# Lecture Notes on Regularity of Area Minimizing Currents for the Summer School "Geometric Measure Theory and Optimal Trasport" ICTP Trieste July 2013

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# Introduction

The solution of the Plateau problem in higher codimension may show new singular behavior with respect to the case of area minimizing hypersurfaces. A simple way to illustrate this phenomenon, together with the main analytical challenges, is to proceed via examples.

**0.1. Branch points in complex curves.** As discovered by Federer in the '70s, every complex variety in $\mathbb{C}^n$ is an example of a locally area minimizing integer rectifiable current. In particular, we can consider the case of complex curves in $\mathbb{C}^2$, for instance

$$\mathcal{W} := \big\{(z,w) \in \mathbb{C}^2 : z^3 = w^2\big\}.$$

If we identify $\mathbb{C}$ with $\mathbb{R}^2$, it is simple to see that $\mathcal{W}$ is an immersed (real) 2-dimensional submanifold, globally parametrized, for instance, by the map

$$u : \big\{(\rho,\theta) : \rho > 0,\ \theta \in [0,2\pi)\big\} \longrightarrow \mathbb{R}^4$$

given by

$$(\rho,\theta) \longmapsto \big(\rho\cos\theta, \rho\sin\theta, \rho^{\frac{3}{2}}\cos(3\theta), \rho^{\frac{3}{2}}\sin(3\theta)\big).$$

(Verify as exercise that $u$ is an immersion).

Nevertheless, $\mathcal{W}$ is not an embedded submanifold in a neighborhood of the origin, because it is not a graph over the plane $\Pi := Du(0)(\mathbb{R}^2) = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^4$ in any neighborhood of 0. Indeed, for every $z = (\rho,\theta) \in \Pi \simeq \mathbb{C}$, there exist two points

$$w_1 = \left(\rho^{\frac{3}{2}}\cos\left(\frac{3}{2}\theta\right), \rho^{\frac{3}{2}}\sin\left(\frac{3}{2}\theta\right)\right),$$
$$w_2 = \left(\rho^{\frac{3}{2}}\cos\left(\frac{3}{2}\theta + \pi\right), \rho^{\frac{3}{2}}\sin\left(\frac{3}{2}\theta + \pi\right)\right) = -w_1,$$

such that $(z,w_1),(z,w_2) \in \mathcal{W}$. The point $0 \in \mathcal{W}$ is a typical example of *branch point*: i.e., said informally, the multiple valued function $z \mapsto \{w_1(z), w_2(z)\}$ is discontinuous whenever going around an arbitrary small circuit around the origin.

**0.2.** The most challenging aspect of the branch points in minimal surface theory is that, unless the case of codimension 1, they may be "flat" points in terms of the excess. Since the pioneering work by De Giorgi [**4**], it is known that the main regularity parameter for minimizing currents is the *Excess*, i.e. a integral norm of the oscillation of the tangent plane to a current $T$ (we denote by $m$ its dimension):

$$\mathbf{E}(T, B_r(p)) = \min_{\vec{\pi}} \frac{1}{\omega_m\, r^m} \int_{B_r(p)} |\vec{T} - \vec{\pi}|^2 \, d\|T\|,$$

where the minimum is taken among all oriented $m$-plane $\vec{\pi}$. The celebrated Allard's $\varepsilon$-regularity theorem (proved for minimizing hypersurfaces first by De Giorgi) says that the regular point of a stationary current $T$ of codimension 1 are all and only the points $p \in \mathrm{spt}(T)$ for which there exists $r > 0$ such that $\mathbf{E}(T, B_r(p)) \leq \varepsilon$ with $\varepsilon > 0$ a dimensional constant.

In higher codimension (the minimizing current $\mathcal{W}$ is an example), this criterium of regularity fails in general, and the Excess cannot be used any more in the same way for understanding the singular behavior of branch points. In fact this failure, more than being a simple technical issue, represents one of the main reasons for the differences between the theory of minimizing codimension 1 minimal currents and higher codimension ones (cp. Theorem 0.1 below with the partial regularity for minimizing hypersurfaces).

**0.3. Almgren's big regularity paper.** Starting from this considerations, Almgren developed in the late '70s a different, new approach to the study of higher codimension area minimizing currents, which culminated in the proof of his celebrated partial regularity result [**1**].

Theorem 0.1 (Almgren). *Let $T$ be an $m$-dimensional, locally area minimizing current in $\mathbb{R}^{m+n}$ and without boundary. Then, $T$ is represented by the integration over a smooth $m$-dimensional submanifold $M$ such that $\bar{M} \setminus M$ has Hausdorff dimension at most $m-2$.*

In particular, in order to deal with the fact that around a branch point the current cannot be parametrized by a function, he introduced a new object which has played a fundamental rôle in the development of the theory.

**0.4. The linear theory: multiple valued functions.** If it is true that $\mathcal{W}$ is not a graph of a function in any neighborhood of the origin, nevertheless it is a graph of a 2-valued function from the right reference plane $\Pi$. This behavior is typical to any branch point in a complex varieties: locally, one can always find a plane of the dimension of the variety such that latter can be written as a graph of multiple valued function *with a fix number of values*. This point of view, moreover, is particularly favorable for the questions in geometric measure theory, where the notion of slice allows to represent (some directions of) a currents in terms of an integral of lower dimensional currents.

Therefore, for a nonparametric theory of branch points, Almgren introduced the notion of multiple valued functions with fixed number of values, called $Q$-valued functions, and, being the area functional defined for maps with a weak gradient (with the right summability), he also developed the theory of Sobolev multiple valued functions and of the analogous of harmonic functions.

**0.5. Frequency Function.** One of the most important discoveries of Almgren in this context is a new monotonic quantity, called *Frequency Function*. This function is given the ratio between the Dirichlet energy and the $L^2$ norm of the boundary and is at the very base of the estimate on the dimension of the singular set. Loosely speaking, it express the fact that energy of an harmonic function is decreasing in relation to the norm of the function itself as we zoom around any given point; and this turns out to reduce the analysis of the singularities to that of a special class of functions, namely the radially homogeneous ones.

**0.6. The general case: a blow-up proof.** The attempt to transfer the regularity of the linearized equations to the case of minimizing currents passes through many different conceptual difficulties. In particular, as in all the regularity theories for minimal surfaces, one has to face the problem of the approximation of the surface via graphs (in this case multiple valued!) which in principle are not everywhere pointwise related to the original currents (thus implying that they do not solve any variational principle or any partial differential equation).

Remarkably Almgren succeeded to carry on the entire program, giving a proof of Theorem 0.1 by a blow-up argument, which linked the estimate of the singular set of a current to those of appropriate approximate solutions of the linearized problem. In this process many new analytical challenges emerged, one of which is briefly commented in this lecture notes and related with the choice of a proper system of coordinates (which is able to "center" the different sheets of a minimal surface around a branch point).

## 1. Plan of the lectures

In these lecture notes I want to present some of the main analytical issues about the regularity of higher codimension minimal surfaces. In particular, I will focus mainly on the linear theory, trying to discuss some of the main analytical tools used in the analysis. More in details, this is a plan of the lectures.

- In the first chapter I introduce the formalism of multiple valued functions, together with the basic notion of differentiability;
- In Chapter 2, we prove the existence and the continuity of the solution to the linearized problem, namely of Dirichlet minimizing multiple valued functions;
- In Chapter 3, we will go through the estimate of the singular set of multiple valued functions, which uses the fundamental estimate on the monotonicity of the Frequency Function;
- Finally, in the last chapter we study a very special case of the centering problem mentioned in the previous paragraph. This is the only part of the notes which deals with the nonlinear theory of minimizing currents.

The material of these notes is taken from the following papers written in collaboration with Camillo De Lellis (University of Zurich): [**6, 5**].

CHAPTER 1

# $Q$-valued functions

## 1. Space of $Q$-points

In order to give a representation of branch points in terms of graph of functions we consider multiple valued functions. Roughly speaking, a $Q$-valued function is a map taking $Q$ values in $\mathbb{R}^n$ which are not necessarily distinct and, most importantly, *unordered*. A simple way to formalize this idea is considering maps with values in the set of positive atomic measures of mass $Q$.

Definition 1.1 (Unordered $Q$-tuples). We denote by $[\![P_i]\!]$ the Dirac mass in $P_i \in \mathbb{R}^n$ and we define the space of $Q$-points as

$$\mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] \,:\, P_i \in \mathbb{R}^n \text{ for every } i = 1, \dots, Q \right\}.$$

For the sake of simplicity, we sometimes write $\mathcal{A}_Q$ and $\sum_i [\![P_i]\!]$ when $n$ and $Q$ are clear from the context. Note that the points $P_i$ are not distinct (for example, $Q\,[\![P]\!] \in \mathcal{A}_Q(\mathbb{R}^n)$).

**1.1.** Note that $\mathcal{A}_Q(\mathbb{R}^n)$ is noting else than the quotient of $(\mathbb{R}^n)^Q$ via the action of the group of permutations of $Q$ indexes $\mathcal{P}_Q$. In other words, defining the equivalence relation

$$(P_1, \dots, P_Q) \sim (P_{\sigma(1)}, \dots, P_{\sigma(Q)}) \quad \forall\, \sigma \in \mathcal{P}_Q,$$

it is clear that $\mathcal{A}_Q \simeq (\mathbb{R}^n)^Q/_\sim$. It follows then that the space of $Q$-points, though it is a singular space, inherits many properties from the Euclidean space (cp. §3). Nevertheless, for many purposes it is more convenient to discard the combinatorics involved in this representation and view it as a genuinely abstract metric space.

**1.2.** This is indeed what we are going to do next, namely we endow $\mathcal{A}_Q(\mathbb{R}^n)$ with a metric which makes it a complete metric space (the completeness is an elementary exercise left to the reader).

Definition 1.2. For every $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n)$, with $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!]$, we define

$$\mathcal{G}(T_1, T_2) := \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum_i \left|P_i - S_{\sigma(i)}\right|^2},$$

where we recall that $\mathcal{P}_Q$ denotes the group of permutations of $Q$ indexes $\{1, \dots, Q\}$.

Remark 1.3. Note that $\mathcal{G}$ coincides with the $L^2$-Wasserstein distance on the space of positive measures with finite second moment (see for instance [**3**] and [**12**]).

## 2. Sobolev $Q$-valued functions

The metric theory of $Q$-valued functions starts from this remark. Let for the rest of the paper $\Omega \subset \mathbb{R}^m$ be a bounded open subset with smooth boundary.

DEFINITION 1.4. A $Q$-valued function is simple a map $u : \Omega \to (\mathcal{A}_Q(\mathbb{R}^n), \mathcal{G})$. We say that $u$ is continuous (resp. Lipschitz, Hölder and measurable) is it is so as function between metric spaces. Similarly, $u \in L^p(\Omega, \mathcal{A}_Q)$ if $x \mapsto \mathcal{G}(u(x), Q\,[\![0]\!]) \in L^p(\Omega)$ (observe that, since $\Omega$ is bounded, this is equivalent to ask that $\|\mathcal{G}(u,T)\|_{L^p}$ is finite for every $T \in \mathcal{A}_Q$).

**2.1.** It is simple to show that every measurable $Q$-valued function can be written as the "sum" of $Q$ measurable functions.

PROPOSITION 1.5 (Measurable selection). *Let $B \subset \mathbb{R}^m$ be a measurable set and let $f : B \to \mathcal{A}_Q$ be a measurable function. Then, there exist $f_1, \ldots, f_Q$ measurable functions such that*

$$f(x) = \sum_i [\![f_i(x)]\!] \quad \text{for a.e. } x \in B. \tag{1.1}$$

EXERCISE 1. Prove Proposition 1.5 by induction on the number of values $Q$.

We use this result only to simplify our notation. Obviously, such a choice is far from being unique, but, in using notation (1.1).

**2.2.** We now introduce the Sobolev spaces of functions taking values in the metric space of $Q$-points. Here we use the approach in [**6**], which goes back to the pioneering work by Ambrosio [**2**] for more general metric spaces.

DEFINITION 1.6. A measurable function $f : \Omega \to \mathcal{A}_Q$ is in the Sobolev class $W^{1,p}$ ($1 \le p \le \infty$) if there exists a positive $\varphi \in L^p(\Omega)$ such that, $\forall\, T \in \mathcal{A}_Q$,

(i) $x \mapsto \mathcal{G}(f(x), T) \in W^{1,p}(\Omega)$;
(ii) $|D\,\mathcal{G}(f,T)| \le \varphi$ a.e. in $\Omega$.

EXERCISE 2. Prove that $u \in W^{1,p}(\Omega)$ if and only if there exists $\psi \in L^p(\Omega)$ such that, for every $F : \mathcal{A}_Q \to \mathbb{R}$ Lipschitz,

$$F \circ u \in W^{1,p}(\Omega) \quad \text{and} \quad |D(F \circ u)| \le \mathrm{Lip}(F)\,\psi \ \text{ a.e. in } \Omega$$

**2.3.** The above definition allows to give a notion of "modulus" of the gradient of a Sobolev function, depending only on the metric structure of the target. Various choices are possible, depending on the kind of problems considered. For what concerns the harmonic energy, the correct one is the following: fix a countable dense subset $\{T_i\}_{i\in\mathbb{N}}$ of $\mathcal{A}_Q$ and, for every $j = 1, \ldots, m$, set

$$|\partial_j f| = \sup_{i\in\mathbb{N}} |\partial_j\, \mathcal{G}(f, T_i)| \quad \text{and} \quad |Df|^2 := \sum_{j=1}^m |\partial_j f|^2. \tag{1.2}$$

EXERCISE 3. Show that the definition of $|\partial_j f|$ in (1.2) is well-posed (i.e. does not depend on the choice of the countable set) and is characterized by the following property:

(*i*) $|\partial_j \mathcal{G}(f,T)| \leq |\partial_j f|$ a.e. for every $T \in \mathcal{A}_Q$;
(*ii*) if $\varphi_j \in L^p$ is such that $|\partial_j \mathcal{G}(f,T)| \leq \varphi_j$ for all $T \in \mathcal{A}_Q$, then $|\partial_j f| \leq \varphi_j$ a.e.

## 3. Pointwise differential

In the special case $Q$-valued maps, it is also possible to give a notion of pointwise derivative. This is possible due to the "nonlinear Euclidean" structure of the space of $Q$-points shortly described in §1.1.

Definition 1.7. Let $f : \Omega \to \mathcal{A}_Q$ and $x_0 \in \Omega$. We say that $f$ is differentiable at $x_0$ if there exist $Q$ matrices $L_i$ satisfying:

(i) $\mathcal{G}(f(x), T_{x_0} f) = o(|x - x_0|)$, where

$$T_{x_0} f(x) := \sum_i \llbracket L_i \cdot (x - x_0) + f_i(x_0) \rrbracket\,; \tag{1.3}$$

(ii) $L_i = L_j$ if $f_i(x_0) = f_j(x_0)$.

The $Q$-valued map $T_{x_0} f$ will be called the *first-order approximation* of $f$ at $x_0$. The point $\sum_i \llbracket L_i \rrbracket \in \mathcal{A}_Q(\mathbb{R}^{n\times m})$ will be called the differential of $f$ at $x_0$ and is denoted by $Df(x_0)$.

In the sequel we fix the notation $Df_i$ for $L_i$ in Definition 1.7. Note that by (ii) this is unambiguous: namely, if $g_1, \ldots, g_Q$ is a different selection for $f$, $x_0$ a point of differentiability and $\pi$ a permutation such that $g_i(x_0) = f_{\pi(i)}(x_0)$ for all $i \in \{1, \ldots, Q\}$, then $Dg_i(x_0) = Df_{\pi(i)}(x_0)$. When the $f_i$'s are a smooth selection, then $f$ is differentiable and the $Df_i$'s coincide with the classical differentials.

If $D$ is the set of points of differentiability of $f$, the map $x \mapsto Df(x)$ is a $Q$-valued map, which we denote by $Df$. In a similar fashion, we define the directional derivatives

$$\partial_\nu f(x) = \sum_i \llbracket Df_i(x) \cdot \nu \rrbracket\,,$$

and establish the notation $\partial_\nu f = \sum_i \llbracket \partial_\nu f_i \rrbracket$.

**3.1.** Point wise derivatives exists almost everywhere, as implied by the following generalized version of Rademacher's Theorem for multiple valued functions.

Theorem 1.8. *Let $f : \Omega \to \mathcal{A}_Q$ be a Lipschitz function. Then, $f$ is differentiable almost everywhere in $\Omega$.*

Proof. We prove the theorem for $Q = 2$ and leave the details of the general case to the readers (cp. [**6**, Theorem 1.13]). Let $\tilde\Omega$ be the set of points where $f$ takes a single value with multiplicity 2:

$$\tilde\Omega = \big\{x \in \Omega \,:\, f_1(x) = f_2(x)\big\}.$$

Note that $\tilde\Omega$ is closed. In $\Omega \setminus \tilde\Omega$, $f_1$ and $f_2$ are differentiable almost everywhere by the classical Rademacher's Theorem (see, for instance, 3.1.2 of [**10**]), and

$$T_{x_0} f(x) := \llbracket Df_1 \cdot (x - x_0) + f_1(x_0) \rrbracket + \llbracket Df_2 \cdot (x - x_0) + f_2(x_0) \rrbracket$$

is first order approximation for $f$ at almost every point $x_0 \in \Omega \setminus \tilde\Omega$.

