General two-point function:

$$C^{xx}(l_1, t_1; l_2, t_2) = \langle \Psi_0^{(0)} | \sigma_{l_1}^x(t_1) \sigma_{l_2}^x(t_2) | \Psi_0^{(0)} \rangle$$

Equal-time correlation function:

$$C_t^{xx}(l_1, l_2) \equiv C^{xx}(l_1, t; l_2, t)$$

for large separation:

$$C_t^{xx}(l_1, l_2) = m_{l_1}(t) m_{l_2}(t)$$

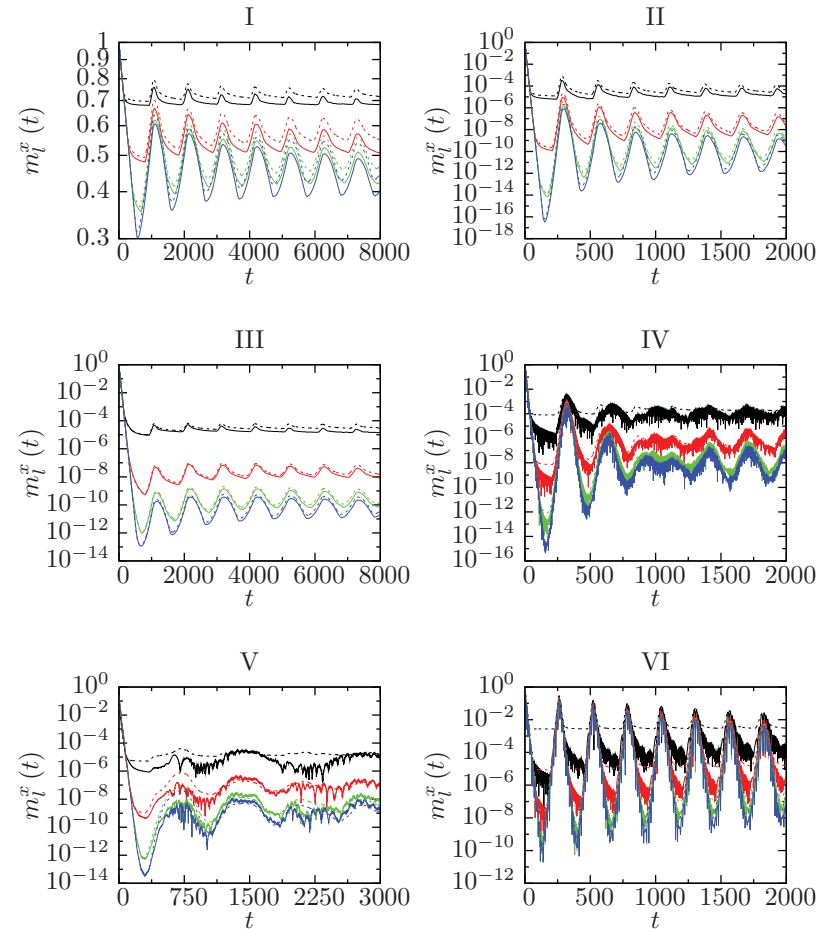
local magnetization:

$$m_l(t) = \lim_{b \rightarrow 0^+} b \langle \Psi_0^{(0)} | \sigma_l^x(t) | \Psi_0^{(0)} \rangle_b$$

b : strength of a longitudinal field

time regimes:

- starting exponential decrease
- stationary regime
- exponential reconstruction
- (quasi)periodic repetition



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local magnetization for the 6 quenches
 ($L = 256, l = 32 : -$, $64 : -$, $96 : -$, $128 : -$).

