Isoperimetric inequalities and the structure of metric spaces - Part 2

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July 20, 2013

Overview - Part 2

Review and aim

X metric space, $c\colon S^1 \to X$ Lipschitz curve

$$\text{Fillarea}_0(c) = \inf\left\{\text{Area}(\varphi) : \varphi\colon D \to X \text{ Lip}, \varphi|_{S^1} = c\right\}.$$

Filling area function in X:

$$\boxed{\text{FA}_0^X(r) = \sup\left\{\text{Fillarea}_0(c) : \text{L}(c) \leq r\right\}.}$$

Aim is to prove

Theorem (W.)

X geodesic metric space. If there exists $\varepsilon > 0$ such that

$$\text{FA}_0^X(r) \leq \frac{1-\varepsilon}{4\pi} r^2 \quad \forall r \gg 1$$

then $\text{FA}_0^X(r) \preceq r$.

. . . and . . .

Theorem (W.)

There exists nilpotent Lie group G of step 2 such that $\mathrm{FA}_0^G(r) \not\asymp r^\alpha$ for any $\alpha \in \mathbb{R}$.

Basic idea in both proofs:

1. Rescale metric d in X by factors $r_n \searrow 0$.

   Sequence $(X, r_n d)$ has limit $\rightsquigarrow$ asymptotic cones X_ω.

2. Would like:

$$\boxed{\mathrm{FA}_0^X(r) \preceq r^2 \Rightarrow \mathrm{FA}_0^{X_\omega}(r) \leq Cr^2 \ \forall r \geq 0.}$$

True? True for homological version $\mathrm{FA}^{X_\omega}(r)$ of $\mathrm{FA}_0^{X_\omega}(r)$.

Ambrosio-Kirchheim currents: *m*-forms in metric spaces

X complete metric space

Idea (De Giorgi): Use $(m+1)$-tuples

$$(f, \pi_1, \dots, \pi_m)$$

of Lipschitz functions on X as substitute for m-forms.

If $X = \mathbb{R}^N$ and f, π_i smooth then think

$$\boxed{(f, \pi_1, \dots, \pi_m) \quad \leftrightsquigarrow \quad f\, d\pi_1 \wedge \dots \wedge d\pi_m.}$$

Note:

- $d(f\, d\pi_1 \wedge \dots \wedge d\pi_m) = df \wedge d\pi_1 \wedge \dots \wedge d\pi_m$
- If $\varphi : \mathbb{R}^M \to \mathbb{R}^N$ smooth then

$$\varphi^*(f\, d\pi_1 \wedge \dots \wedge d\pi_m) = f \circ \varphi\, d(\pi_1 \circ \varphi) \wedge \dots \wedge d(\pi_m \circ \varphi)$$

Definition

For X complete metric space and $m \geq 0$:

$$\mathscr{D}^m(X) := \mathrm{Lip}_b(X) \times \mathrm{Lip}(X)^m.$$

Notation:

- $\mathrm{Lip}(X) := \{f\colon X \to \mathbb{R} \ : \ f \text{ Lipschitz}\}$
- $\mathrm{Lip}_b(X) := \{f\colon X \to \mathbb{R} \ : \ f \text{ Lipschitz and bounded}\}$
- $\mathscr{B}^\infty(X) := \{f\colon X \to \mathbb{R} \ : \ f \text{ Borel and bounded}\}$
- For $f : X \to Y$ set

$$\mathrm{Lip}(f) := \sup_{x \neq x'} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}$$

Currents in metric spaces (Ambrosio-Kirchheim)

Definition: A function $T\colon \mathscr{D}^m(X) \to \mathbb{R}$ is called m-current if:

(i) T is multi-linear.

(ii) If $\pi_i^n \in \operatorname{Lip}(X)$ with $\sup_n \operatorname{Lip}(\pi_i^n) < \infty$ and $\pi_i^n \to \pi_i$ then

$$T(f, \pi_1^n, \dots, \pi_m^n) \to T(f, \pi_1, \dots, \pi_m).$$

(iii) If π_i is constant on $\operatorname{spt} f$ for some i then

$$T(f, \pi_1, \dots, \pi_m) = 0.$$

(iv) $\exists$ finite Borel measure μ, concentrated on σ-cpt set, such that

$$|T(f, \pi_1, \dots, \pi_m)| \leq \prod_{i=1}^m \operatorname{Lip}(\pi_i) \int_X |f| d\mu$$

for all $(f, \pi_1, \dots, \pi_m) \in \mathscr{D}^m(X)$.

