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RENORMALIZATION OF HENON MAPS: A SURVEY

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Contents

1. Introduction

Since the universality discoveries, made in the mid-1970's by Feigenbaum [F1, F2] and, independently, by Coullet and Tresser [CT, TC], these fundamental phenomena have attracted a great deal of attention from mathematicians, pure and applied, and physicists. In particular, universality and the corresponding geometric rigidity of the attractors at the transition from regular behavior to chaotic behavior are central themes in one-dimensional dynamics. Coullet and Tresser conjectured that the universal geometry at transition to chaos in one-dimensional dynamics will also be observed in higher dimensional systems. This conjecture has been confirmed by many numerical and physical experiments.[1]

A rigorous study of universality and rigidity has been surprisingly difficult and technically sophisticated and so far has only been thoroughly carried out in the case of one-dimensional maps, on the interval or the circle (see [FMP, He, L, Ma2, McM, S, VSK, Y] and references

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[1]This conjecture should be taken with caution as not every transition to chaos is related to a transition in one-dimensional dynamics.

therein). The study of universality and rigidity is in essence the study of a corresponding renormalization operator. This operator replaces a system by another which describes the original systems on a smaller scale. It acts like a microscope.

A frequently observed transition to chaos in one-dimensional dynamics is the so-called *period doubling cascade to chaos*. The corresponding renormalization operator has a unique hyperbolic fixed point. The dynamics of the renormalization fixed point, which is itself a one-dimensional system, and the behavior of the renormalization operator around this fixed point determine the asymptotic small scale geometry of systems at transition and the asymptotic small scale properties around the boundary of chaos. This explains the observed universality.

A rigorous exploration of universality for dissipative higher dimensional systems was begun in an article by Collet, Eckmann and Koch [CEK]. It is shown in this article that the one-dimensional renormalization fixed point is also a hyperbolic fixed point for renormalization of strongly dissipative higher-dimensional maps close to the one-dimensional renormalization fixed point: this explained the parameter universality observed in families of such systems.

A subsequent paper by Gambaudo, van Strien and Tresser [GST] demonstrates that, similarly to the one-dimensional situation, infinitely renormalizable two-dimensional maps which are close to the one-dimensional renormalization fixed point have an attracting Cantor set which is, up to topological equivalence, the same as the attractor of the renormalization fixed point.

Observations in physical and numerical experiments indicate that universality and rigidity are also playing a crucial role in higher dimensional dynamics. This survey discusses two-dimensional strongly dissipative Hénon maps at transition to chaos. These are maps which are infinitely renormalizable of period doubling type. Indeed, there is still universality and rigidity but in a much more delicate form.

The Cantor attractors of infinitely renormalizable maps of period doubling type cannot be understood geometrically in terms of their one-dimensional counterpart. Though they lie on a rectifiable curve, the geometry of the Cantor attractors in small scales essentially differs from their one-dimensional counterpart. In fact, a typical map in a family will have *unbounded geometry*. However, almost everywhere the Cantor attractors have the same geometry as their one-dimensional counterpart, so we encounter here a new phenomenon of *probabilistic universality* and *probabilistic rigidity*.

The topology of infinitely renormalizable maps of period doubling type also differs from the one-dimensional equivalent. These differences come from the bifurcations in the heteroclinic web of the maps in question. The *average Jacobian* of such a map, which is an ergodic theoretical invariant, is closely related to the observed differences in topology and geometry. By changing the average Jacobian one changes the topology of the heteroclinic web and the geometry of the non-universal part of the Cantor attractor.

This survey is based on the series of articles [CLM, LM2, LM3, HLM, C]. The results of section §3, §4, and parts of §5 are generalized by P. Hazard, to maps of more general periodic renormalization types and can be found in [H].

2. Unimodal Renormalization

A unimodal map is a smooth map of the interval with only one critical point. The critical point is non-degenerate. A smooth unimodal map $f \in \mathcal{U}$ is *renormalizable* if it contains two disjoint intervals which are exchanged by the map. The two smallest intervals which are exchanged form the first renormalization cycle, $\mathcal{C}_1 = \{I_0^1, I_1^1\}$, where I_0^1 contains the critical point c of f. Let $\mathcal{U}_0$ be the collection of renormalizable maps. The renormalization of $f \in \mathcal{U}_0$ is an affinely rescaled version of the first return map to I_0^1, $f^2 : I_0^1 \to I_0^1$. This defines an operator

$$R_c : \mathcal{U}_0 \to \mathcal{U}$$

Similarly, one can rescale the first return map to I_1^1, the interval which contains the critical value v of f. This defines the second renormalization operator

$$R_v : \mathcal{U}_0 \to \mathcal{U}.$$

The intervals I_0^1 and I_1^1 are called *renormalization domains*.

These renormalization operators are microscopes used to the study the small scale geometry of the dynamics. In particular, $R_c f$ is a unimodal map which describes the dynamics on one scale lower in I_0^1. Similarly $R_v f$ describes the geometry one scale smaller in I_1^1. The strength of renormalization is expressed by the Coullet-Tresser-Feigenbaum Conjecture whose proof has a long history (see [FMP, He, Ma2, McM, S, VSK, Y] and references therein) and was finally obtained in [L].

Theorem 2.1. *There is a unique fixed point f_* of R_c. It is a hyperbolic fixed point with codimension one stable manifold and a one dimensional unstable manifold. The operator R_v also has a unique fixed point f_v.*

It is also hyperbolic and its stable manifold coincides with the stable manifold of R_c.

Remark 2.1. The hyperbolicity of the renormalization operators depends on the smoothness class of the unimodal maps. The hyperbolicity holds on the class of $C^{2+\alpha}$ unimodal maps with non degenerate critical point. Compare [CMMT], and [FMP].

Remark 2.2. The relative length of I_c^1 of f_* in the domain of f_* is called the *universal scaling ratio*. It is denoted by $\sigma < 1$.