Semiclassical theory - correlations

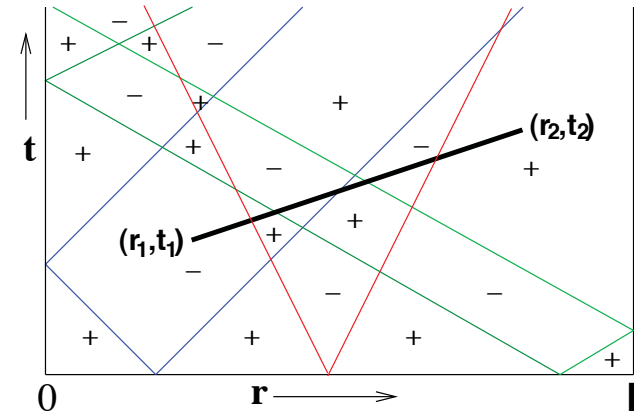
- The QPs in the spin basis are (for small h and $1 - \gamma$)
 wave packets: $\eta_p^\dagger |\Psi_0^{(0)}\rangle \rightarrow \sum_k a_k |k\rangle$
 superposition of kinks at position k
 $|k\rangle = |++\cdots+-\cdots-\rangle$
 $a_k \propto \sin(k\pi/L)$, $k = 1, \dots, L$.
- if the space-time trajectory of a kink passes the line: $(\mathbf{r}_1, \mathbf{t}_1; \mathbf{r}_2, \mathbf{t}_2)$ the correlation function changes sign
- the correlation function has its initial value if even number of kinks has passed the line
- calculation of the correlation function
 - denote by $Q(r_1, t_1; r_2, t_2)$ the probability that a given kink has passed odd times the line
 - the probability that a given set of n kinks has passed (each odd times): $Q^n(1 - Q)^{L-n}$

– summing over all possibilities

$$\frac{C(r_1, t_1; r_2, t_2)}{C_{\text{eq}}(r_1, r_2)}$$

$$= \sum_{n=0}^L (-1)^n Q^n (1 - Q)^{L-n} \frac{L!}{n!(L-n)!}$$

$$= (1 - 2Q)^L \approx \boxed{\exp[-2Q(r_1, t_1; r_2, t_2)L]}$$



Typical semiclassical contribution to the correlation function $C(r_1, t_1; r_2, t_2)$.

Semiclassical theory - magnetization

$$m_l(t) = m_l^{\text{eq}} \exp[-2q(t,l)L]$$

Here

$$q(t,l) = \frac{1}{2\pi} \int_0^\pi dp f_p q_p(t,l)$$

and

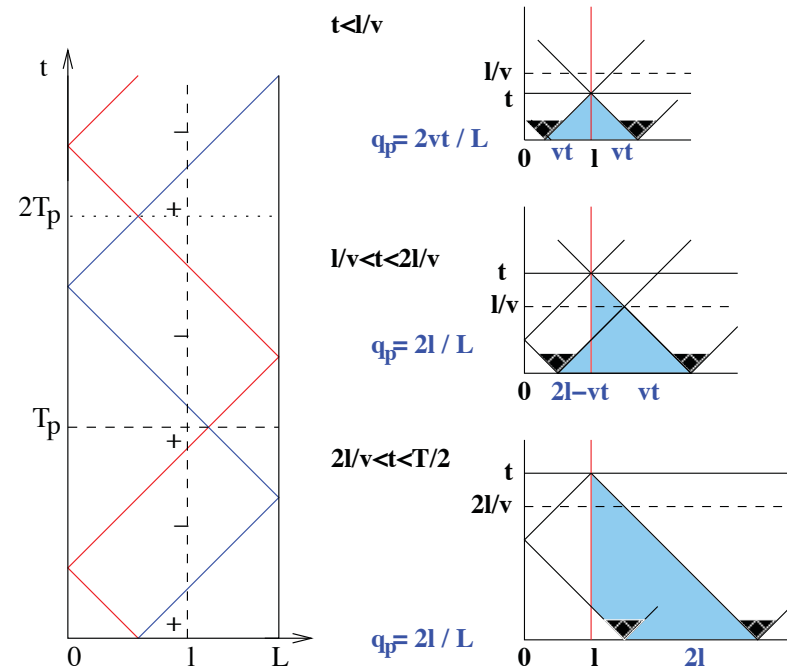
$$q_p(t,l) = \frac{1}{L} \int_0^L dx_0 q_p(x_0, t, l)$$