We only need to show that $f$ is differentiable a.e. in $\tilde\Omega$. Since $f_1|_{\tilde\Omega}$ is a Lipschitz function, we can consider a Lipschitz extension of it to all $\Omega$, denoted by $g$. We claim that $f$ is differentiable in all the points $x_0 \in \tilde\Omega$ where

(a) $\tilde\Omega$ has density 1;

(b) $g$ is differentiable,

with the first order expansion given by $T_{x_0} f(y) := 2\,[\![Dg(x_0)\cdot(y-x_0)+g(x_0)]\!]$. Our claim would conclude the proof, because both conditions (a) and (b) are verified on a subset of full measure of $\tilde\Omega$.

In order to show the claim, for every $y\in\mathbb{R}^m$, let $r=|y-x_0|$ and choose $y^*\in\tilde\Omega\cap\overline{B_{2r}(x_0)}$ such that

$$|y-y^*| = \mathrm{dist}\left(y, \tilde\Omega\cap\overline{B_{2r}(x_0)}\right).$$

Being $f$, $g$ and $Tg$ Lipschitz with constant at most $\mathrm{Lip}(f)$, we infer that

$$\begin{aligned}\mathcal{G}\big(f(y),T_{x_0}f(y)\big) &\leq \mathcal{G}\big(f(y),f(y^*)\big)+\mathcal{G}\big(f(y^*),T_{x_0}f(y^*)\big)+\mathcal{G}\big(T_{x_0}f(y^*),T_{x_0}f(y)\big)\\ &\leq \mathrm{Lip}(f)\,|y-y^*| + Q\,\mathrm{Lip}(f)\,|y-y^*| +\\ &\quad + \mathcal{G}\big(2\,[\![g(y^*)]\!]\,, 2\,[\![Dg(x_0)\cdot(y^*-x_0)+g(x_0)]\!]\,\big)\\ &\leq (Q+1)\,\mathrm{Lip}(f)\,|y-y^*| + o\big(\,|y^*-x_0|\,\big),\end{aligned} \tag{1.4}$$

where we used the differentiability of $g$ at $x_0$. Since $|y^*-x_0|\leq 2\,r = 2\,|y-x_0|$, it remains to estimate $\rho := |y-y^*|$. Note that the ball $B_\rho(y)$ is contained in $B_r(x_0)$ and does not intersect $\tilde\Omega$. Therefore

$$|y-y^*| = \rho \leq C\,\left|B_{2r}(x_0)\setminus\tilde\Omega\right|^{1/m} \leq C(m)\,r\left(\frac{|B_{2r}(x_0)\setminus\tilde\Omega|}{|B_{2r}(x_0)|}\right)^{\frac{1}{m}}. \tag{1.5}$$

Since $x_0$ is a point of density 1, we can conclude from (1.5) that $|y-y^*| = |y-x_0|\,o(1)$. Inserting this inequality in (1.4), we conclude that $\mathcal{G}(f(y),T_{x_0}f(y)) = o(|y-x_0|)$, which shows that $T_{x_0}f$ is the first order expansion of $f$ at $x_0$. □

**3.2.** It is now very simple to prove that every Sobolev $Q$-valued function is in fact approximately pointwise differentiable at almost every point.

Definition 1.9. A $Q$-valued function $f$ is approximately differentiable in $x_0$ if there exists a measurable subset $\tilde\Omega\subset\Omega$ containing $x_0$ such that $\tilde\Omega$ has density 1 at $x_0$ and $f|_{\tilde\Omega}$ is differentiable at $x_0$.

Indeed, as for the classical theory one can prove that a Lusin-type approximation holds.

Proposition 1.10. *There exists a constant $C=C(m,\Omega,Q)$ with the following property. For every $f\in W^{1,p}(\Omega,\mathcal{A}_Q)$ and every $\lambda>0$, there exists a set $\Omega_\lambda$ such that $\mathrm{Lip}\,(f|_{\Omega_\lambda})\leq C\,\lambda$ and*

$$|\Omega\setminus\Omega_\lambda| \leq \frac{C\||Df|\|^p_{L^p}}{\lambda^p}. \tag{1.6}$$

*In particular, $f$ is approximate differentiable at almost every point.*

PROOF. We consider the case $1 \leq p < \infty$ ($p = \infty$ is immediate - why?) and we set

$$\Omega_\lambda = \big\{x \in \Omega \,:\, M(|Df|) \leq \lambda\big\},$$

where $M$ is the Maximal Function Operator. By rescaling, we can assume $\||Df|\|_{L^p} = 1$. Notice that, for every $T \in \mathcal{A}_Q$ and every $j \in \{1, \ldots, m\}$,

$$M\big(\,|\partial_j \mathcal{G}(f,T)|\,\big) \leq M(|Df|) \leq \lambda \quad \text{in } \Omega_\lambda.$$

By standard calculation (see, for example, 6.6.3 in [**10**]), we deduce that, for every $T$, $\mathcal{G}(f,T)$ is $(C\,\lambda)$-Lipschitz in $\Omega_\lambda$, with $C = C(m)$. Therefore,

$$\big|\mathcal{G}(f(x),T) - \mathcal{G}(f(y),T)\big| \leq C\,\lambda\,|x-y| \quad \forall\, x, y \in \Omega_\lambda \text{ and } \forall\, T \in \mathcal{A}_Q. \tag{1.7}$$

From (1.7), we get a Lipschitz estimate for $f|_{\Omega_\lambda}$ by setting $T = f(x)$. The standard weak $(p-p)$ estimate for maximal functions yields

$$|\Omega \setminus \Omega_\lambda| \leq \frac{C}{\lambda^p} \int_{\Omega\setminus\Omega_{\lambda/2}} |Df|^p.$$

□

**3.3.** A generalized Kirzsbraun's Theorem holds for $Q$-valued functions (cp. [**6**, Theorem 1.7]).

THEOREM 1.11. *Let $B \subset \mathbb{R}^m$ and $f : B \to \mathcal{A}_Q(\mathbb{R}^n)$ be Lipschitz. Then, there exists an extension $\bar{f} : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ of $f$, with* $\mathrm{Lip}(\bar{f}) \leq C(m,Q)\,\mathrm{Lip}(f)$.

In particular, a simple corollary is the following extension of Proposition 1.10.

PROPOSITION 1.12. *Let $f$ be in $W^{1,p}(\Omega, \mathcal{A}_Q)$. For every $\lambda > 0$, there exists a Lipschitz $Q$-function $f_\lambda$ such that* $\mathrm{Lip}\,(f_\lambda) \leq \lambda$ *and*

$$\big|\big\{x \in \Omega \,:\, f(x) \neq f_\lambda(x)\big\}\big| \leq \frac{C}{\lambda^p} \int_{\Omega\setminus\Omega_{\lambda/2}} |Df|^p, \tag{1.8}$$

*where the constant $C$ depends only on $Q$, $m$ and $\Omega$.*

**3.4.** We can now prove that the "modulus" of the gradient as defined in §2.3 and the pointwise differential are linked.

PROPOSITION 1.13. *For every $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ and every $j = 1, \ldots, m$, we have*

$$|Df|^2 = \sum_i |Df_i|^2 \quad a.e. \tag{1.9}$$

*(where $|L|$ denotes the Hilbert-Schmidt norm of the matrix $L$).*

PROOF. By Proposition 1.12, we assume from now on that $f$ is Lipschitz. Moreover, by the definition of $|Df|$, it suffices to show that

$$|\partial_j f|^2 = \sum_i |\partial_j f_i|^2 \quad \text{a.e.} \quad \forall\, j = 1, \ldots, m. \tag{1.10}$$

We recall the definition of $|\partial_j f|$: for a countable dense set $\{T_l\}_{l\in\mathbb{N}}\subset\mathcal{A}_Q$,

$$|\partial_j f| = \sup_{l\in\mathbb{N}} |\partial_j \mathcal{G}(f,T_l)|\,.$$

On the set $E_l=\{x\in\Omega : f(x)=T_l\}$ both $|\partial_j f|$ and $\sum_i|\partial_j f_i|^2$ vanish a.e. Hence, it suffices to show (1.10) on any point $x_0$ where $f$ and all $\mathcal{G}(f,T_l)$ are differentiable and $f(x_0)\notin\{T_l\}_{l\in\mathbb{N}}$.

Fix such a point, which, without loss of generality, we can assume to be the origin, $x_0=0$. Let $T_0 f$ be the first oder approximation of $f$ at 0. Since $\mathcal{G}(\cdot,T_l)$ is a Lipschitz function, we have $\mathcal{G}(f(y),T_l)=\mathcal{G}(T_0f(y),T_l)+o(|y|)$. Therefore, $g(y):=\mathcal{G}(T_0f(y),T_l)$ is differentiable at 0 and $\partial_j g(0)=\partial_j\mathcal{G}(f,T_l)(0)$.

We assume, without loss of generality, that $\mathcal{G}(f(0),T_l)^2=\sum_i|f_i(0)-P_i|^2$, where $T_l=\sum_i[\![P_i]\!]$. Next, we consider the function

$$h(y):=\sqrt{\sum_i|f_i(0)+Df_i(0)\cdot y-P_i|^2}.$$

Then, $g\le h$. Since $h(0)=g(0)$, we conclude that $h-g$ has a minimum at 0. Recall that both $h$ and $g$ are differentiable at 0 and $h(0)=g(0)$. Thus, we conclude $\nabla h(0)=\nabla g(0)$, which in turn yields the identity

$$\partial_j\,\mathcal{G}(f,T_l)(0)=\partial_j g(0)=\partial_j h(0)=\sum_i\frac{(f_i(0)-P_i)\cdot\partial_j f_i(0)}{\sqrt{\sum_i|f_i(0)-P_i|^2}}. \tag{1.11}$$

Using the Cauchy-Schwartz inequality and (1.11), we deduce that

$$|\partial_j f|(0)^2=\sup_{l\in\mathbb{N}}|\partial_j\mathcal{G}(f,T_l)(0)|^2\le\sum_i|\partial_j f_i(0)|^2\,. \tag{1.12}$$

If the right hand side of (1.12) vanishes, then we clearly have equality. Otherwise, let $Q_i=f_i(0)+\lambda\,\partial_j f_i(0)$, where $\lambda$ is a small constant to be chosen later, and consider $T=\sum_i[\![Q_i]\!]$. Since $\{T_l\}$ is a dense subset of $\mathcal{A}_Q$, for every $\varepsilon>0$ we can find a point $T_l=\sum_i[\![P_i]\!]$ such that

$$P_i=f_i(0)+\lambda\,\partial_j f_i(0)+\lambda\,R_i,\quad \text{with } |R_i|\le\varepsilon \text{ for every } i.$$

Now we choose $\lambda$ and $\varepsilon$ small enough to ensure that $\mathcal{G}(f(0),T_l)^2=\sum_i|f_i(0)-P_i|^2$ (indeed, recall that, if $f_i(0)=f_k(0)$, then $\partial_j f_i(0)=\partial_j f_k(0)$). So, we can repeat the computation above and deduce that

$$\partial_j\,\mathcal{G}(f,T_l)(0)=\sum_i\frac{(f_i(0)-P_i)\cdot\partial_j f_i(0)}{\sqrt{\sum_i|f_i(0)-P_i|^2}}=\sum_i\frac{(\partial_j f_i(0)+R_i)\cdot\partial_j f_i(0)}{\sqrt{\sum_i|\partial_j f_i(0)+R_i|^2}}.$$

Hence,

$$|\partial_j f|(0)\ge\sum_i\frac{(\partial_j f_i(0))^2+\varepsilon|\partial_j f_i(0)|}{\sqrt{\sum_i(|\partial_j f_i(0)|+\varepsilon)^2}}.$$

Letting $\varepsilon\to 0$, we obtain the inequality $|\partial_j f|(0)\ge\sum_j(\partial_j f_i(0))^2$. □

CHAPTER 2

# Dir-minimizing $Q$-valued functions

The generalized *Dirichlet energy* of a $Q$-valued function $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ is given by

$$\mathrm{Dir}(f,\Omega) := \int_\Omega |Df|^2 = \sum_i \int_\Omega |Df_i|^2.$$

We say that a function is *Dir-minimizing* if $\mathrm{Dir}(f,\Omega) \leq \mathrm{Dir}(g,\Omega)$, for all $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$ such that

$$\mathcal{G}(f,T)|_{\partial\Omega} = \mathcal{G}(g,T)|_{\partial\Omega} \quad \text{for all} \quad T \in \mathcal{A}_Q, \tag{2.1}$$

where the last inequality is intended in the sense of traces. When (2.1) holds, we write simply $f|_{\partial\Omega} = g|_{\partial\Omega}$.

## 1. Existence of Dir-minimizers

The first issue to be settle is the existence of Dir-minimizing functions.

THEOREM 2.1. *Let $g \in W^{1,2}(\Omega, \mathcal{A}_Q)$. Then, there exists a* Dir*-minimizing function $f \in W^{1,2}(\Omega, \mathcal{A}_Q)$ such that $f|_{\partial\Omega} = g|_{\partial\Omega}$.*

The proof of this result follows by the application of the direct method in the calculus of variations.

PROOF. For what concerns the $Q$-valued functions and the generalized Dirichlet energy, the same results as in the classical setting hold, namely

(a) the sequentially weak compactness;
(b) the continuity of the trace under weak convergence;
(c) and the weak sequentially lower semicontinuity of the energy.

We can then argue as follows. Let $f_k \in W^{1,2}(\Omega, \mathcal{A}_Q)$ be a minimizing sequence of functions, i.e. $f_k|_{\partial\Omega} = g|_{\partial\Omega}$ and

$$\lim_{k\to+\infty} \mathrm{Dir}(f_k,\Omega) = \min\left\{\mathrm{Dir}(h,\Omega) \,:\, h \in W^{1,2}(\Omega, \mathcal{A}_Q) \quad \text{with} \quad h|_{\partial\Omega} = g|_{\partial\Omega}\right\}.$$

Then,

(a) by sequentially weak compactness, there exists a subsequence $(f_{k_l})_l$ which is $L^2$ converging to some function $f$:

$$\lim_{l\to+\infty} \|\mathcal{G}(f_{k_l}, f)\|_{L^2(\Omega)} = 0;$$

(b) by continuity of the trace under weak convergence, $f|_{\partial\Omega} = g|_{\partial\Omega}$;

(c) and by the lower semicontinuity of the Dirichlet energy we finally infer that

$$\begin{aligned}\mathrm{Dir}(f,\Omega) &\leq \lim_{l\to+\infty} \mathrm{Dir}(f_k,\Omega)\\ &= \min\left\{\mathrm{Dir}(h,\Omega)\,:\, h\in W^{1,2}(\Omega,\mathcal{A}_Q) \quad \text{with} \quad h|_{\partial\Omega}=g|_{\partial\Omega}\right\},\end{aligned}$$

thus proving that $f$ is a minimizer. □

**1.1.** The continuity of the trace operator is a simple consequence of the definitions and the results for classical Sobolev functions.

Proposition 2.2. *The following set is closed under the weak convergence (i.e. $f_k \rightharpoonup f$ if $f_k \to f$ in $L^2(\Omega)$ and $\sup_k \mathrm{Dir}(f_k,\Omega)<+\infty$)*

$$W^{1,2}_g(\Omega,\mathcal{A}_Q) := \left\{f\in W^{1,2}(\Omega,\mathcal{A}_Q)\,:\, f|_{\partial\Omega}=g|_{\partial\Omega}\right\}.$$

Proof. Let $f_k\in W^{1,2}_g(\Omega,\mathcal{A}_Q)$ be weakly convergent to $f$. Clearly $f_k \rightharpoonup f$ if and only if $\varphi\circ f_k \rightharpoonup \varphi\circ f$ for any Lipschitz function $\varphi:\mathcal{A}_Q\to\mathbb{R}$. Therefore, by the continuity of the trace under the classical weak convergence, we have that $\varphi\circ f|_{\partial\Omega}=\varphi\circ g|_{\partial\Omega}$ for every Lipschitz $\varphi$. Since two functions that coincide under the left composition with every Lipschitz functions are the same (prove this statement as exercise), it follows that $f|_{\partial\Omega}=g|_{\partial\Omega}$. □

**1.2.** For what concerns the sequential weak compactness, we prove the following special case of Sobolev embeddings.

Proposition 2.3. *If $(f_k)_k$ is a sequence such that*

$$\|\mathcal{G}(f_k, Q\,[\![0]\!])\|_{L^2}+\sum_j \||\partial_j f_k|\|_{L^2}\leq C<+\infty,$$

*then there exists $f\in L^2(\Omega,\mathcal{A}_Q)$ and a subsequence $(f_{k_j})_j$ such that $f_{k_j}\to f$ in $L^2$.*

Proof. For every $l\in\mathbb{N}$, let $f_{k,l}$ be the function given by Proposition 1.12 choosing $\lambda=l$. From the Ascoli–Arzelà Theorem and a diagonal argument, we find a subsequence $f_{k_j}$ such that, for any fixed $l$, $\{f_{k_j,l}\}_k$ is a Cauchy sequence in $C^0$. We now use this to show that $f_k$ is a Cauchy sequence in $L^2$. Indeed,

$$\|\mathcal{G}(f_k,f_{k'})\|_{L^2}\leq \|\mathcal{G}(f_k,f_{k,l})\|_{L^2}+\|\mathcal{G}(f_{k,l},f_{k',l})\|_{L^2}+\|\mathcal{G}(f_{k',l},f_{k'})\|_{L^2}. \tag{2.2}$$

We claim that the first and third terms are bounded by $C\,l^{1/q-1/p^*}$. It suffices to show it for the first term. By Proposition 1.12, we infer

$$\begin{aligned}\|\mathcal{G}(f_{k_j},f_{k_j,l})\|^2_{L^2} &\leq C\int_{\{f_{k_j}\neq f_{k_j,l}\}} \left[\mathcal{G}(f_{k_j},Q\,[\![0]\!])^2+\mathcal{G}(f_{k_j,l},Q\,[\![0]\!])^2\right]\\ &\leq \Big(\|\mathcal{G}(f_{k_j},[\![0]\!])\|^2_{L^{2^*}}+\|\mathcal{G}(f_{k_j,l},[\![0]\!])\|^2_{L^{2^*}}\Big)|\{f_k\neq f_{k_j,l}\}|^{1-2/2^*}\leq C l^{2/2^*-1},\end{aligned}$$

where in the last line we have used the Sobolev embedding for the functions $\mathcal{G}(f_{k_j},[\![0]\!])$ and $\mathcal{G}(f_{k_j,l},[\![0]\!])$ (in the critical case $2^*$) and the Hölder inequality.

