Isoperimetric inequalities and the structure of metric spaces

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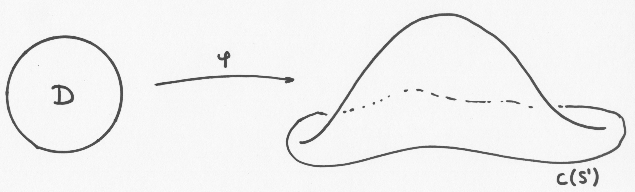
Overview - Part 1

Area of Lipschitz discs

X metric space, e.g. Riemannian manifold



For $\varphi\colon D \to X$ Lipschitz define

$$\boxed{\operatorname{Area}(\varphi) = \int_X N(\varphi, x)\, d\mathscr{H}^2(x),}$$

where $N(\varphi, x) = \#\{z \in D : \varphi(z) = x\}$.

Filling area function for metric spaces

Remarks:

- If φ injective then $\mathrm{Area}(\varphi) = \mathscr{H}^2(\varphi(D))$.
- If X Riemannian manifold then $\mathrm{Area}(\varphi) = \int_D \mathbf{J}_2(d_z\varphi)dz$.

For $c\colon S^1 \to X$ Lipschitz define

$$\boxed{\mathrm{Fillarea}_0(c) = \inf\left\{\mathrm{Area}(\varphi) : \varphi\colon D \to X \text{ Lip}, \ \varphi|_{S^1} = c\right\}}$$

Definition

The filling area function (isoperimetric function) in X is

$$\mathrm{FA}_0^X(r) = \sup\left\{\mathrm{Fillarea}_0(c) : \mathrm{L}(c) \leq r\right\}$$

for $r \geq 0$.Here, $\mathrm{L}(c)$ denotes length of c.

$\mathrm{FA}_0(r)$ and non-positive curvature

Examples:

1. Isoperimetric inequality in $\mathbb{R}^n$ for $n \geq 2$:
$$\mathrm{FA}_0^{\mathbb{R}^n}(r) = \frac{1}{4\pi} r^2 \quad \forall r \geq 0.$$
2. X 1-connected Riem. manifold of sectional curvature $\leq \kappa$:

Theorem (Bol, Fiala, Huber, ...)

- *If* $\kappa = 0$ *then* $\mathrm{FA}_0^M(r) \leq \frac{1}{4\pi} r^2$ *for all* $r \geq 0$.
- *If* $\kappa < 0$ *then* $\mathrm{FA}_0^M(r) \leq |\kappa|^{-\frac{1}{2}} r$ *for all* $r \geq 0$.

Quadratic versus linear growth!

Questions about $\mathrm{FA}_0(r)$

Questions:

1. Possible growth of $\mathrm{FA}_0^X(r)$ as $r \to \infty$?
    - For X general?
    - For X in a given class, e.g. nilpotent Lie groups?
2. What does growth of $\mathrm{FA}_0^X(r)$ tell about X?

Growth of functions:

For $f, g : [0, \infty) \to [0, \infty)$ non-decreasing define

- $f \preceq g$ if for some $C > 0$ and all $r \geq 0$
$$f(r) \leq Cg(Cr + C) + Cr + C.$$
- $f \simeq g$ if $f \preceq g$ and $g \preceq f$.

Goal of lecture course

Note:

- $\simeq$ is equivalence relation
- if $\alpha, \beta \geq 1$ then $r^\alpha \simeq r^\beta$ if and only if $\alpha = \beta$
- $e^{2r} \simeq e^{3r}$

Goal of lecture course:

Study growth of $\mathrm{FA}_0^X(r)$ as $r \to \infty$ using

Geometric Measure Theory in metric spaces.

Main tools:

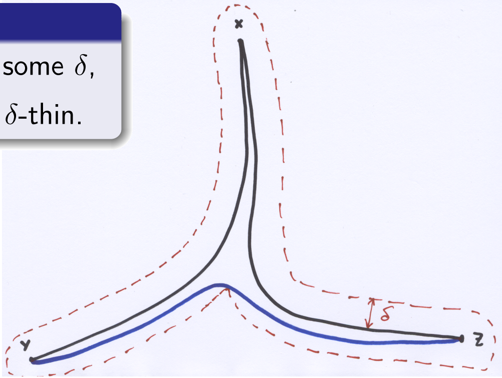
- Currents in metric spaces (Ambrosio-Kirchheim)
- Differentiability of Lipschitz maps to metric spaces (Kirchheim)

Gromov hyperbolic spaces

X geodesic metric space

Definition

X is Gromov hyperbolic if, for some δ, every geodesic triangle in X is δ-thin.



Gromov hyperbolicity

$\leftrightsquigarrow$

coarse negative curvature

Gromov hyperbolicity and $\mathrm{FA}_0(r)$

Examples:

1. X geodesic space with $\operatorname{diam} X < \infty$.
2. X metric tree.
3. X simply-connected Riem. manifold with $\sec_X \leq \kappa < 0$.
4. $X := [0,1] \times \mathbb{R}$ with Euclidean metric.