$q_p(x_0, l, t) = 1$, if the p kink-pair of initial position x_0 pass the site l an odd number of times before t

$q_p(x_0, l, t) = 0$, otherwise

$$Lq_p(t) = \begin{cases} 2v_p t & \text{for } t \leq t_1 \\ 2l & \text{for } t_1 \leq t \leq t_2 \\ 2 - 2v_p t & \text{for } t_2 \leq t < T_p \end{cases}$$

with $t_1 = l/v_p$, $T_p = L/v_p$ and $t_2 = T_p - t_1$.



Left: Typical semi-classical contribution to the time dependence of the local magnetization $m_l(t)$. **Right:** Sketch of the trajectories of kink pairs that flip the spin at position l exactly once for times $t < T_p/2$. q_p is the fraction of the marked intervals on the $t = 0$ -axis.

Half-infinite system: $L \rightarrow \infty$

- Local magnetization

$$m_l(t) = m_l(0) \exp \left(-t \frac{2}{\pi} \int_0^\pi dp v_p f_p \theta(l - v_p t) - l \frac{2}{\pi} \int_0^\pi dp f_p \theta(v_p t - l) \right)$$

relaxation time:

$$\tau_{\text{mag}}^{-1} = \frac{2}{\pi} \int_0^\pi dp v_p f_p$$

correlation length:

$$\xi_{\text{mag}}^{-1} = \frac{2}{\pi} \int_0^\pi dp f_p$$

- Correlation function

$$C_t^{xx}(r) = C_0^{xx}(r) \exp \left(-t \frac{4}{\pi} \int_0^\pi dp v_p f_p \theta(r - 2v_p t) - l \frac{2}{\pi} \int_0^\pi dp f_p \theta(2v_p t - r) \right)$$

- The characteristic time and length scales are:

$$\tau_{\text{corr}} = \tau_{\text{mag}}/2, \quad \xi_{\text{corr}} = \xi_{\text{mag}}.$$

- Excellent agreement for quenches deep in the ferromagnetic phase.

- For quenches close to the critical point use the modified occupation probability:

$$f_p \rightarrow \tilde{f}_p = -\frac{1}{2} \ln |\cos \Delta_p|$$

- exact in the thermodynamic limit (Calabrese, Essler, Fagotti 2012).

- Excellent agreement for finite systems, too.

Generalized Gibbs ensemble (GGE)

Question of thermalization

Compare the equilibrium correlation length at $T > 0$ temperature, ξ_T , with the nonequilibrium correlation length after the quench

- semiclassical theory (Sachdev & Young)

$$\xi_T^{-1} = \frac{2}{\pi} \int_0^\pi dp \exp\left(-\frac{\varepsilon(p)}{T}\right)$$

to be compared with:

$$\xi^{-1} = \frac{2}{\pi} \int_0^\pi dp f_p$$

from this follows

$$f_p = \exp\left(-\frac{\varepsilon(p)}{T_{\text{eff}}(p)}\right)$$

- exact result (Barouch & McCoy)

$$\xi_T^{-1} = -\frac{1}{\pi} \int_0^\pi dp \ln \left| \tanh \frac{\varepsilon(p)}{2T} \right|$$

to be compared with:

$$\xi^{-1} = \frac{2}{\pi} \int_0^\pi dp \tilde{f}_p$$

from this follows

$$f_p = \frac{1}{\exp\left(\frac{\varepsilon(p)}{T_{\text{eff}}(p)}\right) + 1}$$