Let $\varepsilon$ be a given positive number. Then, we can choose $l$ such that the first and third term in (2.2) are both less than $\varepsilon/3$, independently of $j$. On the other hand, since $\{f_{k_j,l}\}$ is a Cauchy sequence in $C^0$, there is an $N$ such that $\|\mathcal{G}(f_{k_j,l}, f_{k_{j'},l})\|_{L^q} \leq \varepsilon/3$ for every $j, j' > N$. Clearly, for $j, j' > N$, we then have $\|\mathcal{G}(f_{k_j}, f_{k_{j'}})\| \leq \varepsilon$. This shows that $\{f_k\}$ is a Cauchy sequence in $L^2$ and hence completes the proof. □

**1.3.** Finally, establishing the lower semicontinuity of the Dirichlet energy completes the proof of the existence of Dir-minimizing functions.

PROPOSITION 2.4. *Let $f_k \rightharpoonup f$ in $W^{1,2}(\Omega, \mathcal{A}_Q)$. Then,*

$$\mathrm{Dir}(f, \Omega) \leq \liminf_{k\to\infty} \mathrm{Dir}(f_k, \Omega). \tag{2.3}$$

PROOF. Let $\{T_l\}_{l\in\mathbb{N}}$ be a dense subset of $\mathcal{A}_Q$ and recall that $|\partial_j f|^2 = \sup_l \big(\partial_j \mathcal{G}(f, T_l)\big)^2$. Thus, if we set

$$h_{j,N} = \max_{l\in\{1,\ldots,N\}} \big(\partial_j \mathcal{G}(f, T_l)\big)^2,$$

we conclude that $h_{j,N} \uparrow |\partial_j f|^2$. Next, for every $N$, denote by $\mathcal{P}_N$ the collections $P = \{E_l\}_{l=1}^N$ of $N$ disjoint measurable subsets of $\Omega$. Clearly, it holds

$$h_{j,N} = \sup_{P\in\mathcal{P}} \sum_{E_l\in P} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \mathbf{1}_{E_l}.$$

By the Monotone Convergence Theorem, we conclude

$$\mathrm{Dir}(f, \Omega) = \sum_{j=1}^m \sup_N \int h_{j,N}^2 = \sum_{j=1}^m \sup_N \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2.$$

Fix now a partition $\{F_1, \ldots, F_N\}$ such that, for a given $\varepsilon > 0$,

$$\sum_l \int_{F_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \geq \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 - \varepsilon.$$

Then, we can find compact sets $\{K_1, \ldots, K_N\}$ with $K_l \subset F_l$ and

$$\sum_l \int_{K_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \geq \sup_{P\in\mathcal{P}_N} \sum_{E_l\in P} \int_{E_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 - 2\varepsilon.$$

Since the $K_l$'s are disjoint compact sets, we can find disjoint open sets $U_l \supset K_l$. So, denote by $\mathcal{O}_N$ the collections of $N$ pairwise disjoint open sets of $\Omega$. We conclude

$$\mathrm{Dir}(f, \Omega) = \sum_{j=1}^m \sup_N \int h_{j,N}^2 = \sum_{j=1}^m \sup_N \sup_{P\in\mathcal{O}_N} \sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2. \tag{2.4}$$

Note that, since $\mathcal{G}(f_k, T_l) \to \mathcal{G}(f, T_l)$ strongly in $L^2(\Omega)$, then $\partial_j \mathcal{G}(f_k, T_l) \rightharpoonup \partial_j \mathcal{G}(f, T_l)$ in $L^2(U)$ for every open $U \subset \Omega$. Hence, for every $N$ and every $P \in \mathcal{O}_N$, we have

$$\sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f, T_l)\big)^2 \leq \liminf_{k\to+\infty} \sum_{U_l\in P} \int_{U_l} \big(\partial_j \mathcal{G}(f_k, T_l)\big)^2 \leq \liminf_{k\to\infty} \int_\Omega |\partial_j f_k|^2.$$

Taking the supremum in $\mathcal{O}_N$ and in $N$, and then summing in $j$, in view of (2.4), we achieve (2.3). $\square$

## 2. Hölder continuity

A fundamental result in the theory of higher codimension area minimizing current is the Hölder continuity of Dir-minimizing $Q$-valued functions.

THEOREM 2.5. *There exist constants $\alpha = \alpha(m,Q) \in ]0,1[$ and $C = C(m,n,Q,\delta) > 0$ with the following property. If $f : B_1 \to \mathcal{A}_Q$ is* Dir*-minimizing, then*

$$[f]_{C^{0,\alpha}(\overline{B_\delta})} = \sup_{x,y\in\overline{B_\delta}} \frac{\mathcal{G}(f(x),f(y))}{|x-y|^\alpha} \leq C\,\mathrm{Dir}(f,\Omega)^{\frac{1}{2}} \quad \text{for every } 0<\delta<1.$$

The proof of Theorem 2.5 consists of two parts: the first is stated in the following proposition which gives the crucial estimate; the second is a standard application of the Campanato–Morrey estimates.

PROPOSITION 2.6. *Let $f \in W^{1,2}(B_r,\mathcal{A}_Q)$ be* Dir*-minimizing and suppose that*

$$g = f|_{\partial B_r} \in W^{1,2}(\partial B_r, \mathcal{A}_Q).$$

*Then, we have that*

$$\mathrm{Dir}(f,B_r) \leq C(m)\,r\,\mathrm{Dir}(g,\partial B_r), \tag{2.5}$$

*where $C(2) = Q$ and $C(m) < (m-2)^{-1}$.*

Assuming Proposition 2.6, we proceed with the proof of Theorem 2.5.

PROOF OF THEOREM 2.5. Set $\gamma(m) := C(m)^{-1} - m + 2$, where $C(m)$ is the constant in (2.5). Define $h(r) = \int_{B_r} |Df|^2$. Note that $h$ is absolutely continuous and that

$$h'(r) = \int_{\partial B_r} |Df|^2 \geq \mathrm{Dir}(f,\partial B_r) \quad \text{for a.e. } r, \tag{2.6}$$

where $\mathrm{Dir}(f,\partial B_r)$ is given by

$$\mathrm{Dir}(f,\partial B_r) = \int_{\partial B_r} |\partial_\tau f|^2,$$

with $|\partial_\tau f|^2 = |Df|^2 - \sum_{i=1}^Q |\partial_\nu f_i|^2$. Here $\partial_\tau$ and $\partial_\nu$ denote, respectively, the tangential and the normal derivatives. Therefore, (2.6) and (2.5) imply

$$(m-2+\gamma)\,h(r) \leq r\,h'(r). \tag{2.7}$$

Integrating this differential inequality, we obtain

$$\int_{B_r} |Df|^2 = h(r) \leq r^{m-2+\gamma}\,h(1) = r^{m-2+\gamma} \int_{B_1} |Df|^2.$$

By the Campanato–Morrey estimates (easily generalized to $Q$-valued via the Lipschitz compositions) we conclude the Hölder continuity of $f$ with exponent $\alpha = \frac{\gamma}{2}$. $\square$

**2.1.** We are now ready for the proof of Proposition 2.6.

PROOF OF PROPOSITION 2.6. We give the in the case of planar domains (i.e. $m = 2$) and $Q = 2$. The arguments for general $m = 2$ and general $Q$ are a simple generalizations of those below, while the case of higher dimension $m \geq 3$ needs different ideas (cp. [**6**, Chapter 3]).

It is enough to prove (2.5) for $r = 1$, because the general case follows from an easy scaling argument. Note that there exist two possibilities:

(a) either there exists $x_0 \in \partial B_1$ such that $g(x_0) = 2\,[\![P]\!]$ for some $P \in \mathbb{R}^n$;
(b) or $\mathrm{card}(\mathrm{spt}(g)) \equiv 2$.

In case (a) there exists two functions $\gamma_1, \gamma_2 : \partial B_1 \to \mathbb{R}^n$ such that $g = [\![\gamma_1]\!] + [\![\gamma_2]\!]$. If we consider the harmonic extensions $\zeta_1, \zeta_2$, by a simple computation on planar harmonic functions, it is easy to see that

$$\int_{B_1} |D\zeta|^2 \leq \int_{\partial B_1} |\partial_\tau \gamma|^2 . \tag{2.8}$$

We can then conclude by the minimizing property of $f$ that

$$\mathrm{Dir}(f, B_1) \leq \sum_{i=1,2} \mathrm{Dir}\left(\zeta_i, B_1\right) \overset{(2.8)}{\leq} \sum_{i=1,2} \int_{\partial B_1} |\partial_\tau \gamma_1|^2 = \mathrm{Dir}(g, \partial B_1).$$

In case (b), by the uniform continuity of functions $W^{1,2}(\partial B_1, \mathcal{A}_Q)$ (easily proven from the definition), there exists a function $\gamma : \partial B_1 \to \mathbb{R}^n$ such that

$$g(x) = \sum_{z^2 = x} [\![\gamma(z)]\!] ,$$

where we identified the plane $\mathbb{R}^2$ with $\mathbb{C}$. Consider now the harmonic extension $\zeta$ of $\gamma$ in $B_1$, and set

$$\tilde{f}(x) = \sum_{z^2 = x} [\![\zeta(z)]\!] .$$

The 2-valued function $\tilde{f}$ is an admissible competitor for $f$, because $\tilde{f}|_{\mathbb{S}^1} = f|_{\mathbb{S}^1}$. Since the Dirichlet energy is invariant under conformal transformations (cp. Exercise 4), from (2.8), we easily conclude (2.5):

$$\mathrm{Dir}(f, B_1) \leq \mathrm{Dir}\left(\tilde{f}, B_1\right) = \int_{B_1} |D\zeta|^2 \overset{(2.8)}{\leq} \int_{\partial B_1} |\partial_\tau \gamma|^2 = 2\,\mathrm{Dir}(g, \partial B_1).$$

□

EXERCISE 4. Let $\zeta \in W^{1,2}(\mathbb{D}, \mathbb{R}^n)$ and consider the $Q$-valued function $f$ defined by

$$f(x) = \sum_{z^Q = x} [\![\zeta(z)]\!] .$$

Then, the function $f$ belongs to $W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ and

$$\mathrm{Dir}(f, \mathbb{D}) = \int_{\mathbb{D}} |D\zeta|^2 . \tag{2.9}$$

Moreover, if $\zeta|_{\mathbb{S}^1} \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^n)$, then $f|_{\mathbb{S}^1} \in W^{1,2}(\mathbb{S}^1, \mathcal{A}_Q)$ and

$$\mathrm{Dir}(f|_{\mathbb{S}^1}, \mathbb{S}^1) = \frac{1}{Q} \int_{\mathbb{S}^1} |\partial_\tau \zeta|^2 . \tag{2.10}$$

CHAPTER 3

# Singular points

In this chapter we study the regularity properties of Dir-minimizing $Q$-valued functions. In particular, we introduce the following natural definition of regular and singular points.

Definition 3.1 (Regular and singular points). A $Q$-valued function $f$ is regular at a point $x \in \Omega$ if there exist $r > 0$ and $Q$ analytic functions $f_i : B_r(x) \to \mathbb{R}^n$ such that

$$f(y) = \sum_i [\![f_i(y)]\!] \quad \text{for every } y \in B_r(x)$$

and either $f_i(y) \neq f_j(y)$ for every $y \in B_r(x)$ or $f_i \equiv f_j$. The singular set $\Sigma_f$ of $f$ is the complement of the set of regular points.

Note that the regular set is by definition open and the singular set relatively closed. The aim of this chapter is to explain some of the key ideas in the proof of partial regularity result. In particular, we will always consider the case of multiple valued functions minimizing the Dirichlet energy and show the following analogous of the partial regularity result for area minimizing currents.

Theorem 3.2 (Estimate of the singular set). *Let $f$ be a* Dir*-minimizing function. Then, the singular set $\Sigma_f$ has Hausdorff dimension at most $m-2$, and it is at most countable if $m = 2$.*

## 1. First variations

**1.1. Chain rules.** Given $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$, one can consider the following different compositions:

(a) for $\Phi : \tilde{\Omega} \to \Omega$ be differentiable, we set $f \circ \Phi : \tilde{\Omega} \to \mathcal{A}_Q(\mathbb{R}^n)$ given by

$$f \circ \Phi(x) := \sum_i [\![f_i(\Phi(x))]\!]\,;$$

(b) for $\Psi : \Omega \times \mathbb{R}^n \to \mathbb{R}^k$ differentiable, we set $\Psi(x, f) : \Omega \to \mathcal{A}_Q(\mathbb{R}^k)$ given by

$$\Psi(x, f) := \sum_i [\![\Psi(x, f_i(x))]\!]\,;$$

(c) for $F : (\mathbb{R}^n)^Q \to \mathbb{R}^k$ differentiable such that

$$F(y_1, \dots, y_Q) = F\left(y_{\pi(1)}, \dots, y_{\pi(Q)}\right) \quad \forall\,(y_1, \dots, y_Q) \in (\mathbb{R}^n)^Q\,,\ \forall\, \pi \in \mathcal{P}_Q, \tag{3.1}$$

we define the function $F \circ f : \Omega \to \mathbb{R}^k$ as

$$F \circ f(x) := F(f_1(x), \dots, f_Q(x)).$$

Using the definition of first order approximation for multiple valued functions, it is easy to prove the following.

PROPOSITION 3.3. *Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be differentiable at $x_0$ and $\Phi$, $\Psi$ and $F$ as above. Then, the following differentials exist and the identities below hold:*

$$D(f \circ \Phi)(y_0) = \sum_i [\![ Df_i(x_0) \cdot D\Phi(y_0) ]\!] \quad \text{for } \Phi(y_0) = x_0;$$

$$D\Psi(x, f))(x_0) = \sum_i [\![ D_u\Psi(x_0, f_i(x_0)) \cdot Df_i(x_0) + D_x\Psi(x_0, f_i(x_0)) ]\!] \,;$$

$$D(F \circ f)(x_0) = \sum_i D_{y_i} F(f_1(x_0), \dots, f_Q(x_0)) \cdot Df_i(x_0).$$

EXERCISE 5. Prove the proposition.

**1.2.** By the use of the compositions above, we can now introduce the following variations for Dir-minimizing $Q$-valued functions:

(IV) *Inner Variations*: given $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$, for $\varepsilon$ sufficiently small, we consider the diffeomorphism $x \mapsto \Phi_\varepsilon(x) = x + \varepsilon\varphi(x)$ which leaves $\partial\Omega$ fixed, and compute

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |D(f \circ \Phi_\varepsilon)|^2 \tag{3.2}$$

$$= 2 \int \sum_i \big\langle Df_i : Df_i \cdot D\varphi \big\rangle - \int |Df|^2 \operatorname{div} \varphi. \tag{3.3}$$

(OV) *Outer Variations*: given $\psi \in C^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that $\operatorname{spt}(\psi) \subset \Omega' \times \mathbb{R}^n$ for some $\Omega' \subset\subset \Omega$, and

$$|D_u\psi| \le C < \infty \quad \text{and} \quad |\psi| + |D_x\psi| \le C\,(1 + |u|)\,, \tag{3.4}$$

we set $\Psi_\varepsilon(x) = \sum_i [\![ f_i(x) + \varepsilon\psi(x, f_i(x)) ]\!]$ and derive

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |D\,\Psi_\varepsilon|^2 \tag{3.5}$$

$$= \int \sum_i \big\langle Df_i(x) : D_x\psi(x, f_i(x)) \big\rangle \, dx + \int \sum_i \big\langle Df_i(x) : D_u\psi(x, f_i(x)) \cdot Df_i(x) \big\rangle \, dx.$$

**1.3.** Testing (3.2) and (3.5) with suitable $\varphi$ and $\psi$, we get two key identities. In what follows, $\nu$ will always denote the outer unit normal on the boundary $\partial B$ of a given ball.