Theorem (Gromov)

If X is Gromov hyperbolic and $\mathrm{FA}_0^X(r) < \infty$ for all $r \geq 0$ then

$$\mathrm{FA}_0^X(r) \preceq r.$$

There is a converse:

If $\mathrm{FA}_0^X(r) \preceq r$ then X is Gromov hyperbolic.

Gap in the isoperimetric spectrum

Theorem (Gromov, Bowditch, Drutu, Papasoglu, Short)

X geodesic metric space. If there exists $r_0 \geq 0$ such that

$$\mathrm{FA}_0^X(r) \leq \frac{1}{4000} r^2$$

for all $r \geq r_0$ then X is Gromov hyperbolic. In particular,

$$\mathrm{FA}_0^X(r) \preceq r.$$

Consequence: No X has

$$r^\alpha \preceq \mathrm{FA}_0^X(r) \preceq r^\beta$$

for some $1 < \alpha \leq \beta < 2$.

Gromov hyperbolicity and the sharp bound

Strengthening of Gromov's theorem:

Theorem (W.)

X geodesic metric space. If there exist $\varepsilon > 0$ and $r_0 \geq 0$ such that

$$\mathrm{FA}_0^X(r) \leq \frac{1-\varepsilon}{4\pi} r^2 \tag{1}$$

for all $r \geq r_0$ then X is Gromov hyperbolic.

Remarks:

1. Theorem is optimal because $\mathrm{FA}_0^{\mathbb{R}^2}(r) = \frac{1}{4\pi} r^2$.
2. If (1) holds for all $r \geq 0$ then X is metric tree.
3. Best known constant for Riem. manifolds was $\frac{1}{16\pi}$ (Gromov).
4. What if (1) holds with $\varepsilon = 0$?

Borderline case $\varepsilon = 0$

Theorem (Lytchak-W.)

If X is proper geodesic metric space with

$$\mathrm{FA}_0^X(r) \leq \frac{1}{4\pi} r^2$$

for all $r \geq 0$ then X is $\mathrm{CAT}(0)$, i.e. has non-positive curvature.

Remark: X is $\mathrm{CAT}(0)$ if geodesic Δ in X are thinner than in $\mathbb{R}^2$.

Example: X simply-connected Riem. manifold with $\sec_X \leq 0$.

Back to isoperimetric spectrum

Already seen:

There is nothing between

linear and quadratic growth for $\mathrm{FA}_0(r)$.

Super-quadratic growth of $FA_0(r)$

Question: Other gaps in the isoperimetric spectrum? $\rightsquigarrow$ No!

Theorem (Grimaldi-Pansu)

Let f be smooth with $f' > 0$ and such that for every $k \in \mathbb{N}$

$$f(k \cdot r) \geq k \cdot f(r) \quad \text{for all } r \gg 1.$$

If $f(r) \succeq r^2$ then there exists a surface of revolution M with

$$FA_0^M(r) \simeq f(r).$$

Nilpotent Lie groups

G connected, 1-connected Lie group, with Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$.

Definition

G is nilpotent if $\exists\, k \geq 1$ (called the step of G) with

$$\mathfrak{g} =: \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(k)} \supset \mathfrak{g}^{(k+1)} = \{0\},$$

where

$$\mathfrak{g}^{(i+1)} = \left[\mathfrak{g}^{(i)}, \mathfrak{g}\right] = \operatorname{span}\left\{[v, w] : v \in \mathfrak{g}^{(i)}, w \in \mathfrak{g}\right\}.$$

G will always be endowed with left-invariant Riemannian metric d_0.

Difficult open problem: What is the possible growth of $\mathrm{FA}_0^X(r)$ for

$$X = (G, d_0)?$$

Heisenberg groups

Example: The n-th Heisenberg group is $H^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with

$$(x, y, z) \star (x', y', z') = (x + x', y + y', z + z' + \langle x, y' \rangle).$$

Basis of left-invariant vector fields on H^n:

$$X_i = \frac{\partial}{\partial x_i}, \quad Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

$[X_i, Y_j] = \delta_{ij} Z$, $[X_i, Z] = [Y_j, Z] = 0 \Rightarrow H^n$ is nilpotent of step 2.

Proposition (Thurston)

The first Heisenberg group H^1 satisfies

$$\mathrm{FA}_0^{H^1}(r) \simeq r^3.$$

Heisenberg groups

Theorem (Gromov, Allcock)

For $n \geq 2$ the n-th Heisenberg group H^n satisfies

$$\mathrm{FA}_0^{H^n}(r) \simeq r^2.$$

Generalization of Thurston's result

Theorem (Baumslag-Miller-Short, Pittet, Gersten)

If G is a free nilpotent Lie group of step k then

$$\mathrm{FA}_0^G(r) \simeq r^{k+1}$$

Example: First Heisenberg group H^1 is free nilpotent.