Space of m-currents in X:

$$\boxed{\mathbf{M}_m(X) := \left\{ m\text{-currents in } X \right\}.}$$

Remark: Each $T \in \mathbf{M}_m(X)$ extends uniquely to

$$T : \mathscr{B}^\infty(X) \times \mathrm{Lip}(X)^m \to \mathbb{R}.$$

Can show it satisfies same properties (i) – (iv) with $f \in \mathscr{B}^\infty(X)$.

Proposition

Given $T \in \mathbf{M}_m(X)$ there exists a smallest Borel measure on X satisfying (iv), denoted $\|T\|$. For $U \subset X$ open

$$\|T\|(U) = \sup \left\{ \sum_n |T(f_n, \pi^n)| : \mathrm{Lip}(\pi_i^n) \leq 1, \sum |f_n| \leq 1_U \right\}.$$

Define mass of T by

$$\boxed{\mathbf{M}(T) := \|T\|(X).}$$

Lower semi-continuity of mass:

Corollary

If $(T_n) \subset \mathbf{M}_m(X)$ converges weakly (i.e. pointwise) to $T \in \mathbf{M}_m(X)$ then

$$\mathbf{M}(T) \leq \liminf_{n\to\infty} \mathbf{M}(T_n).$$

Remark: $\mathbf{M}_m(X)$ becomes a complete metric space with

$$d_M(T, S) := \mathbf{M}(T - S).$$

Constructions for currents

Constructions for $T \in \mathbf{M}_m(X)$:

(i) If $m \geq 1$ define

$$\partial T(f, \pi_1, \dots, \pi_{m-1}) := T(1, f, \pi_1, \dots, \pi_{m-1}).$$

$\Rightarrow$ ∂T satisfies axioms (i) – (iii) for currents; and $\partial\partial T = 0$.

(ii) If $\varphi : X \to Y$ Lipschitz define

$$\varphi_\# T(g, \tau_1, \dots, \tau_m) := T(g \circ \varphi, \tau_1 \circ \varphi, \dots, \tau_m \circ \varphi).$$

$\Rightarrow$ $\varphi_\# T \in \mathbf{M}_m(Y)$ and $\|\varphi_\# T\| \leq \mathrm{Lip}(\varphi)^m \varphi_\# \|T\|$.

(iii) For $A \subset X$ Borel define

$$(T \llcorner A)(f, \pi_1, \dots, \pi_m) := T(f 1_A, \pi_1, \dots, \pi_m).$$

$\Rightarrow$ $T \llcorner A \in \mathbf{M}_m(X)$ and $\|T \llcorner A\| = \|T\| \llcorner A$.

Normal currents

Example: For $\theta \in L^1(\mathbb{R}^m)$

$$[\![\theta]\!](f, \pi) := \int_{\mathbb{R}^m} f\theta \det(d\pi) d\mathscr{L}^m$$

defines metric m-current in $\mathbb{R}^m$ and $\|[\![\theta]\!]\| = |\theta| d\mathscr{L}^m$.

Space of normal m-currents in X:

$$\mathbf{N}_m(X) := \{T \in \mathbf{M}_m(X) : \partial T \in \mathbf{M}_{m-1}(X)\}.$$

Theorem

If X is compact and $(T_n) \subset \mathbf{N}_m(X)$ satisfies

$$\sup_n [\mathbf{M}(T_n) + \mathbf{M}(\partial T_n)] < \infty$$

then there exists (T_{n_j}) converging weakly to some $T \in \mathbf{N}_m(X)$.

Integer rectifiable and integral currents

Definition

$T \in \mathbf{M}_0(X)$ is called integer rectifiable if $\exists x_i \in X$ and $m_i \in \mathbb{Z}$, $i = 1, \dots, n$, with

$$T = \sum_{i=1}^{n} m_i [\![x_i]\!].$$

Definition

$T \in \mathbf{M}_m(X)$ with $m \geq 1$ is called integer rectifiable if

(i) $\|T\|$ is concentrated on a countably $\mathscr{H}^m$-rectifiable set.