A map is infinitely renormalizable if it can be renormalized infinitely many times. That means for each $n \geq 1$ $R^n f \in \mathcal{U}_0$. An infinitely renormalizable map has cycles, pairwise disjoint intervals,

$$\mathcal{C}_n = \{I_i^n | i = 0, 1, 2, \ldots, 2^n - 1\},$$

with $f(I_i^n) = I_{i+1}^n$ and

$$\bigcup \mathcal{C}_{n+1} \subset \bigcup \mathcal{C}_n.$$

This nested sequence of dynamical cycles accumulates on a Cantor set.

$$\mathcal{C} = \bigcap \bigcup \mathcal{C}_n.$$

This Cantor set attracts almost every orbit. It is called the Cantor attractor of the map. The only points whose orbits are not attracted to this Cantor set are the periodic point, of period 2^n, and their stable manifolds. The cycle $\mathcal{C}_n$ is centered around a periodic orbit of length 2^{n+1}. It contains all the periodic orbits of period 2^s with $s \geq n+1$.

Every small part of the Cantor attractor $\mathcal{C}$ of some infinitely renormalizable map, say within an interval I_i^n of the $n^{th}-$cycle, can be studied by repeatedly applying one of the renormalization operators R_c or R_v. For each interval in the cycle there is a uniquely defined sequence of length n of choices $w = (c, c, v, c, \ldots, v)$ such that the

$$R_w f = R_c \circ R_c \circ R_v \circ R_c \circ \cdots \circ R_v f$$

describes the dynamics within the given I_i^n. This means that $R_w f$ is an affinely rescaled version of the first return map to I_i^n. Denote the length of a word w by $|w|$. The collection of infinitely renormalizable maps coincides with the stable manifold of the two renormalization operators.

Theorem 2.2. *(Universality) There exists $\rho < 1$ such that for any two infinitely renormalizable maps $f, g \in \mathcal{U}_0$ and any finite word w*

$$dist(R_w f, R_w g) = O(dist(f, g)\rho^{|w|}).$$

The universality means that the Cantor attractors of two infinitely renormalizable maps are asymptotically the same on small scale. However, the actual geometry one observes depends on the place where one zooms in. This universal geometric structure of attractor is far from the well-known middle-third Cantor set, where in every place one recovers the same geometry. In the Cantor attractor there are essentially no two places with the same asymptotic geometry, [BMT].

Given two infinitely renormalizable maps $f, g \in \mathcal{U}$, there exists a homeomorphism h between the domains of the two maps which maps orbits to orbits,

$$h \circ f = g \circ h.$$

The maps are conjugated, the homeomorphism is called a conjugation. The dynamics of two conjugated maps are the same from a topological point of view.

Theorem 2.3. *(Rigidity) The conjugation between two infinitely renormalizable maps is differentiable on the attractor.*

If a conjugation is differentiable, it means that on small scale the conjugation is essentially affine. This means that the microscopic geometrical properties of corresponding parts of the attractor are the same. One can deform an infinitely renormalizable map to another infinitely renormalizable map which will deform the geometry on large scale. However, the microscopic structure of the Cantor attractor is not changed: this is the rigidity phenomenon.

The *topology* of the system determines the *geometry* of the system.

This central idea has been rigorously justified in one-dimensional dynamics. It also holds when the systems are not of the period doubling type described above but have topological characteristics which are tame. We will not discuss the most general statement and omit the precise definition of tameness.

3. Hénon Renormalization

A smooth map $F\colon B \to \mathbb{R}^2$, $B = I^h \times I^v$ is called a *Hénon map* if it maps vertical sections of B to horizontal segments, while the horizontal sections are mapped to parabola-like arcs (i.e., graphs of unimodal functions over the y-axis). Examples of Hénon maps are given by small perturbations of unimodal maps of the form

$$F(x, y) = (f(x) - \varepsilon(x, y),\, x), \tag{3.1}$$

where $f\colon I^h \to \mathbb{R}$ is unimodal and ε is small. Note that, in this case, the Jacobian is

$$\operatorname{Jac} F = |\frac{\partial \varepsilon}{\partial y}|.$$

If $\partial\varepsilon/\partial y \neq 0$ then the vertical sections are mapped diffeomorphically onto horizontal arcs, so that F is a diffeomorphism onto a "thickening" of the graph $\Gamma_f = \{(f(x), x)\}_{x\in I^h}$ (Figure 3.1). In this case F is a

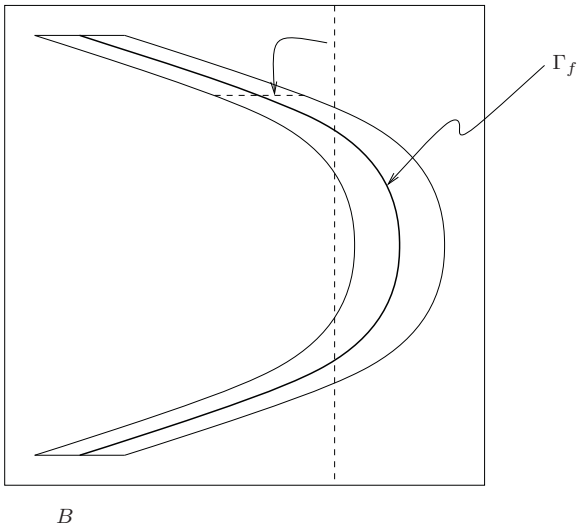

FIGURE 3.1. A Hénon-like map.

diffeomorphism onto its image. The classical Hénon family is obtained, up to affine normalization, by letting $f(x)$ be a quadratic polynomial and $\varepsilon(x, y) = by$. A Hénon map with $\varepsilon = 0$ is called a *degenerate Hénon map*. We will mainly discuss *strongly dissipative* maps, i.e. map with a small ε.

Let $\Omega^h, \Omega^v \subset \mathbb{D}_2 \subset \mathbb{C}$ be neighborhoods of I^h and I^v resp. and $\Omega = \Omega^h \times \Omega^v \subset \mathbb{C}^2$. Let $\mathcal{H}_\Omega$ stand for the class of Hénon maps $F \in \mathcal{H}^\omega$ of form (3.1) such that the unimodal map f admits a holomorphic extension to Ω^h and ε admits a holomorphic extension to Ω. The subspace of maps $F \in \mathcal{H}_\Omega$ with $\|\varepsilon\|_\Omega \leq \bar{\varepsilon}$ will be denoted by $\mathcal{H}_\Omega(\bar{\varepsilon})$.

Realizing a unimodal map f as a degenerate Hénon map F_f with $\varepsilon = 0$ yields an embedding of the space of unimodal maps $\mathcal{U}_{\Omega^h}$ into the space of Hénon maps $\mathcal{H}_\Omega$ making it possible to think of $\mathcal{U}_{\Omega^h}$ as a subspace of $\mathcal{H}_\Omega$.