Theorem (Gromov, Pittet, Gersten-Holt-Riley)

If G is nilpotent of step k and contains a lattice or is Carnot then

$$\mathrm{FA}_0^G(r) \preceq r^{k+1}.$$

Does $FA_0^G(r)$ always grow exactly polynomially?

Question: If G is nilpotent does there exist $n \in \mathbb{N}$ such that

$$FA_0^G(r) \simeq r^n?$$

Answer: No!

Theorem (W.)

There exist nilpotent Lie groups of step 2 such that

$$r^2 \varrho(r) \preceq FA_0^G(r) \preceq r^2 \log r$$

for some function ϱ with $\varrho(r) \to \infty$ as $r \to \infty$.

$\mathrm{FA}_0(r)$ for nilpotent Lie groups

For nilpotent groups, $\mathrm{FA}_0(r)$ is not well understood.

Motivation for studying $\mathrm{FA}_0(r)$

Motivation 1:

Problem: Given metric spaces X and Y, how to determine whether X and Y look 'alike' from far away?

Trivial observation: if X and Y are biLipschitz homeomorphic then

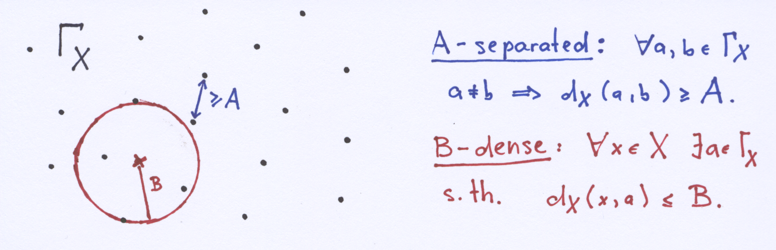
$$\mathrm{FA}_0^X(r) \simeq \mathrm{FA}_0^Y(r).$$

Remains true if X and Y are 'biLipschitz at large scale' and have bounded geometry. More precisely, ...

Quasi-isometries

Two metric spaces X, Y are quasi-isometric if there exist $\Gamma_X \subset X$ and $\Gamma_Y \subset Y$ such that

- $\exists\, \psi : \Gamma_X \to \Gamma_Y$ bijective and biLipschitz;
- Γ_X, Γ_Y are A-separated and B-dense for some $A, B > 0$.



$\mathrm{FA}_0(r)$ is a quasi-isometry invariant

Note: X and Y quasi-isometric $\not\Rightarrow \mathrm{FA}_0^X(r) \simeq \mathrm{FA}_0^Y(r)$.

Theorem (Gromov, Bridson)

Let M, N be the universal covers of closed Riemannian manifolds. If M and N are quasi-isometric then

$$\mathrm{FA}_0^M(r) \simeq \mathrm{FA}_0^N(r).$$

Proof: uses adapted version of Federer-Fleming's Deformation theorem.

Dehn function of a group

Motivation 2:

Let $G = \langle\, S \mid R \,\rangle$ be finitely presented group:

$$G \cong F(S)/N(R),$$

S finite set, $R \subset F(S)$ finite, $F(S)$ free group generated by S.

Example: $\mathbb{Z}^2 = \langle\, a, b \mid aba^{-1}b^{-1} \,\rangle$.

Define word length of $w = s_1 \cdot \dots \cdot s_n \in F(S)$ by

$$|w| = n.$$

Dehn function of a group

For $w \in F(S)$ with $w =_G e$ define 'filling area' of w by

$$\mathrm{A}(w) = \min\left\{k : w = \prod_{i=1}^{k} g_i r_i^{\pm 1} g_i^{-1} \text{ for some } g_i \in F(S),\ r_i \in R\right\}.$$

The Dehn function of G is defined by

$$\boxed{\delta_G(n) = \max\left\{\,\mathrm{A}(w) : w \in F(S), w =_G e, |w| \leq n\right\}.}$$

Theorem (Gromov, Bridson)

If G is the fundamental group of a closed Riem. manifold M then

$$\delta_G(n) \simeq \mathrm{FA}_0^{\tilde{M}}(n).$$

Word problem for groups

Motivation 3:

Let G be a finitely generated group, generators $g_1, \dots, g_k$.

Word:
$$w = g_{j_1}^{\pm 1} \cdot \dots \cdot g_{j_m}^{\pm 1}.$$

Word problem: $\exists$ algorithm which determines whether a given word represents the identity in G?

Theorem

Suppose G is finitely presented and $G = \pi_1(M)$, where M is closed Riemannian manifold. Let $\tilde{M}$ be universal cover of M. Then the word problem in G is solvable if and only if $\mathrm{FA}_0^{\tilde{M}}(r)$ is computable.