PROPOSITION 3.4. *Let $x \in \Omega$. Then, for a.e. $0 < r < \operatorname{dist}(x, \partial\Omega)$, we have*

$$(m-2) \int_{B_r(x)} |Df|^2 = r \int_{\partial B_r(x)} |Df|^2 - 2\,r \int_{\partial B_r(x)} \sum_i |\partial_\nu f_i|^2, \tag{3.6}$$

$$\int_{B_r(x)} |Df|^2 = \int_{\partial B_r(x)} \sum_i \langle \partial_\nu f_i, f_i \rangle. \tag{3.7}$$

PROOF. Without loss of generality, we assume $x = 0$. We test (3.2) with a function $\varphi$ of the form $\varphi(x) = \phi(|x|)\, x$, where $\phi$ is a function in $C^\infty([0,\infty))$, with $\phi \equiv 0$ on $[r,\infty)$, $r < \mathrm{dist}(0, \partial\Omega)$, and $\phi \equiv 1$ in a neighborhood of 0. Then,

$$D\varphi(x) = \phi(|x|)\,\mathrm{Id} + \phi'(|x|)\, x \otimes \frac{x}{|x|} \quad \text{and} \quad \mathrm{div}\,\varphi(x) = m\,\phi(|x|) + |x|\,\phi'(|x|), \tag{3.8}$$

where Id denotes the $m \times m$ identity matrix. Note that

$$\partial_\nu f_i(x) = Df_i(x) \cdot \frac{x}{|x|}.$$

Then, inserting (3.8) into (3.2), we get

$$\begin{aligned} 0 = 2\int |Df(x)|^2\, \phi(|x|)\, dx + 2\int \sum_{i=1}^{Q} |\partial_\nu f_i(x)|^2\, \phi'(|x|)\, |x|\, dx \\ - m \int |Df(x)|^2\, \phi(|x|)\, dx - \int |Df(x)|^2\, \phi'(|x|)\, |x|\, dx. \end{aligned}$$

By a standard approximation procedure, it is easy to see that we can test with

$$\phi(t) = \phi_n(t) := \begin{cases} 1 & \text{for } t \le r - 1/n, \\ n\,(r-t) & \text{for } r - 1/n \le t \le r. \end{cases} \tag{3.9}$$

With this choice we get

$$\begin{aligned} 0 = (2-m)\int |Df(x)|^2\, \phi_n(|x|)\, dx - \frac{2}{n}\int_{B_r \setminus B_{r-1/n}} \sum_{i=1}^{Q} |\partial_\nu f_i(x)|^2\, |x|\, dx \\ + \frac{1}{n}\int_{B_r \setminus B_{r-1/n}} |Df(x)|^2\, |x|\, dx. \end{aligned}$$

Let $n \uparrow \infty$. Then, the first integral converges towards $(2-m)\int_{B_r} |Df|^2$. As for the second and third integral, for a.e. $r$, they converge, respectively, to

$$-r\int_{\partial B_r} \sum_{i=1}^{Q} |\partial_\nu f_i|^2 \quad \text{and} \quad r\int_{\partial B_r} |Df|^2.$$

Thus, we conclude (3.6).

Similarly, test (3.5) with $\psi(x,u) = \phi(|x|)\, u$. Then,

$$D_u\psi(x,u) = \phi(|x|)\,\mathrm{Id} \quad \text{and} \quad D_x\psi(x,u) = \phi'(|x|)\, u \otimes \frac{x}{|x|}\,. \tag{3.10}$$

Inserting (3.10) into (3.5) and differentiating in $\varepsilon$, we get

$$0 = \int |Df(x)|^2\, \phi(|x|)\, dx + \int \sum_{i=1}^{Q} \langle f_i(x), \partial_\nu f_i(x) \rangle\, \phi'(|x|)\, dx.$$

Therefore, choosing $\phi$ as in (3.9), we can argue as above and, for $n \uparrow \infty$, we conclude (3.7). □

## 2. Frequency Function

The analysis of the singularities in higher codimension minimizing currents depends deeply on a new monotonicity formula discovered by Almgren.

Definition 3.5. Let $f$ be a Dir-minimizing function, $x \in \Omega$ and $0 < r < \mathrm{dist}(x, \partial\Omega)$. We define the functions

$$D_{x,f}(r) = \int_{B_r(x)} |Df|^2, \quad H_{x,f}(r) = \int_{\partial B_r} |f|^2 \quad \text{and} \quad I_{x,f}(r) = \frac{r D_{x,f}(r)}{H_{x,f}(r)}. \tag{3.11}$$

$I_{x,f}$ is called the *frequency function*.

When $x$ and $f$ are clear from the context, we will often use the shorthand notation $D(r)$, $H(r)$ and $I(r)$. Note that, by Theorem 2.5, $|f|^2$ is a continuous function. Therefore, $H_{x,f}(r)$ is a well-defined quantity for every $r$. Moreover, if $H_{x,f}(r) = 0$, then, by minimality, $f|_{B_r(x)} \equiv 0$. So, except for this case, $I_{x,f}(r)$ is always well defined.

**2.1.** The reason why $I$ is called *frequency* function can be explained by looking at its value on the planar harmonic functions $f_k(r, \theta) = r^k \cos(k\,\theta)$: it is very simple to compute that $I_{0,f_k}(r) \equiv k$ is the corresponding frequency of the angular parameter.

**2.2.** The most important estimate in the analysis of singular points is the following monotonicity formula.

Theorem 3.6. *Let $f$ be* Dir*-minimizing and $x \in \Omega$. Either there exists $\varrho$ such that $f|_{B_\varrho(x)} \equiv 0$ or $I_{x,f}(r)$ is an absolutely continuous nondecreasing positive function. In particular, in the latter case there exists the limit*

$$I_{x,f}(0) = \lim_{r \to 0} I_{x,f}(r).$$

Proof. Without loss of generality assume $x = 0$. $D$ is an absolutely continuous and

$$D'(r) = \int_{\partial B_r} |Df|^2 \quad \text{for a.e. } r. \tag{3.12}$$

As for $H(r)$, note that $|f|$ is the composition of $f$ with a Lipschitz function, and therefore belongs to $W^{1,2}$. It follows that $|f|^2 \in W^{1,1}$ and hence that $H \in W^{1,1}$. In order to compute $H'$, note that the distributional derivative of $|f|^2$ coincides with the approximate differential a.e. Therefore, Proposition 3.3 justifies for a.e. $r$ the following computation:

$$\begin{aligned} H'(r) &= \frac{d}{dr} \int_{\partial B_1} r^{m-1} \, |f(ry)|^2 dy = (m-1) r^{m-2} \int_{\partial B_1} |f(ry)|^2 dy + \int_{\partial B_1} r^{m-1} \frac{\partial}{\partial r} |f(ry)|^2 \, dy \\ &= \frac{m-1}{r} \int_{\partial B_r} |f|^2 + 2 \int_{\partial B_r} \sum_i \langle \partial_\nu f_i, f_i \rangle. \end{aligned}$$

Using (3.6), we then conclude

$$H'(r) = \frac{m-1}{r} H(r) + 2D(r). \tag{3.13}$$

Note, in passing, that, since $H$ and $D$ are continuous, $H \in C^1$ and (3.13) holds pointwise.

If $H(r) = 0$ for some $r$, then, as already remarked, $f|_{B_r} \equiv 0$. In the opposite case, we conclude that $I \in C \cap W^{1,1}_{loc}$. To show that $I$ is nondecreasing, it suffices to compute its derivative a.e. and prove that it is nonnegative. Using (3.12) and (3.13), we infer that

$$\begin{aligned} I'(r) &= \frac{D(r)}{H(r)} + \frac{r\,D'(r)}{H(r)} - r\,D(r)\frac{H'(r)}{H(r)^2} \\ &= \frac{D(r)}{H(r)} + \frac{r\,D'(r)}{H(r)} - (m-1)\frac{D(r)}{H(r)} - 2r\frac{D(r)^2}{H(r)^2} \\ &= \frac{(2-m)D(r) + r\,D'(r)}{H(r)} - 2\,r\,\frac{D(r)^2}{H(r)^2} \quad \text{for a.e. } r. \end{aligned} \tag{3.14}$$

Recalling (3.6) and (3.7) and using the Cauchy–Schwartz inequality, from (3.14) we conclude that, for almost every $r$,

$$I'(r) = \frac{r}{H(r)^2}\left\{\int_{\partial B_r(x)} |\partial_\nu f|^2 \cdot \int_{\partial B_r(x)} |f|^2 - \left(\int_{\partial B_r(x)} \sum_i \langle \partial_\nu f_i, f_i\rangle\right)^2\right\} \geq 0. \tag{3.15}$$

□

**2.3.** The same computations as above yield the following two important corollaries.

Corollary 3.7. *Let $f$ be* Dir*-minimizing in $B_\varrho$. Then, $I_{0,f}(r) \equiv \alpha$ if and only if $f$ is $\alpha$-homogeneous, i.e.*

$$f(y) = |y|^\alpha f\left(\frac{y\,\varrho}{|y|}\right). \tag{3.16}$$

Proof. Let $f$ be a Dir-minimizing $Q$-valued function. Then, $I(r) \equiv \alpha$ if and only if equality occurs in (3.15) for almost every $r$, i.e. if and only if there exist constants $\lambda_r$ such that

$$f_i(y) = \lambda_r\,\partial_\nu f_i(y), \text{ for almost every } r \text{ and a.e. } y \text{ with } |y| = r. \tag{3.17}$$

Recalling (3.7) and using (3.17), we infer that, for such $r$,

$$\alpha = I(r) = \frac{r\,D(r)}{H(r)} = \frac{r\int_{\partial B_r}\sum_i\langle\partial_\nu f_i, f_i\rangle}{\int_{\partial B_r}\sum_i |f_i|^2} \overset{(3.17)}{=} \frac{r\lambda_r\int_{\partial B_r}\sum_i |f_i|^2}{\int_{\partial B_r}\sum_i |f_i|^2} = r\lambda_r.$$

So, summarizing, $I(r) \equiv \alpha$ if and only if

$$f_i(y) = \frac{\alpha}{|y|}\,\partial_\nu f_i(y) \quad \text{for almost every } y. \tag{3.18}$$

Let us assume that (3.16) holds. Then, (3.18) is clearly satisfied and, hence, $I(r) \equiv \alpha$. On the other hand, assuming that the frequency is constant, we now prove (3.16). To this aim, let $\sigma_y = \{r\,y \,:\, 0 \leq r \leq \varrho\}$ be the radius passing through $y \in \partial B_1$. Note that, for almost every $y$, $f|_{\sigma_y} \in W^{1,2}$; so, for those $y$, since $W^{1,2}$ functions on the line are absolutely continuous, we can write $f|_{\sigma_y} = \sum_i [\![f_i|_{\sigma_y}]\!]$, where $f_i|_{\sigma_y} : [0, \varrho] \to \mathbb{R}^n$ are $W^{1,2}$ functions (details left to the reader – cp. [**6**, Proposition 1.2]). By (3.18), we infer that $f_i|_{\sigma_y}$ solves the ordinary differential equation

$$(f_i|_{\sigma_y})'(r) = \frac{\alpha}{r}\, f_i|_{\sigma_y}(r), \quad \text{for a.e. } r.$$

Hence, for a.e. $y \in \partial B_1$ and for every $r \in (0, \varrho]$, $f_i|_{\sigma_y}(r) = r^\alpha\, f(y)$, thus concluding (3.16). □

Corollary 3.8. *Let $f$ be* Dir*-minimizing in $B_\varrho$. Let $0 < r < t \le \varrho$ and suppose that $I_{0,f}(r) = I(r)$ is defined for every $r$ (i.e. $H(r) \neq 0$ for every $r$). Then,*

(*i*) *for almost every $r \le s \le t$,*

$$\left(\frac{r}{t}\right)^{2I(t)} \frac{H(t)}{t^{m-1}} \le \frac{H(r)}{r^{m-1}} \le \left(\frac{r}{t}\right)^{2I(r)} \frac{H(t)}{t^{m-1}}; \tag{3.19}$$

(*ii*) *if $I(t) > 0$, then*

$$\frac{I(r)}{I(t)} \left(\frac{r}{t}\right)^{2I(t)} \frac{D(t)}{t^{m-2}} \le \frac{D(r)}{r^{m-2}} \le \left(\frac{r}{t}\right)^{2I(r)} \frac{D(t)}{t^{m-2}}. \tag{3.20}$$

Proof. The proof is a straightforward consequence of equation (3.13). Indeed, (3.13) implies, for almost every $s$,

$$\frac{d}{d\,\tau}\Big|_{\tau=s} \left(\frac{H(\tau)}{\tau^{m-1}}\right) = \frac{H'(s)}{s^{m-1}} - \frac{(m-1)\,H(s)}{s^m} \overset{(3.13)}{=} \frac{2\,D(s)}{s^{m-1}}. \tag{3.21}$$

Integrating (3.21) and using the monotonicity of $I$, one obtains (3.19). Similarly, (3.20) follows from (3.19), using the identity $I(r) = \frac{r\,D(r)}{H(r)}$. □

## 3. Blow-up of Dir-minimizing $Q$-valued functions

In this section we work out the necessary technical results in order to perform a blow-up analysis. Since we will look at point of maximal multiplicity, we assume in the sequel that $f$ is a nontrivial (i.e. $\text{Dir}(f, B_\varrho(y)) > 0$ for every $\varrho$) Dir-minimizing $Q$-valued function such that $f(y) = Q\,[\![0]\!]$. There are different possibilities for defining the blow-ups of $f$ at $y$; one is to rescale according to the energy in the following way:

$$f_{y,\varrho}(x) = \frac{\varrho^{\frac{m-2}{2}}\, f(\varrho\, x + y)}{\sqrt{\text{Dir}(f, B_\varrho(y))}}. \tag{3.22}$$

To simplify the notation, we will not display the subscript $y$ in $f_{y,\rho}$ when $y$ is the origin.

The main result of this section is the convergence of blow-ups of Dir-minimizing functions to homogeneous Dir-minimizing functions, which we call *tangent functions*.

Theorem 3.9. *Let $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ be* Dir*-minimizing, with $f(0) = Q\,[\![0]\!]$ and $\text{Dir}(f, B_\varrho) > 0$ for every $\varrho \le 1$. Then, every sequence $\{f_{\varrho_k}\}$ with $\rho_k \downarrow 0$ posses a subsequence converging locally uniformly to a function $g : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ satisfying:*

(*a*) $\text{Dir}(g, B_1) = 1$ *and $g|_\Omega$ is* Dir*-minimizing for any bounded $\Omega$;*

*(b)* $g(x) = |x|^\alpha \, g\left(\frac{x}{|x|}\right)$, *where* $\alpha = I_{0,f}(0) > 0$ *is the frequency of* $f$ *at* $0$.

Theorem 3.9 is a direct consequence of the estimate on the frequency function and of the following convergence result for Dir-minimizing functions.

PROPOSITION 3.10. *Let* $f_k \in W^{1,2}(B_1, \mathcal{A}_Q)$ *be* Dir*-minimizing $Q$-valued functions such that* $\sup_k \mathrm{Dir}(f_k, B_1) < +\infty$ *and* $f_k \to f$ *uniformly. Then, for every* $r < 1$, $f|_{B_r}$ *is* Dir-*minimizing and* $\mathrm{Dir}(f, B_r) = \lim_k \mathrm{Dir}(f_k, B_r)$.

Assuming Proposition 3.10, we prove Theorem 3.9.

PROOF OF THEOREM 3.9. We consider any ball $B_N$ of radius $N$ centered at 0. It follows from estimate (3.20) that $\mathrm{Dir}(f_\varrho, B_N)$ is uniformly bounded in $\varrho$. Hence, the functions $f_\varrho$ are all Dir-minimizing and Theorem 2.5 implies that the $f_{\varrho_k}$'s are locally equi-Hölder continuous. Since $f_\varrho(0) = Q\,[\![0]\!]$, the $f_\varrho$'s are also locally uniformly bounded and the Ascoli–Arzelà theorem yields a subsequence (not relabeled) converging uniformly on compact subsets of $\mathbb{R}^m$ to a continuous $Q$-valued function $g$. We can apply Proposition 3.10 and conclude $(a)$ (note that $\mathrm{Dir}(f_\varrho, B_1) = 1$ for every $\varrho$).