(ii) $\|T\|$ vanishes on $\mathscr{H}^m$-negligible Borel sets.

(iii) $\forall \varphi : U \to \mathbb{R}^m$ Lipschitz, $U \subset X$ open, $\exists \theta \in L^1(\mathbb{R}^m, \mathbb{Z})$ with

$$\varphi_\#(T \llcorner U) = [\![\theta]\!].$$

Space of integer rectifiable m-currents in X:

$$\boxed{\mathcal{I}_m(X) := \{T \in \mathbf{M}_m(X) : T \text{ integer rectifiable}\}.}$$

Representation of integer rectifiable currents:

Theorem

If $T \in \mathcal{I}_m(X)$ with $m \geq 1$ then $\exists\, \psi_i : K_i \to X$ biLipschitz, $K_i \subset \mathbb{R}^m$ cpt, and $\exists\, \theta_i \in L^1(K_i, \mathbb{Z})$ such that

$$T = \sum_{i=1}^{\infty} \psi_{i\#}[\![\theta_i]\!] \quad \text{and} \quad \mathbf{M}(T) = \sum_{i=1}^{\infty} \mathbf{M}(\psi_{i\#}[\![\theta_i]\!]).$$

Integral currents and Closure Theorem

Space of integral m-currents in X:

$$\mathbf{I}_m(X) := \mathcal{I}_m(X) \cap \mathbf{N}_m(X).$$

Closure Theorem:

Theorem

If $T \in \mathbf{N}_m(X)$ is the weak limit of a sequence $(T_n) \subset \mathbf{I}_m(X)$ with

$$\sup_n \left[\mathbf{M}(T_n) + \mathbf{M}(\partial T_n)\right] < \infty$$

then $T \in \mathbf{I}_m(X)$.

Ultralimits of metric spaces

How to define limit of (arbitrary) sequence of metric spaces?

Let (X_n) be sequence of metric spaces $X_n = (X_n, d_n)$.

For given base points $p_n \in X_n$ define

$$\hat{X} = \left\{ (x_n)_{n\in\mathbb{N}} \ : \ x_n \in X_n, \sup_n d_n(x_n, p_n) < \infty \right\}.$$

For $(x_n), (x'_n) \in \hat{X}$ want to define distance by

$$\lim_{n\to\infty} d_n(x_n, x'_n)$$

if limit exists.

$\exists$ device making consistent "choice" of convergent subsequences:

Non-principal ultrafilters

Let ω be a non-principal ultra-filter on $\mathbb{N}$:

- $\omega : 2^{\mathbb{N}} \to \{0, 1\}$ finitely additive;
- $\omega(\mathbb{N}) = 1$ and $\omega(A) = 0$ for all $A \subset \mathbb{N}$ finite.

Existence: Zorn's lemma.

Proposition

If Z is compact Hausdorff and $(z_n) \subset Z$ then $\exists! \, z \in Z$ such that for all $U \subset Z$ open with $z \in U$

$$\omega(\{n \in \mathbb{N} : z_n \in U\}) = 1.$$

Notation: $\lim_\omega z_n := z$.

For $(x_n), (x'_n) \in \hat{X}$ define

$$d_\omega((x_n), (x'_n)) := \lim_\omega d_n(x_n, x'_n).$$

$\Rightarrow d_\omega$ is pseudo-metric on $\hat{X}$.

Definition

Ultralimit of (X_n, d_n, p_n) with respect to ω:

$$(X_n, d_n, p_n)_\omega := \left(\hat{X}/_\sim, d_\omega, [(p_n)]\right),$$

where $(x_n) \sim (x'_n) :\Leftrightarrow d_\omega((x_n), (x'_n)) = 0$.

Examples:

1. X proper, $p \in X \Rightarrow (X, p)_\omega \cong (X, p)$.
2. X_n proper, $(X_n, p_n) \xrightarrow{\text{pt. GH}} (X_\infty, p_\infty) \Rightarrow (X_n, p_n)_\omega \cong (X_\infty, p_\infty)$.
3. E ∞-dim. Banach $\Rightarrow (E, 0)_\omega \not\cong (E, 0)$.