3.1. **Renormalizable Hénon maps.** An orientation preserving Hénon map is *renormalizable* if it has two saddle fixed points — a *regular* saddle β_0, with positive eigenvalues, and a *flip* saddle β_1, with negative eigenvalues — such that the unstable manifold $W^u(\beta_0)$ intersects the stable manifold $W^s(\beta_1)$ at a single orbit, see Figure 3.2.

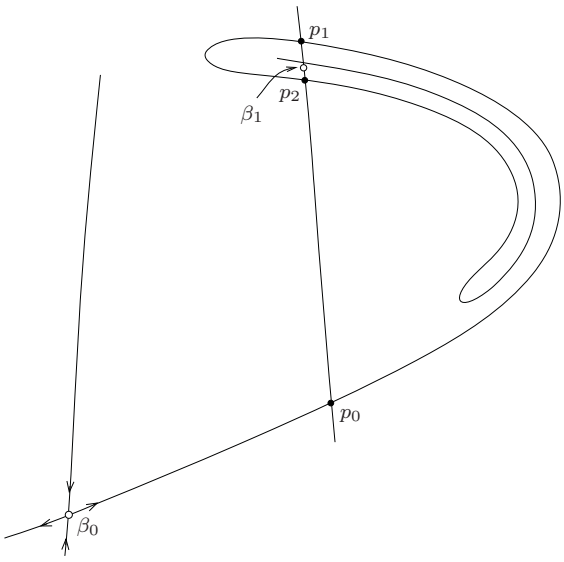

FIGURE 3.2. A renormalizable Hénon map.

For example, if f is a twice renormalizable unimodal map then a small Hénon perturbation of type (3.1) is a renormalizable Hénon-like map.

Given a renormalizable map F, consider an intersection point $p_0 \in W^u(\beta_0) \cap W^s(\beta_1)$, and let $p_n = F^n(p_0)$. Let D be the topological disk bounded by the arcs of $W^s(\beta_1)$ and $W^u(\beta_0)$ with endpoints at p_0 and p_1. The disk D is invariant under F^2. The map $F^2|D$ is called a *pre-renormalization* of F.

The topological notion of pre-renormalization is not convenient for the analysis of renormalizable maps. The Hénon renormalization operator introduced in [CLM] has three non-conventional aspects. The renormalization domain has a geometric definition, not a topological definition. The renormalization domain is a neighborhood of the *tip*. The tip plays the role of critical value, it will be discussed in more detail. Finally, the *rescalings* are not affine maps, they are carefully chosen diffeomorphisms. A crucial part of the analysis of the Hénon

renormalization operator deals with the repeated rescalings: the composed rescalings have a universal asymptotic shape. This asymptotic shape relates the asymptotic geometry of the renormalization to the actual asymptotic small scale geometry of the dynamics of the original map.

3.2. **The Hénon renormalization operator.** Hénon maps take vertical lines and map them into horizontal segments. The second iterate of such a map does not have this property. In general one can not find an affine coordinate change such that F^2 after this coordinate change is again a Hénon map. There is essentially only one diffeomorphic coordinate change which does bring F^2 back to the Hénon form.

Consider the map $F(x, y) = (f(x) - \varepsilon(x, y), x)$ with the norm of ε small. The second iterate F^2 maps curves of the foliation defined by

$$f(x) - \varepsilon(x, y) = \text{Const} \tag{3.2}$$

into horizontal segments. The leave of this foliation through a point (x, y) with x away from the critical point of f is an almost vertical curve. The map is renormalizable if there exists a domain bounded by two curves of the foliation and two horizontal lines which is mapped into itself by F^2. The renormalization domain will be the smallest domain with this property. Denote this domain by B^1_v, see Figure 4.1.

The renormalization domain B^1_v is foliated by almost vertical curves of the form (3.2). The diffeomorphism $H : B^1_v \to \mathbb{R}^2$ defined by

$$H(x, y) = (f(x) - \varepsilon(x, y), y)$$

straightens the leaves, it maps them into vertical straight lines. It maps horizontal lines into horizontal lines, it is a *horizontal diffeomorphism*. The image $H(B^1_v)$ is a rectangle, in fact almost a square. Define $G : H(B^1_v) \to H(B^1_v)$ by

$$G = H \circ F^2 \circ H^{-1}.$$

Let Λ be the dilation which maps $H(B^1_v)$ to a rectangle with unit horizontal size and define the renormalization of F by

$$RF = \Lambda \circ G \circ \Lambda^{-1}.$$

The coordinate change which conjugates RF with $F^2|B^1_v$ is denoted by

$$\psi = (\Lambda \circ H)^{-1}.$$

The renormalization domain of a degenerate map $F_f = (f, x)$ where f is a renormalizable unimodal map with critical point c, $I^1_0 = [f^4(c), f^2(c)]$ and $I^1_1 = [f(c), f^3(c)]$ is

$$B^1_v = I^1_1 \times I^1_0$$

and the renormalization of F is the degenerate Hénon map corresponding to the unimodal renormalization $R_c f$. Indeed, the Hénon renormalization operator extends the unimodal renormalization operator R_c.

3.3. **Hyperbolicity of the Hénon renormalization operator.** Let $\mathcal{I}_\Omega(\bar{\varepsilon})$ denote the subspace of infinitely renormalizable Hénon maps (including degenerate ones) of classes $\mathcal{H}_\Omega(\bar{\varepsilon})$. By the unimodal renormalization theory, the fixed point f_* is a quadratic-like map on some domain $\Omega_* \subset \mathbb{C}$, see e.g., [B] and references therein. Moreover, f_* is a hyperbolic fixed point of R_c in any space $\mathcal{U}_V$ with $V \Subset \Omega_*$. The corresponding degenerate Hénon map is denoted by F_*.

Theorem 3.1. *Assume $\Omega^h \Subset \Omega_*$. Then the map F_* is the hyperbolic fixed point for the Hénon renormalization operator R acting on $\mathcal{H}_\Omega$, with one-dimensional unstable manifold $\mathcal{W}^u(F_*) = \mathcal{W}^u(f_*)$ contained in the space of unimodal maps. Moreover, the differential $DR(F_*)$ has vanishing spectrum on the quotient $T\mathcal{H}_\Omega/T\mathcal{U}_{\Omega^h}$.*

The set $\mathcal{I}_\Omega(\bar{\varepsilon})$ of infinitely renormalizable Hénon maps coincides with the stable manifold

$$\mathcal{W}^s(F_*) = \{F \in \mathcal{H}_\Omega(\bar{\varepsilon})\colon\ R^n F \to F_* \text{ as } n \to \infty\},$$

which is a codimension-one real analytic submanifold in $\mathcal{H}_\Omega(\bar{\varepsilon})$.