For $(b)$ we can proceed as follows: from

$$I_{0,g}(r) = \frac{r\,\mathrm{Dir}(g, B_r)}{\int_{\partial B_r} |g|^2} = \lim_{\varrho\to 0} \frac{r\,\mathrm{Dir}(f_\varrho, B_r)}{\int_{\partial B_r} |f_\varrho|^2} = \lim_{\varrho\to 0} \frac{\varrho\, r\,\mathrm{Dir}(f, B_{\varrho r})}{\int_{\partial B_{\varrho r}} |f|^2} = I_{0,f}(0)\ \forall\, r > 0,$$

and Corollary 3.7, we conclude that $f$ is $I_{0,f}(0)$-homogeneous. Moreover, if $I_{0,f}(0) = 0$, then the blowups $f_\varrho$ converge to a continuous 0-homogeneous function $g$, i.e. to a constant, against $\mathrm{Dir}(g, B_1) = 1$. □

PROOF OF PROPOSITION 3.10. Set $D_r = \liminf_k \mathrm{Dir}(f_k, B_r)$ and assume by contradiction that $f|_{B_r}$ is not Dir-minimizing or $\mathrm{Dir}(f, B_r) < D_r$ for some $r < 1$. Therefore, there exists $r_0 > 0$ such that, for every $r \geq r_0$, we find $g \in W^{1,2}(B_r, \mathcal{A}_Q)$ with

$$g|_{\partial B_r} = f|_{\partial B_r} \quad \text{and} \quad \gamma_r := D_r - \mathrm{Dir}(g, B_r) > 0. \tag{3.23}$$

By Fatou's Lemma, $\liminf_k \mathrm{Dir}(f_k, \partial B_r)$ is finite for almost every $r$:

$$\int_0^1 \liminf_{k\to+\infty} \mathrm{Dir}(f_k, \partial B_r)\, dr \leq \liminf_{k\to+\infty} \int_0^1 \mathrm{Dir}(f_k, \partial B_r)\, dr \leq C < +\infty.$$

Hence, passing to a subsequence, we can fix a radius $r \geq r_0$ such that

$$\mathrm{Dir}(f, \partial B_r) \leq \lim_{k\to+\infty} \mathrm{Dir}(f_k, \partial B_r) \leq M < +\infty. \tag{3.24}$$

Let now $0 < \delta < r/2$ to be fixed later and consider the functions $\tilde{f}_k$ on $B_r$ defined by

$$\tilde{f}_k(x) = \begin{cases} g\left(\frac{r\,x}{r-\delta}\right) & \text{for } x \in B_{r-\delta}, \\ h_k(x) & \text{for } x \in B_r \setminus B_{r-\delta}, \end{cases}$$

where the $h_k$ is a suitable interpolations between $f_k \in W^{1,2}(\partial B_r, \mathcal{A}_Q)$ and $g\left(\frac{r\,x}{r-\delta}\right) \in W^{1,2}(B_{r-\delta}, \mathcal{A}_Q)$. One can prove (and we refer to [**6**, Lemma 2.5] for the details) that the interpolation functions $h_k$ satisfy the same estimates as the linear interpolation, namely

$$\begin{aligned}\mathrm{Dir}\big(\tilde f_k, B_r\big) \leq& \mathrm{Dir}\big(\tilde f_k, B_{r-\delta}\big) + C\,\delta\left[\mathrm{Dir}\big(\tilde f_k, \partial B_{r-\delta}\big) + \mathrm{Dir}\big(f_k, \partial B_r\big)\right] + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}\big(f_k, \tilde f_k\big)^2\\ &\leq \mathrm{Dir}(g, B_r) + C\,\delta\,\mathrm{Dir}(g, \partial B_r) + C\,\delta\,\mathrm{Dir}(f_k, \partial B_r) + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}(f_k, g)^2.\end{aligned}$$

Choose now $\delta$ such that $4\,C\,\delta\,(M+1) \leq \gamma_r$, where $M$ and $\gamma_r$ are the constants in (3.24) and (3.23). Using the uniform convergence of $f_k$ to $f$, we conclude, for $k$ large enough,

$$\begin{aligned}\mathrm{Dir}\big(\tilde f_k, B_r\big) &\overset{(3.23),\,(3.24)}{\leq} D_r - \gamma_r + C\,\delta\,M + C\,\delta\,(M+1) + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}(f_k, f)^2,\\ &\leq D_r - \frac{\gamma_r}{2} + \frac{C}{\delta}\int_{\partial B_r} \mathcal{G}(f_k, f)^2 < D_r - \frac{\gamma_r}{4},\end{aligned}$$

Thus contradicting the minimizing property of $f$. □

## 4. Estimate of the singular set

We come finally to the estimate on the Hausdorff dimension of the singular set of a Dir-minimizing $Q$-valued function. The key point of the estimate is to control the size of the set of singular points with multiplicity $Q$.

PROPOSITION 3.11. *Let $\Omega$ be connected and $f \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ be* Dir-*minimizing. Then, either $f = Q\,[\![\zeta]\!]$ with $\zeta : \Omega \to \mathbb{R}^n$ harmonic in $\Omega$, or the set*

$$\Sigma_{Q,f} = \{x \in \Omega \,:\, f(x) = Q\,[\![y]\!]\,,\ y \in \mathbb{R}^n\}$$

*(which is relatively closed in $\Omega$) has Hausdorff dimension at most $m-2$ and it is locally finite for $m = 2$.*

Theorem 3.2 follows by an easy induction argument on $Q$.

PROOF OF THEOREM 3.2. We argue by induction on the number of values. For $Q = 1$ there is nothing to prove, since Dir-minimizing "1-valued" functions are classical harmonic functions. Next, we assume that the theorem holds for every $Q^*$-valued functions, with $Q^* < Q$, and prove it for $Q$-valued functions. If $f = Q\,[\![\zeta]\!]$ with $\zeta$ harmonic, then $\Sigma_f = \emptyset$ and the proposition is proved. If this is not the case, we consider first $\Sigma_{Q,f}$ the set of points of multiplicity $Q$: it is a subset of $\Sigma_f$ and we know from Proposition 3.11 that it is a closed subset of $\Omega$ with Hausdorff dimension at most $m-2$ and at most countable if $m = 2$. Then, we consider the open set $\Omega' = \Omega \setminus \Sigma_{Q,f}$. Thanks to the continuity of $f$, we can find countable open balls $B_k$ such that $\Omega' = \cup_k B_k$ and $f|_{B_k}$ can be decomposed as the sum of two multiple-valued Dir-minimizing functions $f|_{B_k} = [\![f_{k,Q_1}]\!] + [\![f_{k,Q_2}]\!]$ with $Q_1 < Q$, $Q_2 < Q$ and

$$\mathrm{spt}(f_{k,Q_1}(x)) \cap \mathrm{spt}(f_{k,Q_2}(x)) = \emptyset \quad \text{for every } \ x \in B_k.$$

Clearly, it follows from this last condition that $\Sigma_f \cap B_k = \Sigma_{f_{k,Q_1}} \cup \Sigma_{f_{k,Q_2}}$. Moreover, $f_{k,Q_1}$ and $f_{k,Q_2}$ are both Dir-minimizing and, by inductive hypothesis, $\Sigma_{f_{k,Q_1}}$ and $\Sigma_{f_{k,Q_2}}$ are closed subsets of $B_k$ with Hausdorff dimension at most $m-2$. We conclude that

$$\Sigma_f = \Sigma_{Q,f} \cup \bigcup_{k\in\mathbb{N}} \left(\Sigma_{f_{k,Q_1}} \cup \Sigma_{f_{k,Q_2}}\right)$$

has Hausdorff dimension at most $m-2$ and it is at most countable if $m=2$. □

The final part of this section is devoted to the proof of Proposition 3.11.

## 5. Persistence of singularities: Centering

One of the main issue in the blow-up analysis of singularities is the persistence of singularities in the limiting function. An example can illustrate the phenomenon.

**5.1.** As for area-minimizing currents, complex varieties which are multiple covering of some given linear space provide examples of Dir-minimizing functions. For instance, consider the comple variety:

$$\mathcal{V} := \left\{(z,w) \in \mathbb{C}^2 : (w-z^2)^2 = z^5\right\} \subset \mathbb{C}^2 \simeq \mathbb{R}^4.$$

It is simple to see that $\mathcal{V}$ is the graph of the 2-valued function

$$\mathbb{R}^2 \simeq \mathbb{C} \ni z \quad \stackrel{f}{\longmapsto} \quad \sum_{\zeta^2=z} [\![\zeta^2+\zeta^5]\!] \in \mathcal{A}_2(\mathbb{C}) \simeq \mathcal{A}_2(\mathbb{R}^2).$$

Moreover, the function $f$ is Dir-minimizing in any compact set of $\mathbb{R}^2$ (cp. [**11**]). Now, it is very simple to verify via direct computation that the rescaled functions in (3.22) converge uniformly to a regular 2-valued function:

$$f_\rho(z) \longrightarrow g(z) = 2\left[\!\!\left[\frac{z^2}{2\sqrt{\pi}}\right]\!\!\right] \quad L^\infty_{\mathrm{loc}}(\mathbb{R}^2).$$

This shows that, althogh the origin is a singular point for $f$, the limiting blow-up is regular, thus excluding the possibility to estimate the size of the singular set of $f$ looking at the one for $g$.

**5.2.** The solution to such a problem is to perform a sort of change of coordinate and cut the regular part of the blow-up. Heuristically, in the example in § 3.2 one should look at $\mathcal{V}$ as a multiple valued map on the *center manifold* $\mathcal{M} := \{w = z^2\}$. In this case, parametrizing $\mathcal{M}$ by the $z$-plane, we would in fact look at the map $z \mapsto \sum_{\zeta^2=z} [\![\zeta^5]\!]$. The blow-up of such map is the map itself, and the singular point 0 persists in the limit.

Although the solution for the linearized problem of Dir-minimizing functions is very elementary, for the nonlinear case of minimizing currents this issue represents one of the main difficulties in the analysis of singularities for which we will give a short account in the next chapter. For the time being, we treat the case of Dir-minimizing function. To this aim, we introduce the map $\boldsymbol{\eta} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^n$ which takes each measure $T = \sum_i [\![P_i]\!]$ to its center of mass,

$$\boldsymbol{\eta}(T) = \frac{\sum_i P_i}{Q}.$$

LEMMA 3.12. *Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be* Dir*-minimizing. Then,*

(a) *the function $\boldsymbol{\eta} \circ f : \Omega \to \mathbb{R}^n$ is harmonic;*
(b) *for every $\zeta : \Omega \to \mathbb{R}^n$ harmonic, $g := \sum_i [\![f_i + \zeta]\!]$ is as well* Dir*-minimizing.*

PROOF. The proof of $(a)$ follows from plugging $\psi(x,u) = \zeta(x) \in C_c^\infty(\Omega, \mathbb{R}^n)$ in the variations formula (3.5). Indeed, from the composition in § 1.1 of type (c), one infers easily that $Q\, D(\boldsymbol{\eta} \circ f) = \sum_i Df_i$. By (3.5) we get $\int \langle D(\boldsymbol{\eta} \circ f) : D\zeta \rangle = 0$, which by the arbitrariness of $\zeta \in C_c^\infty(\Omega, \mathbb{R}^n)$ gives (a).

To show $(b)$, let $h$ be any $Q$-valued function with $h|_{\partial\Omega} = f|_{\partial\Omega}$: we need to verify that, if $\tilde{h} := \sum_i [\![h_i + \zeta]\!]$, then $\mathrm{Dir}(g, \Omega) \leq \mathrm{Dir}(\tilde{h}, \Omega)$. By writing the energy as in Proposition 1.13, we get

$$\begin{aligned}
\mathrm{Dir}(g, \Omega) &= \int_\Omega \sum_{i,j} |\partial_j g_i|^2 = \int_\Omega \sum_{i,j} \left\{ |\partial_j f_i|^2 + |\partial_j \zeta|^2 + 2\, \partial_j f_i\, \partial_j \zeta \right\} \\
&\overset{\text{min. of } f}{\leq} \int_\Omega \sum_{i,j} \left\{ |\partial_j h_i|^2 + |\partial_j \zeta|^2 \right\} + 2 \int_\Omega D(\boldsymbol{\eta} \circ f) \cdot D\zeta \\
&= \mathrm{Dir}(\tilde{h}, \Omega) + 2 \int_\Omega \left\{ D(\boldsymbol{\eta} \circ f) - D(\boldsymbol{\eta} \circ h) \right\} \cdot D\zeta. \qquad (3.25)
\end{aligned}$$

Since $\boldsymbol{\eta} \circ f$ and $\boldsymbol{\eta} \circ h$ have the same trace on $\partial\Omega$ and $\zeta$ is harmonic, the last integral in (3.25) vanishes. □

**5.3.** Next we need a lemma on cylindrical blow-ups of homogeneous functions. This is the analogous of the cylindrical blow-ups for minimizing currents and the proof follows the very same line.

LEMMA 3.13 (Cylindrical blow-up). *Let $g : B_1 \to \mathcal{A}_Q(\mathbb{R}^n)$ be an $\alpha$-homogeneous,* Dir*-minimizing function with* $\mathrm{Dir}(g, B_1) > 0$*. Suppose that $g(z) = Q\,[\![0]\!]$ and set $\beta = I_{z,g}(0)$ for $z = e_1/2$. Then, the tangent functions $h$ to $g$ at $z$ are $\beta$-homogeneous with* $\mathrm{Dir}(h, B_1) = 1$ *and satisfy:*

(a) *$h(s\, e_1) = Q\,[\![0]\!]$ for every $s \in \mathbb{R}$;*
(b) *$h(x_1, x_2, \ldots, x_m) = \hat{h}(x_2, \ldots, x_m)$, where $\hat{h} : \mathbb{R}^{m-1} \to \mathcal{A}_Q(\mathbb{R}^n)$ is Dir-minimizing on any bounded open subset of $\mathbb{R}^{m-1}$.*

PROOF. The first part of the proof follows from Theorem 3.9, while $(a)$ is straightforward. We need only to verify $(b)$. To simplify notations, we pose $x' = (0, x_2, \ldots, x_m)$: we show that $h(x') = h(s\, e_1 + x')$ for every $s$ and $x'$. This is an easy consequence of the homogeneity of both $g$ and $h$. Recall that $h$ is the local uniform limit of $g_{z,\varrho_k}$ for some

$\rho_k \downarrow 0$ and set $C_k := \mathrm{Dir}(g, B_{\varrho_k}(z))^{-1/2}$, $\beta = I_{z,g}(0)$ and $\lambda_k := \frac{1}{1-2\varrho_k\, s}$, where $z = e_1/2$. Hence, we have

$$\begin{aligned} h(s\,e_1 + x') &\overset{\text{hom. of } h}{=} \lim_{k\uparrow\infty} C_k \,\frac{g_{z,\varrho_k}\left(s\lambda_k\, e_1 + \lambda_k x'\right)}{\lambda_k^\beta} = \lim_{k\uparrow\infty} C_k\, \frac{g\left(\lambda_k\, z + \lambda_k\, \varrho_k\, x'\right)}{\lambda_k^\beta} \\ &\overset{\text{hom. of } g}{=} \lim_{\varrho\to 0} C_k\, \frac{{\lambda_k}^\alpha\, g_{z,\varrho_k}\left(x'\right)}{\lambda_k^\beta} = h(x'), \end{aligned}$$

where we used $\lambda_k z + \lambda_k\, \varrho_k\, x' = z + s\lambda_k\, \varrho_k\, e_1 + \lambda_k\, \varrho_k\, x'$ and $\lim_{k\uparrow\infty} \lambda_k = 1$.

The minimizing property of $\hat{h}$ is a consequence of the Dir-minimality of $h$. It suffices to show it on every ball $B \subset \mathbb{R}^{m-1}$ for which $\hat{h}|_{\partial B} \in W^{1,2}$. To fix ideas, assume $B$ to be centered at 0 and to have radius $R$. Assume the existence of a competitor $\tilde{h} \in W^{1,2}(B)$ such that $\mathrm{Dir}(\tilde{h}, B) \leq D(\hat{h}, B) - \gamma$ and $\tilde{h}|_{\partial B} = \hat{h}|_{\partial B}$. We now construct a competitor $h'$ for $h$ on a cylinder $C_L = [-L, L] \times B_R$. First of all we define

$$h'(x_1, x_2, \ldots, x_n) = \tilde{h}(x_2, \ldots, x_n) \quad \text{for } |x_1| \leq L - 1.$$

It remains to "fill in" the two cylinders $C_L^1 = ]L-1, L[\times B_R$ and $C_L^2 = ]-L, -(L-1)[\times B_R$. Let us consider the first cylinder. We need to define $h'$ in $C_L^1$ in such a way that $h' = h$ on the lateral surface $]L-1, L[\times \partial B_R$ and on the upper face $\{L\} \times B_R$ and $h' = \tilde{h}$ on the lower face $\{L-1\} \times B_R$. Now, since the cylinder $C_L^1$ is biLipschitz to a unit ball, this can be done with a $W^{1,2}$ map.

Denote by $u$ and $v$ the upper and lower "filling" maps in the case $L = 1$ By the $x_1$-invariance of our construction, the maps

$$u_L(x_1, \ldots, x_m) := u(x_1 - L, \ldots, x_m) \quad \text{and} \qquad v_L(x_1, \ldots, x_m) = u(x_1 + L, \ldots, x_m)$$

can be taken as filling maps for any $L \geq 1$. Therefore, we can estimate

$$\begin{aligned} \mathrm{Dir}(h', C_L) - D(h, C_L) &\leq \left(\mathrm{Dir}\left(h', C_L^1 \cup C_L^2\right) - \mathrm{Dir}\left(h, C_L^1 \cup C_L^2\right)\right) - 2\,(L-1)\,\gamma \\ &=: \Lambda - 2\,(L-1)\,\gamma, \end{aligned}$$

where $\Lambda$ is a constant independent of $L$. Therefore, for a sufficiently large $L$, we have $D(h', C_L) < D(h, C_L)$ contradicting the minimality of $h$ in $C_L$. □

**5.4. Proof of Proposition 3.11.** With the help of these two lemmas we prove Proposition 3.11 in the case $m = 2$. For the general case, see [**6**, Section 3.6].