Conclusion

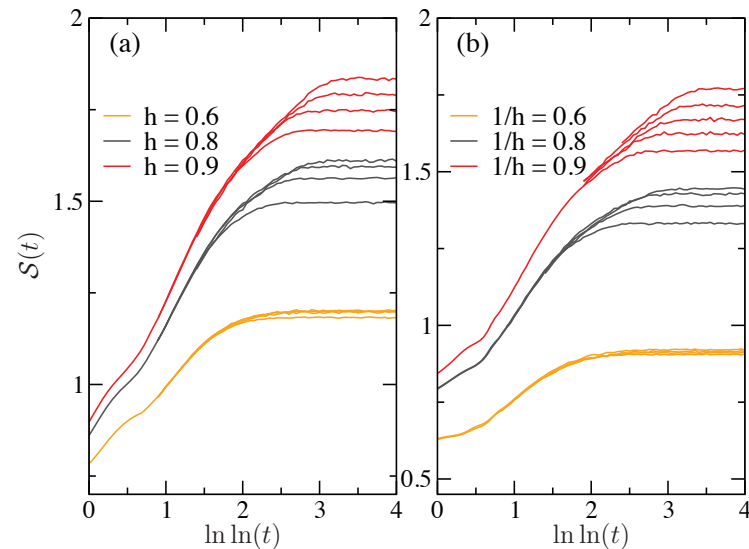
- mode-dependent effective temperature: $T_{\text{eff}}(p)$
- Generalized Gibbs ensemble

SC theory	→	exact results
classical kinks	→	free fermions
Boltzmann distr.	→	Fermi distr.

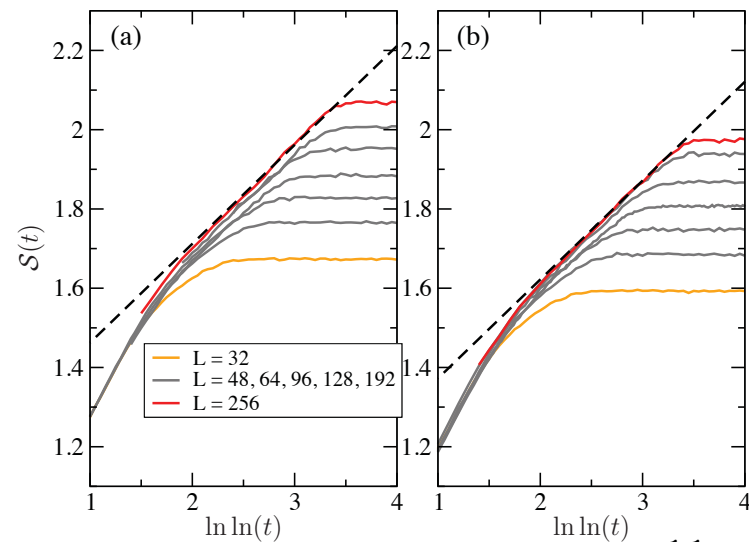
Random transverse Ising chains

$$\mathcal{H} = -\frac{1}{2} \sum_l J_l \sigma_l^x \sigma_{l+1}^x - \frac{1}{2} \sum_l h_l \sigma_l^z$$

- J_l couplings and h_l transverse fields are i.i.d. random numbers
- $p(J)$ uniform distribution in $[0, 1]$
- $\rho(h)$ uniform distribution in $[0, h]$
- phase diagram:
 - $h < 1$ ferromagnetic
 - $h = 1$ critical
 - $h > 1$ paramagnetic
- quasiparticles
 - are localized in the noncritical states
 $x^2(t) \rightarrow cst. \rightarrow S(t) \rightarrow cst.$
 - move with Sinai-diffusion in the critical point
 $x^2(t) \sim \ln^4 t, \quad S \sim \ln x(t) \rightarrow S \sim \ln \ln t$