Corollary 3.2. *For all Ω and $\bar{\varepsilon}$ as above, the intersection of $\mathcal{I}_\Omega(\varepsilon)$ with the Hénon family*

$$F_{a,b}\colon (x,y) \mapsto (a - x^2 - by, x)$$

is a real analytic curve intersecting transversally the one-dimensional slice $b = 0$ at a_*, the parameter value for which $x \mapsto a - x^2$ is infinitely renormalizable.

4. Microscopes for the Invariant Cantor set

4.1. **Design of the microscopes.** The set of n-times renormalizable maps is denoted by $\mathcal{H}^n_\Omega(\overline{\epsilon}) \subset \mathcal{H}_\Omega(\overline{\epsilon})$. If $F \in \mathcal{H}^n_\Omega(\overline{\epsilon})$ we use the notation

$$F_n = R^n F.$$

If F is a renormalizable map then its renormalization RF is well defined on the rectangle with unit horizontal size. The coordinate change $\psi = H^{-1} \circ \Lambda^{-1}$ maps this rectangle onto $A = B^1_v$. If we want to emphasize that some set, say A, is associated with a certain map F we use notation like $A(F)$.

The coordinate change which conjugates $F_k^2|A(F_k)$ to F_{k+1} is denoted by

$$\psi_v^{k+1} = (\Lambda_k \circ H_k)^{-1} : \mathrm{Dom}(F_{k+1}) \to A(F_k). \tag{4.1}$$

Here H_k is the non-affine part of the coordinate change used to define $R^{k+1}F$ and Λ_k is the corresponding dilation.

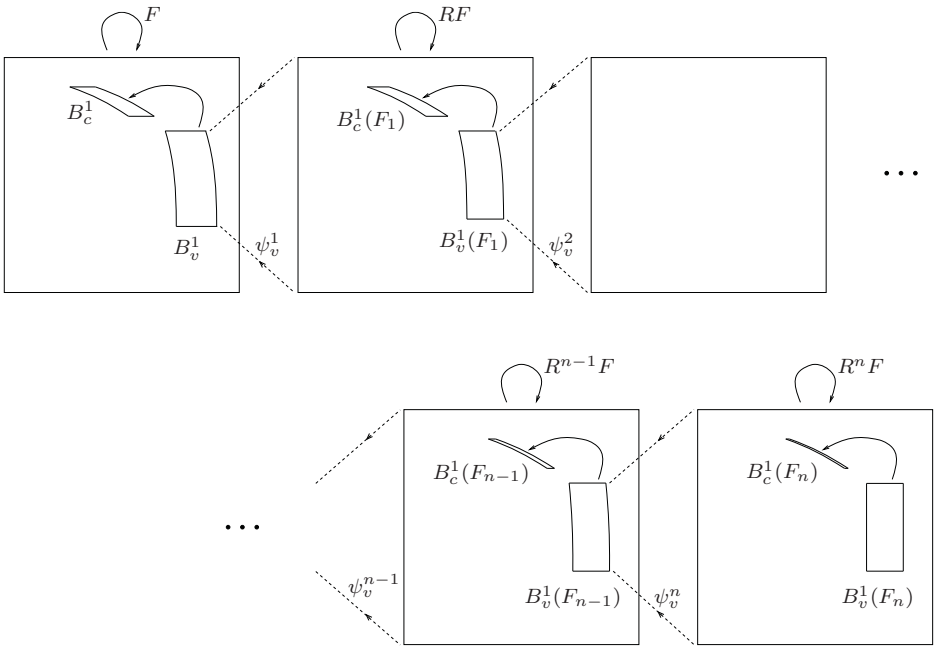

FIGURE 4.1. The renormalization microscope

Recall that the change of coordinates conjugating the renormalization RF to F^2 is denoted by

$$\psi_v^1 := H^{-1} \circ \Lambda^{-1}.$$

To describe the attractor of an infinitely renormalizable Hénon map we also need the map

$$\psi_c^1 = F \circ \psi_v^1.$$

The subscripts v and c indicate that these maps are associated to the critical *value* and the *critical* point, respectively.

If F is twice renormalizable define similarly, ψ_v^2 and ψ_c^2 be the corresponding changes of variable for RF, and let

$$\psi_{vv}^2 = \psi_v^1 \circ \psi_v^2, \quad \psi_{cv}^2 = \psi_c^1 \circ \psi_v^2, \quad \psi_{vc}^2 = \psi_v^1 \circ \psi_c^2, \quad \psi_{cc}^2 = \psi_c^1 \circ \psi_c^2.$$

For an infinitely renormalizable $F \in \mathcal{I}_\Omega(\overline{\epsilon})$ we can proceed this way, and for any $n \geq 0$, we can construct 2^n maps

$$\psi_w^n = \psi_{w_1}^1 \circ \cdots \circ \psi_{w_n}^n, \quad w = (w_1, \ldots, w_n) \in \{v, c\}^n.$$

Consider the domains

$$B_\omega^n = \operatorname{Im} \psi_\omega^n.$$

The coordinate changes ψ^n_ω conjugate R^nF to the first return map $F^{2^n} : B^n_\omega \to B^n_\omega$. In this sense they are *microscopes*. The first return maps to the nested domains

$$B^n_{v^n} = \operatorname{Im} \psi^n_{v^n}$$

correspond to the renormalizations.