First of all we notice that, by Lemma 3.12, it suffices to consider Dir-minimizing function $f$ such that $\boldsymbol{\eta} \circ f \equiv 0$. Under this assumption, it follows that

$$\Sigma_{Q,f} = \{x\, :\, f(x) = Q\, [\![0]\!]\}\,.$$

We prove that, except for the case where all sheets collapse, $\Sigma_{Q,f}$ consists of isolated points. Without loss of generality, let $0 \in \Sigma_{Q,f}$ and assume that $f \not\equiv Q\,[\![0]\!]$ in a neighborhood of 0. Suppose by contradiction that there exist $x_k \to 0$ such that $f(x_k) = Q\,[\![0]\!]$. By Theorem 3.9, the blow-ups $f_{|x_k|}$ converge uniformly, up to a subsequence, to some homogeneous Dir-minimizing function $g$, with $\mathrm{Dir}(g, B_1) = 1$ and $\boldsymbol{\eta} \circ g \equiv 0$. Moreover, since $f(x_k)$ are $Q$-multiplicity points, we deduce that there exists $w \in \mathbb{S}^1$ such that $g(w) = Q\,[\![0]\!]$. Up to rotations, we can assume that $w = e_1$. Considering the blowup of $g$ in the point $e_1/2$, by Lemma 3.13, we find a new tangent function $h$ with the property that $h(0, x_2) = \hat{h}(x_2)$ for some function $\hat{h} : \mathbb{R} \to \mathcal{A}_Q$ which is Dir-minimizing on every interval, $\boldsymbol{\eta} \circ \hat{h} \equiv 0$ and $\hat{h}(0) = Q\,[\![0]\!]$. Moreover, since $\mathrm{Dir}(h, B_1) = 1$, clearly $\mathrm{Dir}\big(\hat{h}, I\big) > 0$, for $I = [-1, 1]$. This is clearly a contradiction. Indeed, by a simple comparison argument, it is easily seen that every Dir-minimizing 1-d function $\hat{h}$ is an affine function of the form $\hat{h}(x) = \sum_i [\![L_i(x)]\!]$ with the property that either $L_i(x) \neq L_j(x)$ for every $x$ or $L_i(x) = L_j(x)$ for every $x$. Since $\hat{h}(0) = Q\,[\![0]\!]$, we would conclude that $\hat{h} = Q\,[\![L]\!]$ for some linear $L$. On the other hand, by $\boldsymbol{\eta} \circ \hat{h} \equiv 0$ we would conclude $L = 0$, contradicting $\mathrm{Dir}(\hat{h}, I) > 0$.

CHAPTER 4

# Center manifold

In this chapter we give some hints to the solution of the centering problem in § 5 of Chapter 3 for the nonlinear problem of the regularity for minimizing currents. This issue is one of the most important among those to be faced in the generalization of the approach for multiple valued functions. Indeed, differently from this case, the average of the different sheets of a current (if ever defined) does not solve in general any given partial differential equation, thus not allowing any simple translation or reparametrization argument.

**0.5.** Heuristically, the issue under consideration is the following. Given a singular point of locally maximal multiplicity, say $Q$, we would like to deduce that the blow-up limit of the rescaled function (eventually) parametrizing the current is a homogeneous multiple valued function that is still singular (in order then to argue in terms of the singular set for such simpler object).

The estimate on the Frequency Function allows to conclude that the limit is not trivial, i.e. identically $Q\,[\![0]\!]$. Nevertheless, what can happen (and do happen, as shown in the example of § in Chapter 3) is that the limiting function may loose all the informations about the singularities, becoming a multiple copy of a smooth surface.

To avoid this phenomenon, one possibilities is to change coordinates in such a way to "cut out" the regular part of the limit and maintain the singular behavior around the point in consideration.

**0.6.** In the case of minimizers of the Dirichlet energy, this change of coordinates is performed by subtracting the average of the values: in fact, such mean leaf turns out to be harmonic and the resulting multiple valued function still Dir-minimizing. In this way, setting the average of our function equal to 0, we have ruled out the possibility of the limiting function to be a multiplicity $Q$ copy of a smooth function, given that the only possibility would then be to get the constant function $Q\,[\![0]\!]$ already excluded by the estimate on the Frequency Function.

Similarly, in the nonlinear case of minimizing currents, the same issue is solved by constructing a regular center manifold that plays the rôle of the average. Clearly, the construction in this case is much more involved and needs several preliminary estimates on minimizing currents which would be too long to present in this notes (cp. [**7, 8, 9**]). For this reason, rather than giving an account of the general case, here we focus our attention on a very special situation in which some of the main ideas still show up.

**0.7. Single sheet currents.** We consider here the case of a current given by the graph of a *single* valued function, $T = \mathrm{Gr}(f)$ for some $f : B_1 \subset \mathbb{R}^m \to \mathbb{R}^n$ Lipschitz continuous. In this case, the average is the same as the function $f$ itself; moreover, since the approximation needs to be as more accurate as the different sheets of the currents get closer, in the extreme case of a single sheet we should be able to approximate $f$ as well as possible, namely we should show that it actually coincides, with a regular function!

**0.8. Important Remark.** By the regularity theory for multiplicity one current developed by Allard (especially in the case of graphs of Lipschitz continuous functions), we know already that $f$ is perfectly smooth (in fact analytic!) The aim of the following construction is to show that a certain degree of regularity of $f$ can be proven via very robust arguments, which eventually can be generalized to the construction of the center manifold in the case of general multiplicity.

## 1. Main result

In this section we settle the hypotheses and the main prerequisites to the construction of the center manifold. In what follows we will assume that $T$ is an (locally) area minimizing $m$-dimensional current in $\mathbb{R}^{m+n}$ which is given by the integration the (oriented) graph of a Lipschitz function $F : \mathbb{R}^m \to \mathbb{R}^n$, with $F(0) = |DF(0)| = 0$.

As remark above, invoking Allard's $\varepsilon$-regularity theorem, we can conclude that $F$ is actually $C^{1,\alpha}$ regular and then, by Schauder regularity, smooth. Nevertheless, in the following we find an alternative way to prove that $F$ is $C^{3,\alpha}$ (note: not just $C^{1,\alpha}$) without using Schauder estimates for the system of the minimal surface equation. The proof is then meant to be generalized to the case of a suitable average of the sheet of a general minimizing current.

THEOREM 4.1. *Given an area-minimizing current $T$ as above, there exists a sequence of functions $h_k : B_1 \subset \mathbb{R}^m \to \mathbb{R}^n$ such that $\sup_k \|h_k\|_{C^{3,\alpha}} \leq C$ for some dimensional constants $\alpha > 0$ and $C$, and the graphs of $h_k$ converge, in the sense of currents, to $T \llcorner (B_1 \times \mathbb{R}^n)$.*

In the proof of Theorem 4.1, we need to use the following known fact about $F$.

**1.1. De Giorgi's excess decay.** The fundamental step in De Giorgi's proof the regularity of minimizing currents is the decay of the quantity usually called "spherical excess":

$$\mathbf{E}(T, B_r(p)) := \min_{\pi} \mathbf{E}(T, B_r(p), \pi),$$

where the minimum is taken over all oriented $m$–planes $\pi$ and

$$\mathbf{E}(T, B_r(p), \pi) := \frac{1}{2} \fint_{B_r(p)} |\vec{T} - \vec{\pi}|^2 d\|T\|.$$

PROPOSITION 4.2. *There is a dimensional constant $C$ with the following property. For every $\delta, \varepsilon_0 > 0$, if $T$ is as above and $\mathbf{E}(T, B_1) = \varepsilon^2 \leq \varepsilon_0^2$, then*

$$\mathbf{E}(T, B_r(p)) \leq C\, \varepsilon^2\, r^{2-2\delta}, \tag{4.1}$$

*for every $r \leq 1/2$ and every $p \in B_{1/2} \cap \operatorname{spt}(T)$.*

Definition 4.3. For later reference, we say that a plane $\pi$ is *admissible* in $p$ at scale $\rho$ (or simply that $(p,\rho,\pi)$ is *admissible*) if

$$\mathbf{E}(T, B_\rho(p), \pi) \leq C_{m,n}\varepsilon^2\,\rho^{2-2\delta}, \tag{4.2}$$

for some fixed (possibly large) dimensional constant $C_{m,n}$.

Proposition 4.2 guarantees that, for every $p$ and $r$ as in the statement, there exists always an admissible plane $\pi_{p,r}$. The following is a straightforward consequence of Proposition 4.2.

Corollary 4.4. *There exists a dimensional constant $C > 0$ with the following property. Under the hypotheses of Proposition 4.2, the following holds:*

(a) *if $(p,\rho,\pi)$ and $(p',\rho',\pi')$ are admissible, then*

$$|\vec{\pi} - \vec{\pi}'| \leq C\,\varepsilon\big(\max\{\rho,\rho',|q-q'|\}\big)^{1-\delta};$$

(b) *there exists a unique tangent plane $\pi_p$ to $T$ at every $p \in \mathrm{spt}(T)\cap B_{1/2}$; moreover, if $(p,\rho,\pi)$ is admissible then $|\pi-\pi_p| \leq C\,\varepsilon\,\rho^{1-\delta}$ and, vice versa, if $|\pi-\pi_p| \leq C\,\varepsilon\,\rho^{1-\delta}$, then $(p,\rho,\pi)$ is admissible;*

(c) *for every $q \in B^m_{1/4}$, there exists a unique $u \in \mathbb{R}^n$ such that $(q,u) \in \mathrm{spt}(T)\cap B_{1/2}$.*

## 2. The approximation scheme

The $C^{3,\alpha}$ regularity of the current $T$ will be deduced from the limit of a suitable approximation scheme. We start by fixing a nonnegative kernel

$$\varphi \in C^\infty_c(B^m_1), \quad \varphi \text{ radial} \quad \text{and} \quad \int \varphi = 1.$$

As usual, for $\tau > 0$, we set $\varphi_\tau(w) := \tau^{-m}\varphi(w/\tau)$. From Corollary 4.4 (b) and (c), it is simple to deduce the following: if $p = (q,u) \in \pi\times\pi^\perp$, $\rho \leq 2^{-6}$ and $\pi$ form an admissible triple $(p, 6\,\rho, \pi)$ with $p \in \mathrm{spt}(T)\cap B_{1/16}$, then

$$T\llcorner\mathcal{C}^\pi_{6\rho}(q) = \mathrm{Gr}(f),$$

for some Lipschitz function $f : B^m_{6\rho}(q) \subset \pi \to \pi^\perp$. Notice that, as a result of De Giorgi's decay we also know that $\mathrm{Lip}(f) \leq \mathbf{E}(T, B_{6\rho})^{\frac{1}{2}}$.

The scheme for the construction of the center manifold starts from the following functions:

($I_1$) $\hat{f} = f * \varphi_\rho$;

($I_2$) $\bar{f}$ such that

$$\begin{cases} \Delta\bar{f} = 0 & \text{on } B^m_{4\rho}(q),\\ \bar{f}|_{\partial B^m_{4\rho}(q)} = \hat{f}; \end{cases}$$

($I_3$) $g : B^m_\rho(q') \subset \pi_0 \to \pi_0^\perp$, with $x(p) = (q',u') \in \pi_0\times\pi_0^\perp$, such that $\mathrm{Gr}_{\pi_0}(g) = \mathrm{Gr}_\pi(\bar{f})$ in the cylinder $\mathcal{C}_\rho(q') \subset \pi_0\times\pi_0^\perp$.

Remark 4.5. In order to proceed further, we need to show the existence of $g$ as in $(I_3)$. This follows from the interior estimates for harmonic functions and the bound on the Lipschitz constant of $f$ implicit in Corollary 4.4 (b) and (c) – details left to the readers.

The function $g$ is the main building block of the construction of this paper. It is called the *$(p,\rho,\pi)$-interpolation of $T$* or, if $\mathbf{E}(T,B_{8\rho}(p)) = \mathbf{E}(T,B_{8\rho}(p),\pi)$, simply the *$(p,\rho)$-interpolation of $T$*.

The main estimates of the paper are contained in the following proposition.

Proposition 4.6. *There are constants $\alpha, C > 0$ such that, if $g, g'$ are respectively $(p,\rho,\pi)$- and $(p',\rho,\pi')$-interpolations, then*

$$\rho^{1-\alpha}\|D^4 g\|_{C^0} + \|g\|_{C^3} \le C, \tag{4.3a}$$

$$\sum_{\ell=0}^{4} \rho^{\ell-3-\alpha}\|D^\ell g(x) - D^\ell g'(x)\|_{C^0} \le C \quad \text{in } B_\rho(p)\cap B_\rho(p'), \tag{4.3b}$$

$$|D^3 g(q) - D^3 g'(q')| \le C|q-q'|^\alpha, \quad \text{with } p=(q,u),\, p'=(q',u'). \tag{4.3c}$$

**2.1. Approximation scheme.** Let $5 < n_0 < k_0$ be natural numbers and consider the cube $Q = [-2^{-n_0}, 2^{-n_0}]^m$. For $k \ge k_0$, we consider the usual subdivision of $\mathbb{R}^m$ into dyadic cubes of size $2\cdot 2^{-k}$, centered at points $c_i = 2^{-k} i \in 2^{-k}\mathbb{Z}^m$. The corresponding closed cubes of the subdivision are then denoted by $Q_i$ and we consider below only those $Q_i$'s which have nonempty intersection with $Q$.

According to Corollary 4.4 and to the previous observations, for every $c_i$ there exists a unique $u_i$ such that $p_i = (c_i, u_i) \in \mathrm{spt}(T)\cap B_{1/16}$. Moreover, for every constant $C$, if $k_0$ is large enough, we can consider the $(p_i, C\,2^{-k})$-interpolation $g_i$ for all $k \ge k_0$.

Let $\psi \in C^\infty_c([-\frac{5}{4},\frac{5}{4}]^m)$ be a nonnegative function such that, if we define $\psi_i(q) := \psi(2^k(q-c_i))$, then

$$\sum_{i\in\mathbb{Z}^m} \psi_i \equiv 1 \text{ in } Q\,.$$

Denote by $\mathcal{A}_i$ the set of indexes $j$ such that $Q_j$ and $Q_i$ are adjacent. Note that the choice of $\psi$ guarantees $\psi_i\,\psi_j = 0$ if $j \notin \mathcal{A}_i$. Moreover the cardinality of $\mathcal{A}_i$ is (bounded by) a dimensional constant independent of $k$ and, if $q \in Q_i$, then in a neighborhood of $q$ we have

$$\sum_{j\in\mathcal{A}_i} \psi_j = 1 \quad \text{and} \quad \sum_{j\in\mathcal{A}_i} D^\ell \psi_j(q) = 0 \quad \text{for all } \ell > 0. \tag{4.4}$$

Assuming Proposition 4.6, we can now prove the main result.

**2.2. Proof of Theorem 4.1.** Consider the functions $h_k : Q \to \mathbb{R}^n$ given by $h_k := \sum_i \psi_i\, g_i$. Given $k$, let $Q_i$ be a cube of the corresponding dyadic decomposition and a point $q \in Q_i$. We already observed that, in a neighborhood of $q$, $h_k = \sum_{j\in\mathcal{A}_i} \psi_j g_j$. Moreover, from the definition, we have that

$$\|D^\ell \psi_j\|_{C^0} = 2^{k\ell}\,\|D^\ell\psi\|_{C^0} = C_\ell\, 2^{k\ell} \quad \text{for every } \ell \in \mathbb{N}. \tag{4.5}$$

The $C^0$ estimate of $h_k$ follows trivially from (4.3a), since

$$|h_k(q)| \leq \sum_{j\in\mathcal{A}_i} \|\psi_j\|_{C^0}\|g_j\|_{C^0} \leq C.$$

As for the $C^1$ estimate, we write the first derivative of $h_k$ as follows,

$$Dh_k(q) = \sum_{j\in\mathcal{A}_i} \big(D\psi_j(q)g_j(q) + \psi_j(q)Dg_j(q)\big) \overset{(4.4)}{=} \sum_{j\in\mathcal{A}} \big(D\psi_j(q)(g_j(q) - g_i(q)) + \psi_j(q)Dg_j(q)\big),$$

from which, using (4.3a), (4.3b) and (4.5), we deduce

$$|Dh_k(q)| \leq \sum_{j\in\mathcal{A}_i} \big(\|D\psi_j\|_{C^0}\|g_i - g_j\|_{C^0} + \|\psi_j\|_{C^0}\|Dg_j\|_{C^0}\big) \leq C.$$

With analogous computations, we obtain

$$\begin{aligned}
|D^2h_k(q)| &\leq \sum_{j\in\mathcal{A}_i} \big(\|D^2\psi_j\|_{C^0}\|g_i - g_j\|_{C^0} + \|D\psi_j\|_{C^0}\|Dg_j - Dg_i\|_{C^0} + \|\psi_j\|_{C^0}\|D^2g_j\|_{C^0}\big) \\
&\leq C, \\
|D^3h_k(q)| &\leq \sum_{j\in\mathcal{A}_i} \big(\|D^3\psi_j\|_{C^0}\|g_j - g_i\|_{C^0} + \|D^2\psi_j\|_{C^0}\|Dg_j - Dg_i\|_{C^0} + \\
&\qquad + \|D\psi_j\|\|D^2g_j - D^2g_i\|_{C^0} + \|\psi_j\|_{C^0}\|D^3g_j\|_{C^0}\big) \leq C, \\
|D^4h_k(q)| &\leq \sum_{j\in\mathcal{A}_i} \big(\|D^4\psi_j\|_{C^0}\|g_j - g_i\|_{C^0} + \|D^3\psi_j\|_{C^0}\|Dg_j - Dg_i\|_{C^0} + \\
&\qquad + \|D^2\psi_j\|\|D^2g_j - D^2g_i\|_{C^0} + \|D\psi_j\|_{C^0}\|D^3g_j - D^3g_i\|_{C^0} + \|\psi_j\|_{C^0}\|D^4g_j\|_{C^0}\big) \\
&\leq C\,2^{k(1-\alpha)},
\end{aligned}$$

where $C$ is a constant independent of $k$.