Quench to noncritical states



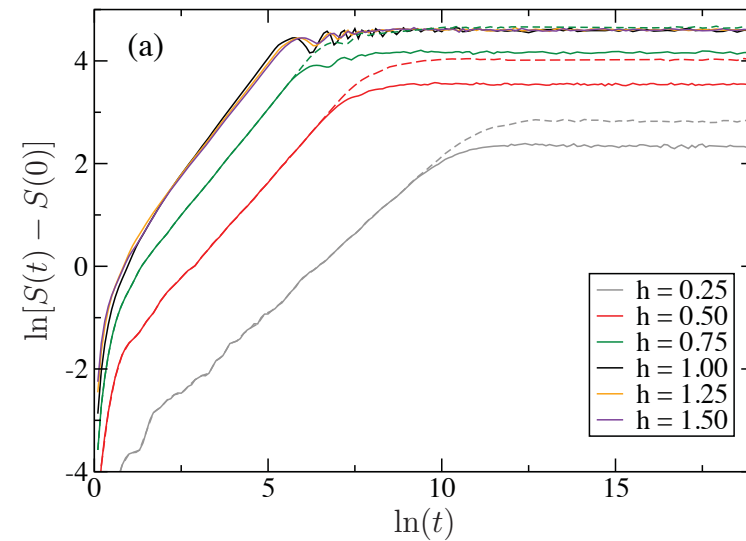
Quench to the critical point¹¹

Quasiperiodic transverse Ising chains

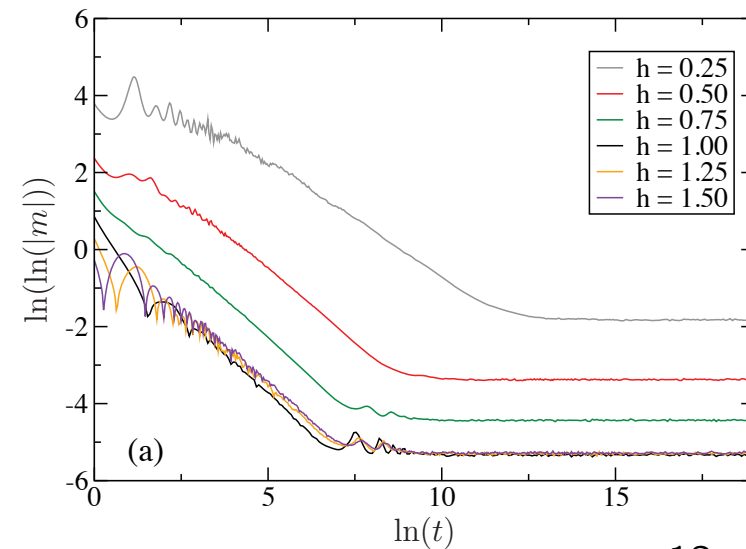
$$\mathcal{H} = -\frac{1}{2} \sum_l J_l \sigma_l^x \sigma_{l+1}^x - \frac{h}{2} \sum_l \sigma_l^z$$

- $J_l = r^{f_l}$
- $r > 0$ amplitude of the inhomogeneity
- $f_l = 0$ or 1 , according to a quasiperiodic sequence
- Fibonacci-chain: $0 \rightarrow 01$ and $1 \rightarrow 0$
- Numerical results:
 - entropy: $S(t) \sim t^\sigma$, $\sigma(r) < 1$
 - magnetization: $m(t) \sim \exp(-Ct^\mu)$, $\mu(r) < 1$
- interpretation through quasiparticles:
 - move with anomalous diffusion

$$x(t) \sim t^D(r), \quad D(r) < 1$$
 - $\sigma(r) \approx \mu(r) \approx D(r)$



entropy: $r = 0.75$ and $h_0 = 0$



magnetization: $r = 0.75$ and $h_0 = 0$

Summary - semiclassical theory

- during the quench QP-s are created
- in homogeneous chains
 - the QP-s are created homogeneously in space
 - the creation probability is p -dependent: f_p
 - they move ballistically with a semiclassical velocity: v_p
 - entanglement is due to correlated QP-pairs
one in the subsystem, its partner in the environment
 - for correlations the QP-s are represented by kinks
 - correlations are reduced (exponentially) due to passing kinks
 - finite width of the kinks is taken into account: $f_p \rightarrow \tilde{f}_p$
 - semiclassical theory is in agreement with the GGE
- in inhomogeneous systems
 - the QP-s are correlated
 - in random chains
 - * QP-s are localised in noncritical states
 - * they move with Sinai diffusion at the critical point
 - in quasicrystals
 - * they move diffusively