The critical point and critical value of a unimodal map plays a crucial role in its dynamics. The counterpart of the critical value for infinitely renormalizable Hénon maps is the *tip*

$$\{\tau_F\} = \bigcap_{n\geq 1} B^n_{v^n}.$$

Each collection $\{B^n_\omega | \omega \in \{c, v\}^n\}$, $n \geq 1$, consists of pairwise disjoint domains. These collections are called *renormalization cycles*. They are nested, as in the unimodal case. An infinitely renormalizable Hénon map has an invariant Cantor set:

$$\mathcal{O}_F = \bigcap_{n\geq 1} \bigcup_{i=0}^{2^n-1} F^i(B^n_{v^n}) = \bigcap_{n\geq 1} \bigcup_{\omega\in\{v,c\}^n} B^n_\omega.$$

The dynamics on this Cantor set is conjugate to an adding machine. Its unique invariant measure is denoted by μ. The *average Jacobian* is

$$b_F = \exp \int \log \operatorname{Jac} F d\mu.$$

4.2. **Universality around the tip.** The convergence of renormalization in the unimodal case is used to study the small scale geometry of the Cantor attractor. This is possible because the coordinate changes used to rescale are affine. In the Hénon case the coordinate changes are not affine. Fortunately, they have a universal asymptotic limit which allows to apply the convergence of renormalization to understand the small scale geometry of the invariant Cantor set.

Let $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ and define

$$\Psi^n_0 = \psi^1_v \circ \psi^2_v \circ \cdots \circ \psi^n_v = \psi^n_{v^n}$$

which is the coordinate change which conjugates the nth-renormalization R^nF to $F^{2^n} : B^n_{v^n} \to B^n_{v^n}$. To describe these maps Ψ^n_0 we will center the coordinate systems around the tips of F and R^nF resp. In these coordinates we introduce the following notation

$$\Psi^n_0 = D^n_0 \circ (\mathrm{id} + \mathbf{S}^n_0)$$

where D_0^n is the derivative of Ψ_0^n at the tip of R^nF and

$$\mathbf{S}_0^n(x,y) = (s_0^n(x,y), 0) = O(\|(x,y)\|^2)$$

near the origin, is the non-linear part of the map. The next Theorem is a crucial tool to study infinitely renormalizable Hénon maps.

Theorem 4.1. *There exists a universal analytic function $v(x)$ and $\rho < 1$ such that the following holds. Given $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ there exists $t_F \asymp -b_F$, a_F, $C_1, C_2 > 0$ such that*

$$D_0^n \sim \begin{pmatrix} 1 & t_F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_1(\sigma^2)^n & 0 \\ 0 & C_2(-\sigma)^n \end{pmatrix}$$

and

$$|x + s_0^n(x,y) - (v(x) + a_F y^2)| = O(\rho^n).$$

Remark 4.1. The coordinate change $\psi_{v^n}^n$ between R^nF and the first return map to $B_{v^n}^n$ around the tip, has well defined asymptotic behavior. In general, the coordinate changes ψ_ω^n between R^nF and the first return map F^{2^n} to B_ω^n do not have such a limit behavior. In fact, there are pieces in the Cantor set where these coordinate changes do degenerate. The study of the general coordinate changes ψ_ω^n and the geometrical consequences is discussed in section 5.

A first consequence of the asymptotic behavior of the coordinate change Ψ_0^n is the following Theorem which describes the universality of the super-exponential convergence of Hénon renormalizations to the unimodal maps, compare Theorem 3.1.

Theorem 4.2 (Universality)**.** *For any $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ with sufficiently small $\bar{\varepsilon}$, we have:*

$$R^nF = (f_n(x) - b^{2^n} a(x)\, y\, (1 + O(\rho^n)),\ x\,),$$

where $f_n \to f_*$ exponentially fast, b is the average Jacobian, $\rho \in (0,1)$, and $a(x)$ is a universal function. Moreover, a is analytic and positive.

5. Geometry of the Invariant Cantor set

The strongly dissipative infinitely renormalizable Hénon maps are small perturbations of unimodal maps. Although the invariant Cantor sets in the one-dimensional context are rigid, the geometry of the Cantor set of an infinitely renormalizable Hénon map differs surprisingly from its one-dimensional counterpart. In particular, it can not be understood within the one-dimensional theory.

Theorem 5.1. *Given an infinitely renormalizable Hénon map F with $b_F > 0$, there are no smooth curves containing $\mathcal{O}_F$.*

The characteristic exponents of the unique invariant measure on $\mathcal{O}_F$ of an infinitely renormalizable Hénon map F with $b_F > 0$, are 0 and $\ln b_F$. The higher dimensional nature of $\mathcal{O}_F$ can be seen more specifically in the following.

Theorem 5.2. *There are no continuous invariant direction fields on $\mathcal{O}_F$ when $b_F > 0$.*

Theorem 5.3. *The map F is not partially hyperbolic on $\mathcal{O}_F$ in the sense that the contracting and neutral line fields corresponding to the characteristic exponents $\log b_F$ and 0 are discontinuous.*

The universality Theorem 4.2 and the universality of the coordinate changes as described in Theorem 4.1 imply that around the tip one recovers geometrical aspects of the one-dimensional Cantor set. Some of the one-dimensional geometric universality survives in the Hénon maps. However, there are parts in the Cantor set of the Hénon maps whose geometry differs from its one-dimensional counterpart: rigidity does not survive.

The tip of a map $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ has a stable manifold. This stable manifold is tilted over t_F, see Theorem 4.1, away from the vertical. Although the tilt t_F is very small, it is proportional to $b_F = O(\bar{\varepsilon})$, it has crucial influence on the geometry of the Cantor set. The pieces $B^n_{v^n}$ are very thin parallelograms aligned along the stable manifold of the tip. They are tilted over an angle proportional to b_F, have horizontal size of the order σ^{2n} and vertical size of the order σ^n, see Theorem 4.1. The two pieces $B^{n+1}_{v^n c}, B^{n+1}_{v^{n+1}}$ contained in $B^n_{v^n}$ will be above each other when

$$b_F \cdot \sigma^n \asymp (\sigma^2)^n.$$

The vertical direction is strongly contracting with a factor of order b_F. One iteration will bring the pieces very close to each other, relative to their size, see Figure 5.1. The corresponding pieces of the one-dimensional renormalization fixed point are small curves next to each other at a distance comparable to their size.

The same distortion phenomenon caused by the tilt happens for the renormalizations. There this distorting effect will be stronger and stronger because the average Jacobian of the renormalizations decays super exponentially. These strong distortions are reflected in the Cantor set of the original map.

This leads to the following Non-Rigidity Theorem.