Now, let $q, q' \in B_{1/2}$ and consider the cubes $Q_i$ and $Q_j$ such that $q \in Q_i$ and $q' \in Q_j$. If the two cubes are adjacent, then we have $|q - q'| \leq C2^{-k}$ and, therefore,

$$|D^3h_k(q) - D^3h_k(q')| \leq \|D^4h_k\|_{C^0}|q - q'| \leq C\,2^{k(1-\alpha)}\,|q - q'| \leq C\,|q - q'|^\alpha.$$

If $Q_i$ and $Q_j$ are not adjacent, then $2\,|q-q'| \geq \max\{|c_i-c_j|, 2^{-k}\}$. Since $\mathrm{spt}(\psi) \subset [-\frac{5}{4}, \frac{5}{4}]^m$, $D^3h_k(c_i) = D^3g_i(c_i)$ for every $i$ and from (4.3c) it follows that

$$\begin{aligned}
|D^3h_k(q) - D^3h_k(q')| &\leq |D^3h_k(q) - D^3h_k(c_i)| + |D^3g_i(c_i) - D^3g_j(c_j)| + \\
&\qquad + |D^3h_k(c_j) - D^3h_k(q')| \\
&\leq C\,2^{-k}\|D^4h_k\|_{C^0} + C|c_j - c_i|^\alpha \leq C2^{-k\alpha} + C|c_i - c_j|^\alpha \leq C|q - q'|^\alpha.
\end{aligned}$$

We finally come to the convergence of the graphs of $h_k$ in the sense of currents. Obviously, by compactness we can assume that a subsequence of $h_k$ (not relabeled) converges in the $C^3(Q)$ norm to some limiting $C^{3,\alpha}$ function $h$. On the other hand, by fact that $\hat{f}$ in $(\mathrm{I}_1)$ is uniformly close to $f$, it follows easily that the support of $F|_Q = h$.

## 3. $L^1$-estimate

The rest of the paper is devoted to the proof of Proposition 4.6. A fundamental point is an estimate for the $L^1$ distance between the harmonic function $\bar f$ introduced in step $(I_2)$ of the approximation scheme and the function $f$ itself. A preliminary step is the following estimate on the Laplacian of $\hat f$, which is a simple consequence of the first variation formula for area-minimizing currents.

Lemma 4.7. *There exists $\delta, \gamma, C, \lambda, \eta > 0$ such that, if $(p, 8\rho, \pi)$ is admissible and $\hat f$ is as in $(I_1)$, then*

$$\|\Delta \hat f\|_{C^0(B^m_{5\rho})} \leq C\rho^{1+\lambda}, \tag{4.6}$$

$$\int_{B_{5\rho}} \left| \int_{B_\rho(w)} Df(z)\cdot D\gamma(w-z)\,dz\right| dw \leq C\,E^{1+\eta}\,\rho^m\,\|D\gamma\|_{L^1}, \quad \forall \gamma \in C^1_c(B_\rho, \mathbb{R}^n), \tag{4.7}$$

*where $E = \mathbf{E}(T, B_{8\rho}(p), \pi)$.*

Proof. Let $\mu$ be the measure defined by $\mu(A) := \|T\|(A\times\pi^\perp)$. We start showing that

$$\left|\int Df\cdot D\kappa\right| \leq C\int |D\kappa|\,|Df|^3\,dx \quad \forall\, \kappa \in C^1_c(B_{6\rho}, \mathbb{R}^n). \tag{4.8}$$

Consider the vector field $\chi(x, y) = (0, \kappa(x))$. From the minimality of the current $T$, we infer that the first variation of the mass in direction $\chi$ vanishes, $\delta T(\chi) = 0$ and

$$\left|\int Df\cdot D\varphi\right| \leq \left|\int Df\cdot D\varphi - \delta T(\chi)\right|. \tag{4.9}$$

The first variation $\delta T(\chi)$ is given by the formula

$$\begin{aligned}\int_{\mathcal{C}_{6\rho}} \mathrm{div}_{\vec T}\chi\, d\|T\| &= \frac{d}{ds}\Big|_{s=0}\int_{B_{6\rho}} \sqrt{1+|Df+sD\kappa|^2+\textstyle\sum_{|\alpha|\geq 2} M_\alpha(Df+sD\kappa)^2}\,dx\\ &= \int_{B_{6\rho}} \frac{Df\cdot D\kappa + \sum_{|\alpha|\geq 2} M_\alpha(Df)\,\frac{d}{ds}\big|_{s=0} M_\alpha(Df+sD\kappa)}{\sqrt{1+|Df|^2+\sum_{|\alpha|\geq 2} M_\alpha(Df)^2}}.\end{aligned}$$

It follows then that

$$\begin{aligned}\left|\int_{\mathcal{C}_{6\rho}} \mathrm{div}_{\vec T}\chi\, d\|T\| - \int_{B_{6\rho}} Df\cdot D\kappa\right| \leq& \int_{B_{6\rho}} |Df||D\kappa|\left(\sqrt{1+|Df|^2+\textstyle\sum_{|\alpha|\geq 2} M_\alpha(Df)^2}-1\right)+\\ &+\left|\int_{B_{6\rho}} \textstyle\sum_{|\alpha|\geq 2} M_\alpha(Df)\,\frac{d}{ds}\big|_{s=0} M_\alpha(Df+sD\kappa)\right|\\ \leq& C\int_{B_{6\rho}} |D\kappa|\,|Df|^3.\end{aligned}$$

We now come to the proof of (4.6). From (4.8) and the Lipschitz continuity of $f$, it follows straightforwardly that

$$\left|\int_{B_{6\rho}} Df \cdot D\kappa\right| \leq C\, E^{1+\eta}\,\rho^m\,\|D\kappa\|_{L^\infty}, \quad \text{for every } \ \kappa \in C_c^1(B_{6\rho}, \mathbb{R}^n). \tag{4.10}$$

Then, putting together the previous estimates, we conclude that

$$\begin{aligned}\|\Delta \hat{f}\|_{L^\infty(B_{5\rho})} &= \sup_{\gamma \in C_c^1(B_{5\rho}), \|\gamma\|_{L^1} \leq 1} \int D\hat{f} \cdot D\gamma = \sup_{\gamma \in C_c^1(B_{5\rho}), \|\gamma\|_{L^1} \leq 1} \int Df \cdot D(\gamma * \varphi_\rho) \\ &\stackrel{(4.10)}{\leq} \sup_{\gamma \in C_c^1(B_{5\rho}), \|\gamma\|_{L^1} \leq 1} C\, E^{1+\eta} \rho^m\, \|D(\gamma * \varphi_\rho)\|_{L^\infty} \leq C\, E^{1+\eta}\,\rho^m\,\|D\varphi_\rho\|_{L^\infty} \\ &\leq C\, E^{1+\eta}\,\rho^{-1} \leq C\,\rho^{(2-2\delta)(1+\eta)-1}.\end{aligned}$$

Therefore, (4.6) follows choosing $\delta$ sufficiently small with respect to $\eta$.

For the proof of (4.7), it is enough to notice that, from (4.8), we get

$$\begin{aligned}&\int_{B_{5\rho}} \left|\int_{B_\rho(w)} Df(z) \cdot D\gamma(w-z)\, dz\right| dw \\ \leq &C \int_{B_{5\rho}} |D\gamma| * |Df|^3 + C \int_{B_{5\rho}} |D\gamma| * \mathbf{1}_{\mathbb{R}^m \setminus K} + C \int_{B_{5\rho}} |D\gamma| * \big(\mu \llcorner (\mathbb{R}^m \setminus K)\big) \\ \leq &\|D\gamma\|_{L^1} \left(C E^\eta \int_{B_{6\rho}} |Df|^2 + |B_{6\rho} \setminus K| + \mu(B_{6\rho} \setminus K)\right) \leq C\, E^{1+\eta}\,\rho^m\,\|D\gamma\|_{L^1}.\end{aligned}$$

□

Now we come to the $L^1$-estimate for the harmonic approximation $\bar{f}$.

Proposition 4.8. *Let $(p, 8\rho, \pi)$ be admissible and $\bar{f}$ be as in (I$_2$). Then, there exists $\alpha > 0$ such that*

$$\|\bar{f} - f\|_{L^1(B_{4\rho})} \leq C\,\rho^{m+3+\alpha}. \tag{4.11}$$

Proof. First we estimate the $L^1$ distance between $\bar{f}$ and $\hat{f}$. Using the Poincaré inequality and a simple integration by parts, we infer that

$$\|\bar{f} - \hat{f}\|_{L^1(B_{4\rho})}^2 \leq C\,\rho^{m+2}\,\|\nabla(\bar{f} - \hat{f})\|_{L^2(B_{4\rho})}^2 = C\,\rho^{m+2} \int_{B_{4\rho}} \Delta\hat{f}\,(\bar{f} - \hat{f}),$$

from which

$$\|\hat{f} - \bar{f}\|_{L^1(B_{4\rho})} \leq C\,\rho^{2+m}\,\|\Delta \hat{f}\|_\infty \stackrel{(4.6)}{\leq} C\,\rho^{m+3+\lambda}.$$

In order to prove (4.11), then it is enough to prove the following inequality,

$$\|\hat{f} - f\|_{L^1(B_{4\rho})} \leq C\rho^{m+3+\alpha}. \tag{4.12}$$

For every $z \in B_{4\rho}$, from the definition of $\hat{f}$ we have

$$\hat{f}(z) - f(z) = \int \varphi_\rho(z-y)(f(y) - f(z))\, dy. \tag{4.13}$$

To simplify the notation assume $z = 0$ and rewrite (4.13) as

$$\begin{aligned}
\hat{f}(0) - f(0) &= \int \varphi_\rho(y) \int_0^{|y|} \frac{\partial f}{\partial r}\left(\tau \frac{y}{|y|}\right) d\tau\, dy = \int \varphi_\rho(y) \int_0^{|y|} \nabla f\left(\tau \frac{y}{|y|}\right) \cdot \frac{y}{|y|}\, d\tau\, dy \\
&= \int \varphi_\rho(y) \int_0^1 \nabla f(\sigma y) \cdot y\, d\sigma\, dy = \int \int_0^1 \varphi_\rho\left(\frac{w}{\sigma}\right) \nabla f(w) \cdot \frac{w}{\sigma^{m+1}}\, d\sigma\, dw \\
&= \int \nabla f(w) \cdot \underbrace{w \left(\int_0^1 \varphi_\rho\left(\frac{w}{\sigma}\right) \sigma^{-m-1}\, d\sigma\right)}_{=:\Phi(w)} dw.
\end{aligned}$$

More generally, for every $z \in B_{4\rho}$, we have $\hat{f}(z) - f(z) = \int \nabla f(w) \cdot \Phi(w-z)\, dw$ and

$$\|\hat{f} - f\|_{L^1(B_{4\rho})} = \int_{B_{4\rho}} \left| \int \nabla f(w) \cdot \Phi(w-z)\, dw \right| dz.$$

Since $\varphi$ is radial, the function $\Phi$ is a gradient. Indeed, it can be easily checked that, for any $\psi$, the vector field $\psi(|w|)\, w$ is curl-free. Moreover, $\mathrm{spt}(\Phi)$ is compactly contained in $B_\rho$. Hence, we can apply (4.7) and get

$$\|\hat{f} - f\|_{L^1(B_{4\rho})} \leq C\, E^{1+\eta}\, \rho^m\, \|\Phi\|_{L^1}. \tag{4.14}$$

By a simple computation,

$$\|\Phi\|_{L^1} = \int_{\mathbb{R}^m} \int_0^1 |w|\, \varphi\left(\frac{w}{\rho\sigma}\right) \rho^{-m} \sigma^{-m-1}\, d\sigma\, dw = \rho \int_{\mathbb{R}^m} \int_0^1 |y|\, \varphi(y)\, d\sigma\, dy\,.$$

The last integral is a constant which depends only on $\varphi$. Thus, (4.12) follows from (4.14). □

A simple consequence of the $L^1$-estimate is a comparison between harmonic approximations at different scales.

Corollary 4.9. *Assume $(p, 16\, r, \pi)$ is an admissible triple and let $\bar{f}_1$ and $\bar{f}_2$ be as in $(I_2)$, with $\rho = r$ and $\rho = 2\, r$ respectively. Then, if $p = (q, u) \in \pi \times \pi^\perp$,*

$$\sum_{\ell=0}^{4} r^{\ell-3-\alpha} \|D^\ell \bar{f}_1 - D^\ell \bar{f}_2\|_{C^0(B^m_{3r/2}(q))} \leq C. \tag{4.15}$$

Proof. It is enough to show that

$$\|\bar{f}_1 - \bar{f}_2\|_{L^1(B_{2r})} \leq C\, r^{m+3+\alpha}, \tag{4.16}$$

because then the conclusion of the lemma follows easily from the classical mean-value property of harmonic functions. Clearly, from the admissibility of $(p, 16\, r, \pi)$ and Corollary 4.4, it follows that $|\pi - \pi_p| \leq C\, r^{2-2\delta}$. Hence, always by the same corollary $E_2 :=$

$\mathbf{E}(T, B_{16r}(p), \pi) \leq C r^{2-2\delta}$. Then, in view of Proposition 4.8, the conclusion follows straightforwardly. □

## 4. Proof of Proposition 4.6

The proof of (4.3a) in Proposition 4.6 is given by a simple iteration of Corollary 4.9 on dyadic balls.

LEMMA 4.10. *Let $g_1, g_2$ be respectively the $(p, \rho, \pi)$- and the $(p, 2^N\rho, \pi)$-interpolation (under the assumption of admissibility (4.2)). Then, for $p = (q', u') \in \pi_0 \times \pi_0^\perp$, it holds*

$$\|g_1\|_{C^3} + \rho^{1-\alpha}\|D^4 g_1\|_{C^0} \leq C, \tag{4.17}$$

$$|D^3 g_1(q') - D^3 g_2(q')| \leq C(2^N\rho)^\alpha. \tag{4.18}$$

PROOF. By a simple lemma on the rotation of the system of coordinates, it suffices to show (4.17) for the function $\bar f_1$. Let $n_0$ be the biggest integer such that $2^{n_0+3}\rho \leq \frac{1}{2}$ and for every $k \leq n_0 - 1$ set $r_k = 2^k\,\rho$. If $\pi_k$ is such that $\mathbf{E}(T, B_{8r_k}, \pi_k) = \mathbf{E}(T, B_{8r_k})$, then, by Corollary 4.4 (b), $|\pi - \pi_k| \leq C\, r_k^{1-\delta}$. Hence, we conclude that the admissibility condition (4.2) holds with $r = r_k$, so that we can consider the approximation $\bar f_k$ as in $(I_2)$ for $r_k$. From Corollary 4.9, we get

$$\|D^\ell \bar f_k - D^\ell \bar f_{k+1}\|_{C^0(B^m_{3r_k/2}(q))} \leq C r_k^{3+\alpha-\ell} \leq C\, 2^{-(n_0-k)(3+\alpha-\ell)} \quad \text{for} \quad \ell \in \{0,1,2,3,4\}. \tag{4.19}$$

Note that the series $\sum_i 2^{-i(3+\alpha-\ell)}$ is summable for $\ell \leq 3$. Therefore, $\|\bar f_1\|_{C^3} \leq C + \|\bar f_{n_0}\|_{C^3}$. On the other hand, since $r_{n_0} > 1/32$, it is easy to see that $\|\bar f_{n_0}\|_{C^3} \leq C$ for some universal constant $C$, so that $\|\bar f_1\|_{C^3} \leq C$. In the same way we have $\|D^4 \bar f_1\|_{C^0} \leq C\,\rho^{\alpha-1}$.