Asymptotic cones

(X, d) metric space, $(p_n) \subset X$, $r_n \searrow 0$, ω non-principal ultrafilter

Definition

Ultralimit of $(X, r_n d, p_n)$ w.r.t. ω is called asymptotic cone of X.

Notation: $X_\omega := (X, r_n d, p_n)_\omega$.

Theorem (Gromov)

For a geodesic metric space X the following are equivalent:

(i) *X is Gromov hyperbolic.*

(ii) *Every asymptotic cone of X is a metric tree.*

Homological filling area function

X, Y geodesic metric spaces, Y complete, and $X \subset Y$

For $c\colon S^1 \to X$ Lipschitz define

$$\boxed{\operatorname{Fillarea}^Y(c) = \inf\left\{\mathbf{M}(S) : S \in \mathbf{I}_2(Y),\ \partial S = c_\#[\![1_{[0,1]}]\!]\right\}.}$$

Note: $\exists \lambda \geq 1$ such that $\operatorname{Fillarea}^X(c) \leq \lambda \operatorname{Fillarea}_0(c)$ for all c.

Definition

The generalized homological filling area function in X w.r.t. Y is

$$\operatorname{FA}^{X,Y}(r) = \sup\left\{\operatorname{Fillarea}^Y(c) :\ c\colon S^1 \to X \text{ Lip., } \operatorname{L}(c) \leq r\right\}$$

for all $r \geq 0$. Abbreviation: $\operatorname{FA}^X(r) := \operatorname{FA}^{X,X}(r)$.

Remark concerning $\mathrm{FA}^X(r)$

For $T \in \mathbf{I}_1(X)$ with $\partial T = 0$ define

$$\boxed{\mathrm{Fillarea}^Y(T) := \inf\{\mathbf{M}(S) : S \in \mathbf{I}_2(Y), \partial S = T\}.}$$

Definition

Y has quadratic isoperimetric inequality for $\mathbf{I}_1(Y)$ if there exists $C \geq 0$ with

$$\mathrm{Fillarea}^Y(T) \leq C\,\mathbf{M}(T)^2$$

for all $T \in \mathbf{I}_1(Y)$ with $\partial T = 0$.

The following are equivalent:

(i) Y has quadratic isoperimetric inequality for $\mathbf{I}_1(Y)$.

(ii) $\exists D$ such that $\mathrm{FA}^Y(r) \leq Dr^2$ for all $r \geq 0$.

Quadratic filling passes to asymptotic cone

Proposition (W.)

X, Y geodesic, $X \subset Y$, Y complete, at bounded distance of X. If

$$\mathrm{FA}^{X,Y}(r) \preceq r^2$$

then $\exists\, C \geq 0$ such that for every asymptotic cone X_ω of X

$$\mathrm{FA}^{X_\omega}(r) \leq Cr^2$$

for all $r \geq 0$.

Direct consequence:

$$\boxed{\mathrm{FA}_0^X(r) \preceq r^2 \quad \Rightarrow \quad \mathrm{FA}^{X_\omega}(r) \leq Cr^2 \quad \forall\, r \geq 0.}$$

Preparation for proof

Let Y be complete metric space. Suppose Y has quadratic isoperimetric inequality for $\mathbf{I}_1(Y)$ with constant C.

Lemma (Ambrosio-Kirchheim)

Let $T \in \mathbf{I}_1(Y)$ with $\partial T = 0$ and $\varepsilon > 0$. Then $\exists\, S \in \mathbf{I}_2(Y)$ with

(i) $\partial S = T$

(ii) $\mathbf{M}(S) \leq (1+\varepsilon)\,\mathrm{Fillarea}^Y(T)$

(iii) *for each* $x \in \operatorname{spt} S$ *and all* $r \in [0, d(x, \operatorname{spt} T)]$

$$\boxed{\|S\|(B(x,r)) \geq \frac{1}{4C} r^2.}$$

Remark: Also works for higher-dimensional integral currents.