Theorem 5.4 (Non-Rigidity)**.** *Let F and $\tilde{F}$ be two infinitely renormalizable Hénon maps with average Jacobian b and $\tilde{b}$ resp. Assume*

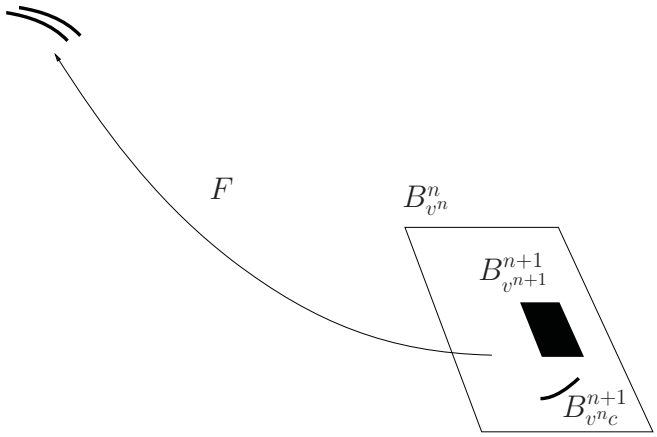

FIGURE 5.1

$b > \tilde{b}$. Let h be a homeomorphism which conjugates $\tilde{F}|_{\mathcal{O}_{\tilde{F}}}$ and $F|_{\mathcal{O}_F}$ with $h(\tau_{\tilde{F}}) = \tau_F$. Then the Hölder exponent of h is at most $\frac{1}{2}(1 + \ln b/\ln \tilde{b})$.

In particular, the conjugation between the Cantor set of a unimodal map and the Cantor set of a Hénon map is not smooth.

Corollary 5.5. *Let F be an infinitely renormalizable Hénon map with the average Jacobian $b_F > 0$ and F_0 be a degenerate infinitely renormalizable Hénon map. Let h be a homeomorphism that conjugates $F_0|_{\mathcal{O}_{F_0}}$ and $F|_{\mathcal{O}_F}$ with $h(\tau_{F_0}) = \tau_F$. Then the Hölder exponent of h is at most $\frac{1}{2}$.*

An infinitely renormalizable Hénon map F has *bounded geometry* if

$$\operatorname{diam}(B^n_{w\nu}) \asymp \operatorname{dist}(B^n_{wv}, B^n_{wc}),$$

for $n \geq 1$ and $w \in \{v, c\}^{n-1}\}$ and $\nu \in \{v, c\}$. A slight modified version of this definition would require

$$\operatorname{diam}(B^n_{w\nu} \cap \mathcal{O}_F) \asymp \operatorname{dist}(B^n_{wv} \cap \mathcal{O}_F, B^n_{wc} \cap \mathcal{O}_F).$$

The one-dimensional renormalization theory relies on the bounded geometry of the Cantor sets. This crucial property fails to hold for typical Hénon maps. The following Theorem, see [HLM], holds for both definitions of bounded geometry:

Theorem 5.6. *There exists G_δ set $G \subset [0, 1]$ of full measure with the following property. Let $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ with sufficiently small $\bar{\varepsilon}$. The map F does not have bounded geometry if $b_F \in G$.*

Consider a renormalization cycle $\{B^n_\omega \cap \mathcal{O}_F | \omega \in \{v,c\}^n\}$ of an infinitely renormalizable map $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$. The non-rigidity theorem implies that the geometry of some of the pieces in this cycle differ from their one-dimensional counterpart. For a typical map the difference can be arbitrary large, see Theorem 5.6.

This phenomenon could restrict tremendously succesfull applications of renormalization in higher dimensions. However, the universal geometrical properties of one-dimensional maps are observed in many higher-dimensional applications. The explanation is that the geometry of most pieces of a renormalization cycle are asymptotically equal to their one-dimensional counterpart. This leads to the notion of *probabilistic universality* and *probabilistic rigidity.*

The precise definition of these probabilistic notions needs some preparation. Consider the degenerate Hénon map $F_* = (f_*(x), x)$, the renormalization fixed point. Observe that for $n \geq 1$ large the pieces B^n_ω are almost straight line segments. The *scaling ratio* of a piece B^n_ω, with $\omega = \omega_0 \nu \in \{v,c\}^n$ is

$$\sigma^*_\omega = \frac{|\pi_2(B^n_\omega)|}{|\pi_2(B^{n-1}_{\omega_0})|},$$

where π_2 is the projection onto the vertical axis. Notice that $B^{n-1}_{\omega_0}$ is the piece of the previous level containing B^n_ω. The function

$$\omega \mapsto \sigma^*_\omega$$

is called the *universal scaling function.*

Consider an infinitely renormalizable map $F \in \mathcal{I}_\Omega(\bar{\varepsilon})$ and a piece B^n_ω. Let us rotate it and then rescale it to horizontal size 1; denote the corresponding linear conformal map by A. Choose the map A to obtain minimal numbers $\delta, \sigma_{\omega c}, \sigma_{\omega v} \geq 0$ such that:

(1) $A(B^n_\omega \cap \mathcal{O}_F) \subset [0,1] \times [0,\delta]$,
(2) $A(B^{n+1}_{\omega c} \cap \mathcal{O}_F) \subset [0,\sigma_{\omega c}] \times [0,\delta]$,
(3) $A(B^{n+1}_{\omega v} \cap \mathcal{O}_F) \subset [1-\sigma_{\omega v},1] \times [0,\delta]$,

where $B^{n+1}_{\omega c}$, and $B^{n+1}_{\omega v}$ are the two pieces of level $n+1$ contained in B^n_ω. We say that B^n_ω is ϵ-*universal* if

$$|\sigma_{\omega c} - \sigma^*_{\omega c}| \leq \epsilon, \quad |\sigma_{\omega v} - \sigma^*_{\omega v}| \leq \epsilon, \quad \text{and} \quad \delta \leq \epsilon.$$

The *precision* of the piece B^n_ω is the smallest $\epsilon > 0$ for which B is ϵ-universal. Let

$$\mathcal{S}^n(\epsilon) \subset \{B^n_\omega\}$$

be the collection of ϵ-universal pieces.