Finally, Corollary 4.9 obviously implies that

$$\int_{B^m_{3r_k/2}(q)} |\bar f_k - \bar f_{k+1}| \leq C r_k^{m+3+\alpha}, \tag{4.20}$$

which in turn implies

$$\int_{B^m_{r_k}(q')} |g_k - g_{k+1}| \leq C r_k^{m+3+\alpha}. \tag{4.21}$$

Let $P_k$ and $P_{k+1}$ be the third order Taylor polynomials at $q'$ of $g_k$ and $g_{k+1}$. From the estimate $\|D^4 g_k\|, \|D^4 g_{k+1}\| \leq C r_k^{\alpha-1}$ and (4.21), we easily infer

$$\int_{B^m_{r_k}(q')} |P_k - P_{k+1}| \leq C\, r_k^{m+3+\alpha}.$$

Hence, applying Lemma 4.11, we then get

$$|D^3 g_k(q') - D^3 g_{k+1}(q')| = |D^3 P_k(q') - D^3 P_{k+1}(q')| \leq C\, r_k^\alpha. \tag{4.22}$$

Arguing as above, the estimate (4.18) follows from (4.22) and a simple iteration. □

LEMMA 4.11. *For every $n, m \in \mathbb{N}$, there exists a constant $C(m,n)$ such that, for every polynomial $R$ of degree at most $n$ in $\mathbb{R}^m$ and every positive $r > 0$,*

$$|D^k R(q)| \leq \frac{C}{r^{m+k}} \int_{B_r(q)} |R| \quad \textit{for all } k \leq n \textit{ and all } q \in \mathbb{R}^m. \tag{4.23}$$

PROOF. We rescale and translate the variables by setting $S(x) = R(rx + q)$. The lemma is then reduced to show that

$$\sum_{k=0}^{n} |D^k S(0)| \leq C \int_{B_1(0)} |S|, \tag{4.24}$$

for every polynomial $S$ of degree at most $n$ in $\mathbb{R}^m$, with $C = C(n,m)$. Consider now the vector space $V^{n,m}$ of polynomials of degree at most $n$ in $m$ variables. $V^{n,m}$ is obviously finite dimensional. Moreover, on this space, the two quantities

$$\|S\|_1 := \sum_{k=0}^{n} |D^k S(0)| \quad \text{and} \quad \|S\|_2 := \int_{B_1(0)} |S|$$

are two norms. The inequality (4.24) is then a corollary of the equivalence of norms on finite-dimensional vector spaces. □

**4.1.** The final step in the proof of Proposition 4.6 consists in comparing two different interpolating functions defined at the same scale but for nearby balls and varying planes $\pi$. We do this in the following two separate lemmas.

LEMMA 4.12. *Let $g_1$ and $g_2$ be the $(p, \rho, \pi)$- and $(p, \rho, \pi')$-interpolating functions where as usual $(p, 8\rho, \pi)$ and $(p, 8\rho, \pi')$ are admissible. Then,*

$$\sum_{\ell=0}^{3} \rho^{\ell-3-\alpha} \|D^\ell g_1 - D^\ell g_2\|_{C^0(B^m_\rho(q))} \leq C. \tag{4.25}$$

PROOF. As before, we first show that

$$\|g_1 - g_2\|_{L^1(B_{3/2\rho}(q))} \leq C\rho^{m+3+\alpha}. \tag{4.26}$$

Denote by $f_1, f_2$ the Lipschitz parametrization in the coordinates associated to $\pi, \pi'$. From Proposition 4.8, we have

$$\|g_i - F\|_{L^1(B_{3/2\rho}(q))} \leq \|f_i - \bar{f}_i\|_{L^1(B_{2\rho}(q_i))} \leq C\,\rho^{m+3+\alpha},$$

where $(q_1, u_1)$, $(q_2, u_2)$ and $(q, u)$ are the coordinates of $p$ in $\pi \times \pi^\perp$, $\pi' \times \pi'^\perp$ and $\pi_0 \times \pi_0^\perp$ respectively. Therefore, (4.26) follows.

From (4.26) we are ready to conclude. Let $x \in B_\rho(q)$ and $P_i$ be the third order Taylor expansions of $g_i$ at $x$. Arguing as in Lemma 4.10, we conclude

$$\|P_1 - P_2\|_{L^1(B_{\rho/2}(x))} \leq C\rho^{m+3+\alpha}.$$

Using Lemma 4.11 we then conclude

$$|D^k P_1(x) - D^k P_2(x)| \leq C\rho^{3-k+\alpha} \quad \text{for } k \in \{0,1,2,3\}. \tag{4.27}$$

On the other hand, since $D^k P_i(x) = D^k g_i(x)$, (4.27) implies the desired estimates. □

Lemma 4.13. *Let $g_1$ and $g_2$ be, respectively, the $(p, \rho, \pi)$- and $(p', \rho, \pi)$-interpolating functions, where $(p, \rho, \pi)$ and $(p', \rho, \pi)$ are admissible. Assume that $p = (q, u)$, $p' = (q', u')$ with $|q - q'| \le \rho/16$. Then,*

$$\sum_{\ell=1}^{4} \rho^{\ell-3-\alpha} \|D^\ell g_1 - D^\ell g_2\|_{C^0(B^n_\rho(q) \cap B^n_\rho(q'))} \le C. \tag{4.28}$$

The proof of this lemma exploits only a portion of the same computations used for Lemma 4.12 and is left to the reader.

The proof of (4.3b) follows straightforwardly from Lemma 4.12 and Lemma 4.13; while the proof of (4.3c) is given below.

Proof of (4.3c). Consider $R := 16\,|q - q'|$ and let $h$, $k$ and $h'$ be the $(q, R, \pi)$-, $(q, R, \pi')$- and $(q', R, \pi')$-interpolations, respectively. By Corollary 4.4, if $|q - q'|$ is small enough, we can apply Lemma 4.12 and Lemma 4.13 to conclude that

$$|D^3 h(q) - D^3 k(q)| + |D^3 k(q') - D^3 h'(q')| \le C R^\alpha.$$

On the other hand, by (4.3a), $\|D^4 k\| \le C R^{\alpha - 1}$, and so $|D^3 h(q) - D^3 h'(q')| \le R^\alpha$. Since by Lemma 4.10 we know that $|D^3 g(q) - D^3 h(q)| \le C R^\alpha$ and $|D^3 g'(q') - D^3 h'(q')| \le C R^\alpha$, the desired conclusion follows. □

CHAPTER 5

# Exercise

Here we give some hints for the solutions to the exercises given in the text.

**Exercise 1.** We prove the exercise by induction on $Q$. The case $Q = 1$ is of course trivial. For the general case, we will make use of the following elementary observation:

(D) if $\bigcup_{i\in\mathbb{N}} B_i$ is a covering of $B$ by measurable sets, then it suffices to find a measurable selection of $f|_{B_i\cap B}$ for every $i$.

Let first $\mathcal{A}_0 \subset \mathcal{A}_Q$ be the closed set of points of type $Q\,[\![P]\!]$ and set $B_0 = f^{-1}(\mathcal{A}_0)$. Then, $B_0$ is measurable and $f|_{B_0}$ has trivially a measurable selection.

Next we fix a point $T \in \mathcal{A}_Q \setminus \mathcal{A}_0$, $T = \sum_i [\![P_i]\!]$. We can subdivide the set of indexes $\{1, \ldots, Q\} = I_L \cup I_K$ into two nonempty sets of cardinality $L$ and $K$, with the property that

$$|P_k - P_l| > 0 \quad \text{for every } l \in I_L \text{ and } k \in I_K. \tag{5.1}$$

For every $S = \sum_i [\![Q_i]\!]$, let $\pi_S \in \mathscr{P}_Q$ be a permutation such that

$$\mathcal{G}(S,T)^2 = \sum_i |P_i - Q_{\pi_S(i)}|^2.$$

If $U$ is a sufficiently small neighborhood of $T$ in $\mathcal{A}_Q$, by (5.1), the maps

$$\tau : U \ni S \mapsto \sum_{l\in I_L} [\![Q_{\pi_S(l)}]\!] \in \mathcal{A}_L, \quad \sigma : U \ni S \mapsto \sum_{k\in I_K} [\![Q_{\pi_S(k)}]\!] \in \mathcal{A}_K$$

are continuous. Therefore, $C = f^{-1}(U)$ is measurable and $[\![\sigma\circ f|_C]\!] + [\![\tau \circ f|_C]\!]$ is a measurable decomposition of $f|_C$. Then, by inductive hypothesis, $f|_C$ has a measurable selection.

According to this argument, it is possible to cover $\mathcal{A}_Q \setminus \mathcal{A}_0$ with open sets $U$'s such that, if $B = f^{-1}(U)$, then $f|_B$ has a measurable selection. Since $\mathcal{A}_Q \setminus \mathcal{A}_0$ is an open subset of a separable metric space, we can find a countable covering $\{U_i\}_{i\in\mathbb{N}}$ of this type. Being $\{B_0\} \cup \{f^{-1}(U_i)\}_{1=1}^{\infty}$ a measurable covering of $B$, from (D) we conclude the proof.

**Exercise 2.** Since the distance function from a point is a Lipschitz map, with Lipschitz constant 1, one implication is trivial. To prove the opposite, consider a Sobolev $Q$-valued function $f$: we claim that (a) and (b) hold with $\psi = \left(\sum_j |\partial_j u|^2\right)^{1/2}$. Indeed, take a Lipschitz function $F \in \mathrm{Lip}(\mathcal{A}_Q)$. By treating separately the positive and the negative part of the function, we can assume, without loss of generality, that $F \geq 0$. If $\{T_i\}_{i\in\mathbb{N}} \subset \mathcal{A}_Q$ is a dense subset and $L = \mathrm{Lip}(F)$, it is a well known fact that $F(T) = \inf_i \left\{F(T_i) + L\,\mathcal{G}(T_i, T)\right\}$. Therefore,

$$F \circ u = \inf_i \left\{F(T_i) + L\,\mathcal{G}(T_i, u)\right\} =: \inf_i g_i. \tag{5.2}$$

Since $u \in W^{1,p}(\Omega, \mathcal{A}_Q)$, each $g_i \in W^{1,p}(\Omega)$ and the inequality $|D(F \circ u)| \leq \sup_i |Dg_i|$ holds a.e. On the other hand, $|Dg_i| = L\,|D\mathcal{G}(u, T_i)| \leq L\sqrt{\sum_j |\partial_j u|^2}$ a.e.

**Exercise 3.** The uniqueness of the functions $|\partial f|_j$ is an obvious corollary of their property $(ii)$. It is enough to prove that $|\partial_j f|$ satisfies $(i)$, because it obviously satisfies $(ii)$. Let $T \in \mathcal{A}_Q$ and $\{T_{i_k}\} \subseteq \{T_i\}$ be such that $T_{i_k} \to T$. Then, $\mathcal{G}(f, T_{i_k}) \to \mathcal{G}(f, T)$ in $L^p$ and, hence, for every $\psi \in C_c^\infty(\Omega)$,

$$\left|\int \partial_j \mathcal{G}(f,T)\,\psi\right| = \lim_{i_k \to +\infty} \left|\int \mathcal{G}(f, T_{i_k})\,\partial_j \psi\right| = \lim_{i_k \to +\infty} \left|\int \partial_j \mathcal{G}(f, T_{i_k})\,\psi\right| \leq \int g_j\,|\psi|. \tag{5.3}$$

Since (5.3) holds for every $\psi$, we conclude $|\partial_j \mathcal{G}(f,T)| \leq g_j$ a.e.

**Exercise 4.** Define the following subsets of the unit disk,

$$\mathcal{D}_j = \left\{r\,e^{i\theta} : 0 < r < 1,\, (j-1)\,2\pi/Q < \theta < j\,2\pi/Q\right\} \text{ and } \mathcal{C} = \left\{r\,e^{i\theta} : 0 < r < 1,\, \theta \neq 0\right\},$$

and let $\varphi_j : \mathcal{C} \to \mathcal{D}_j$ be determinations of the $Q^{th}$-root, i.e.

$$\varphi_j\left(re^{i\theta}\right) = r^{\frac{1}{Q}}\, e^{i\left(\frac{\theta}{Q} + (j-1)\,\frac{2\pi}{Q}\right)}.$$

It is easily recognized that $f|_{\mathcal{C}} = \sum_j [\![\zeta \circ \varphi_j]\!]$. So, by the invariance of the Dirichlet energy under conformal mappings, one deduces that $f \in W^{1,2}(\mathcal{C}, \mathcal{A}_Q)$ and

$$\mathrm{Dir}(f, \mathcal{C}) = \sum_{i=1}^{Q} \mathrm{Dir}(\zeta \circ \varphi_i, \mathcal{C}) = \int_{\mathbb{D}} |D\zeta|^2\,. \tag{5.4}$$

From the above argument and from (5.4), it is straightforward to infer that $f$ belongs to $W^{1,2}(\mathbb{D}, \mathcal{A}_Q)$ and (2.9) holds. Finally, (2.10) is a simple computation left to the reader.

**Exercise 5.** All the formulas are just routine modifications of the classical chain-rule. The proof of the first formula follows easily from Definition 1.7. Since $f$ is differentiable at $x_0$, we have

$$\begin{aligned}\mathcal{G}\left(f \circ \Phi(y), \sum_i [\![Df_i(x_0) \cdot (\Phi(y) - \Phi(y_0)) + f_i(\Phi(y_0))]\!]\right) &= o\left(|\Phi(y) - \Phi(y_0)|\right) \\ &= o\left(|y - y_0|\right),\end{aligned} \tag{5.5}$$

where the last equality follows from the differentiability of $\Phi$ at $y_0$. Moreover, again due to the differentiability of $\Phi$, we infer that

$$Df_i(x_0) \cdot (\Phi(y) - \Phi(y_0)) = Df_i(x_0) \cdot D\Phi(y_0) \cdot (y - y_0) + o(|y - y_0|). \tag{5.6}$$

Therefore, (5.5) and (5.6) imply the conclusion.

For what concerns the second, we note that we can reduce to the case of $\mathrm{card}(f(x_0)) = 1$, i.e.

$$f(x_0) = Q\,[\![u_0]\!] \quad \text{and} \quad Df(x_0) = Q\,[\![L]\!]\,. \tag{5.7}$$

Indeed, since $f$ is differentiable (hence, continuous) in $x_0$, in a neighborhood of $x_0$ we can decompose $f$ as the sum of differentiable multi-valued functions $g_k$, $f = \sum_k [\![g_k]\!]$, such that $\mathrm{card}(g_k(x_0)) = 1$. Then, $\Psi(x, f) = \sum_k [\![\Psi(x, g_k)]\!]$ in a neighborhood of $x_0$, and the differentiability of $\Psi(x, f)$ follows from the differentiability of the $\Psi(x, g_k)$'s. So, assuming (5.7), without loss of generality, we have to show that

$$h(x) = Q\,[\![D_u\,\Psi(x_0, u_0) \cdot L \cdot (x - x_0) + D_x\,\Psi(x_0, u_0) \cdot (x - x_0) + \Psi(x_0, u_0)]\!]$$

is the first-order approximation of $\Psi(x, f)$ in $x_0$. Set

$$A_i(x) = D_u\,\Psi(x_0, u_0) \cdot (f_i(x) - u_0) + D_x\,\Psi(x_0, u_0) \cdot (x - x_0) + \Psi(x_0, u_0).$$

From the differentiability of $\Psi$, we deduce that

$$\mathcal{G}\left(\Psi(x, f), \sum_i [\![A_i(x)]\!]\right) = o\big(|x - x_0| + \mathcal{G}(f(x), f(x_0))\big) = o\left(|x - x_0|\right), \tag{5.8}$$

where we used the differentiability of $f$ in the last step. Hence, we can conclude

$$\begin{aligned}\mathcal{G}\left(\Psi(x, f), h(x)\right) &\leq \mathcal{G}\left(\Psi(x, f), \sum_i [\![A_i(x)]\!]\right) + \mathcal{G}\left(\sum_i [\![A_i(x)]\!], h(x)\right) \\ &\leq o\left(|x - x_0|\right) + \|D_u\,\Psi(x_0, u_0)\|\mathcal{G}\left(\sum_i [\![f_i(x)]\!], Q\,[\![L \cdot (x - x_0) + u_0]\!]\right) \\ &= o\left(|x - x_0|\right).\end{aligned}$$

where $\|D_u\psi(x_0, u_0)\|$ denotes the Hilbert–Schmidt norm of the matrix $D_u\,\Psi(x_0, u_0)$.

Finally, to prove the third formula, fix $x$ and let $\pi$ be such that

$$\mathcal{G}\big(f(x), f(x_0)\big)^2 = \sum_i |f_{\pi(i)}(x) - f_i(x_0)|^2.$$

By the continuity of $f$ and (ii) of Definition 1.7, for $|x - x_0|$ small enough we have

$$\mathcal{G}\big(f(x), T_{x_0}f(x)\big)^2 = \sum_i |f_{\pi(i)}(x) - Df_i(x_0) \cdot (x - x_0) - z_i|^2. \tag{5.9}$$

Set $f_i(x_0) = z_i$ and $z = (z_1, \ldots, z_Q) \in (\mathbb{R}^n)^Q$. The differentiability of $F$ implies

$$\left|F \circ f(x) - F \circ f(x_0) - \sum_i D_{y_i}F(z) \cdot \big(f_{\pi(i)}(x) - z_i\big)\right| = o\left(\mathcal{G}(f(x), f(x_0)\right) = o(|x - x_0|). \tag{5.10}$$

Therefore, for $|x - x_0|$ small enough, we conclude

$$\begin{aligned}&\left|\sum_i D_{y_i}F(z) \cdot \big(f_{\pi(i)}(x) - z_i - Df_i(x_0) \cdot (x - x_0)\big)\right| \leq \\ &\qquad \leq C \sum_i |f_{\pi(i)}(x) - Df_i(x_0) \cdot (x - x_0) - z_i| \overset{(5.9)}{=} o(|x - x_0|),\end{aligned} \tag{5.11}$$

with $C = \sup_i \|D_{y_i}F(z)\|$. Therefore, using (5.10) and (5.11), we conclude.

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