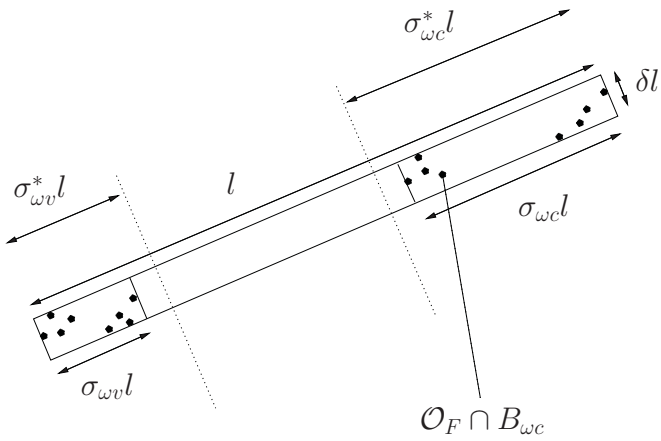

FIGURE 5.2

Definition 5.1. The Cantor attractor $\mathcal{O}_F$ of an infinitely renormalizable Hénon map $F \in \mathcal{H}_\Omega(\overline{\epsilon})$ is *universal in probabilistic sense* if there is $\theta < 1$ such that

$$\mu(\mathcal{S}^n(\theta^n)) \geq 1 - \theta^n, \quad n \geq 1.$$

Theorem 5.7 (Probabilistic Universality)**.** *The Cantor attractor $\mathcal{O}_F$ is universal in probabilistic sense.*

Denote the invariant line field of zero characteristic by

$$T : \mathcal{O}_F \to \mathbb{P}^1.$$

This line field is not continuous, see Theorem 5.3. However, we can determine sets of arbitrary large measure, with respect to the invariant measure on $\mathcal{O}_F$, on which it is continuous. Namely, for each $N \geq 1$ let

$$X_N = \bigcap_{k \geq N} \mathcal{S}_k(\theta^k),$$

where $\theta < 1$ is given by Theorem 5.7 and notice that

$$\mu(X_N) \geq 1 - O(\theta^N).$$

Let

$$X = \bigcup X_N.$$

Theorem 5.8. *There exists $\beta > 0$ such that the restriction $T|X_N$ is β-Hölder*

$$dist(T(x_0), T(x_1)) \leq C_N |x_0 - x_1|^\beta,$$

with $x_0, x_1 \in X_N$.

Theorem 5.9. *The line field T over each X_N consists of β-Hölder tangent lines to $\mathcal{O}_F$. Namely, for each $N \geq 1$ there exists $C_N > 0$ such that*

$$dist(x, T_{x_0}) \leq C_N |x - x_0|^{1+\beta}$$

when $x \in \mathcal{O}_F$, $x_0 \in X_N$.

Remark 5.1. The constants C_N tend to infinity when N becomes large.

Theorem 5.10. *Each set $X_N \subset \mathcal{O}_F$ is contained in a $C^{1+\beta}$-curve.*

Theorem 5.11. *The Cantor set $\mathcal{O}_F$ is contained in a rectifiable-curve.*

Definition 5.2. The attractor $\mathcal{O}_F$ of an infinitely renormalizable Hénon map $F \in \mathcal{H}_\Omega(\overline{\epsilon})$, $\overline{\epsilon} > 0$ small enough, is *rigid in probabilistic sense* if there exists $\beta > 0$ such that for every $\epsilon > 0$ there exists $X \subset \mathcal{O}_F$ with $\mu(X) > 1 - \epsilon$ and such that the restriction $h : X \to h(X)$ of the conjugation $h : \mathcal{O}_F \to \mathcal{O}_{F_*}$, is a $C^{1+\beta}$-diffeomorphism.

Theorem 5.12. *The Cantor attractor $\mathcal{O}_F$ is rigid in probabilistic sense.*

The Hausdorff dimension of a measure μ on a metric space $\mathcal{O}$ is defined as

$$HD_\mu(\mathcal{O}) = \inf_{\mu(X)=1} HD(X).$$

Theorem 5.13. *The Hausdorff dimension is universal*

$$HD_\mu(\mathcal{O}_F) = HD_{\mu_*}(\mathcal{O}_{F_*}).$$

6. Topology of the Attractor

The *global attracting set* of a map F is

$$\mathcal{A}_F = \bigcap_{k \geq 0} F^k(\mathrm{Dom}(F))$$

For a discussion on the concept of attractor see [Mi1] and [Mi3]. The dynamics of an infinitely renormalizable map $F \in \mathcal{H}_\Omega(\overline{\epsilon})$, $\overline{\epsilon} > 0$ small enough, is controlled by its periodic orbits $\boldsymbol{\beta}_n$ of period 2^n, $n \geq 0$, and the invariant Cantor set $\mathcal{O}_F$. The periodic orbits are flip saddles.

Theorem 6.1. *Given an infinitely renormalizable Hénon map $F \in \mathcal{I}_\Omega(\overline{\varepsilon})$ with $\overline{\varepsilon} > 0$ small enough, we have:*

$$\mathcal{A}_F = \overline{W^u(\boldsymbol{\beta}_0)} = \mathcal{O}_F \cup \bigcup_{n \geq 0} W^u(\boldsymbol{\beta}_n).$$

Furthermore, for every point $x \in \mathrm{Dom}(F)$ either $x \in W^s(\boldsymbol{\beta}_n)$ for some $n \geq 0$ or $\omega(x) = \mathcal{O}_F$. The non-wandering set of F is $\Omega_F = \mathcal{P}_F \cup \mathcal{O}_F$.

The second part of Theorem 6.1, concerning the limit sets of points and the non-wandering set, was already obtained in [GST].

The topology of the non-wandering set of an infinitely renormalizable Hénon map is as in the degenerate one-dimensional context. However, the attractors $\mathcal{A}_F$ do have a topology which differs from the one-dimensional situation. The topological differences occur in the *heteroclinic web*

$$W = \bigcup_{k\geq 0} W^u(\boldsymbol{\beta}_k) \cup \bigcup_{k\geq 0} W^s(\boldsymbol{\beta}_k).$$

The topology of the heteroclinic web can be changed by changing the average Jacobian. The reason for this lies in the universal geometry observed around the tip. In particular, the rate of accumulation of stable manifolds corresponding to different periodic orbits towards the stable manifold of the tip is universal. Although the average Jacobian is an ergodic theoretical invariant it also controls the accumulation rate towards the tip of the unstable manifolds of periodic points. This geometrical relation between the invariant manifolds and the average Jacobian leads to

Theorem 6.2. *The average Jacobian is a topological invariant.*

The central idea that the topology of the system determines the geometry might still hold for infinitely renormalizable Hénon like maps. Maps with different average Jacobian do have different geometry, Theorem 5.4, but also different topology, Theorem 6.2. It is still open whether maps with the same average Jacobian have rigid Cantor attractors.

The heteroclinic web is a countable collection of disjoint curves. A point in the web is called *laminar* if it has a matchbox-neighborhood, a neighborhood homeomorph to $(-1,1)\times Q$ where $Q\subset[0,1]$ is a countable set. A heteroclinic tangency is a tangency between some $W^u(\boldsymbol{\beta}_k)$ and $W^s(\boldsymbol{\beta}_n)$.

Remark 6.1. An infinitely renormalizable $F\in\mathcal{H}^n_\Omega(\overline{\varepsilon})$, with $\overline{\varepsilon}>0$ small enough, can not have homoclinic tangencies, a tangency between $W^u(\boldsymbol{\beta}_k)$ and $W^s(\boldsymbol{\beta}_k)$.

Theorem 6.3. *The heteroclinic web is laminar if there are no heteroclinic tangencies.*

There are examples of infinitely renormalizable maps whose heteroclinic webs do not have laminar points at all except along finitely

many unstable manifolds. Given a periodic orbit $\boldsymbol{\beta}_n$ of F, denote the two exponents by $\lambda_n^u < -1$ and $\lambda_n^s \in (-1, 0)$.

Theorem 6.4. *If the map F has a tangency*

$$W^u(\boldsymbol{\beta}_k) \top\!\top W^s(\boldsymbol{\beta}_n),$$

$k < n$, and

$$\frac{\ln|\lambda_k^u|}{\ln|\lambda_n^s|} \notin \mathbb{Q}$$

then no point in

$$\mathcal{A}_F \setminus \bigcup_{l<n} W^u(\boldsymbol{\beta}_l)$$

is laminar.

7. A step towards the Palis Conjecture

A map $F : B \to B$ is *Morse-Smale* if the non-wandering set Ω_F consists of finitely many periodic points, all hyperbolic, and the stable and unstable manifolds of the periodic points are all transversal to each other. The collection $\mathcal{I}_\Omega^n(\overline{\varepsilon}) \subset \mathcal{H}_\Omega^n(\overline{\varepsilon})$ consists of the maps which are exactly n-times renormalizable and have a periodic attractor of period 2^n. The non-wandering set of each map $F \in \mathcal{I}_\Omega^n(\overline{\varepsilon})$, with $\overline{\varepsilon} > 0$ small enough, consists of finitely many periodic points. In particular, a map $F \in \mathcal{I}_\Omega^n(\overline{\varepsilon})$ is Morse-Smale if all its periodic points are hyperbolic and if for every $x, y \in \mathcal{P}_F = \Omega_F$ there are only transverse intersections of $W^u(x)$ and $W^s(y)$.

Theorem 7.1. *Let $\overline{\varepsilon} > 0$ be small enough. The Morse-Smale maps form an open and dense subset of any $\mathcal{I}_\Omega^n(\overline{\varepsilon})$.*

A *Morse-Smale component* is a connected component of the set of non-degenerate Morse-Smale maps in $\mathcal{H}_\Omega(\overline{\varepsilon})$. Morse-Smale maps are structurally stable, see [P1]. Two Morse-Smale components in $\mathcal{I}_\Omega^n(\overline{\varepsilon})$ are of different *type* if the maps in the first component are not conjugate to the maps in the other.

Theorem 7.2. *Let $\overline{\varepsilon} > 0$ be small enough. Then for $n \geq 1$ large enough there are countably many Morse-Smale components of different type in $\mathcal{I}_\Omega^n(\overline{\varepsilon})$. The collection of Morse-Smale components in $\mathcal{I}_\Omega^n(\overline{\varepsilon})$ is not locally finite.*

A finitely renormalizable map $F \in \mathcal{H}_\Omega(\overline{\varepsilon})$ is called *hyperbolic* if its non-wandering set can be decomposed as

$$\Omega_F = \Lambda_F \cup P_F,$$

where P_F is a hyperbolic periodic attractor which attracts almost every point and Λ_F a hyperbolic zero-dimensional set. A closed invariant set is hyperbolic if it has an invariant splitting consisting of one stable direction and one unstable direction. See [PT] for a general discussion of hyperbolicity and invariant splittings. The map is called *hyperbolic with positive entropy* if Λ_F contains a Cantor set which has positive entropy. The Morse-Smale maps discussed in Theorem 7.2 are hyperbolic.

Theorem 7.3. *Let $\gamma \subset \mathcal{H}_\Omega(\overline{\varepsilon})$ be a smooth curve through $F_0 \in \mathcal{I}_\Omega(\overline{\varepsilon})$, $\overline{\varepsilon} > 0$ small enough, which is transversal to $\mathcal{I}_\Omega(\overline{\varepsilon})$. The hyperbolic maps with positive entropy in γ have positive density in F_0.*

A map $F \in \mathcal{H}_\Omega(\overline{\varepsilon})$ is called *regular* if there exists a periodic attractor which attracts almost every point. It is called *stochastic* if there exists an SRB measure which describes the statistics of almost every orbit. Benedicks and Carleson have shown that the stochastic Hénon maps with a fixed, but small Jacobian, form a set of positive one-dimensional measure, [BC]. They discuss Hénon maps in a neighborhood of a specific Misiurewicz unimodal map.

By no means it is a straightforward task to extend their discussion to neighborhoods of unimodal maps with a Collet-Eckmann condition. However, the following result is within reach.

Given a family of unimodal maps, for example the unstable manifold of the period doubling renormalization fixed point. Every Collet-Eckmann map in this family has, for every $\varepsilon > 0$, a neighborhood $U \subset \mathcal{H}_\Omega(\overline{\varepsilon})$ with the following property. Consider a one-parameter family of Hénon maps close enough to the given unimodal family. The fraction of stochastic maps in the part of this Hénon family which crosses the neighborhood U, is at least $1 - \varepsilon$.

Renormalization, the fact that regular and Collet-Eckmann maps have full measure [AM], and this extension would prove the following step towards the Palis Conjecture, [P2].

Let γ be a curve through an infinitely renormalizable map $F \in \mathcal{I}_\Omega(\overline{\varepsilon})$, $\overline{\varepsilon} > 0$ small enough, which is transversal to $\mathcal{I}_\Omega(\overline{\varepsilon})$. The map F is a Lebesgue density point of the regular and stochastic maps in the curve γ.

8. Open Problems

Let us finish with some further questions that naturally arise from the previous discussion.

Problem I:

(1) Prove that F_* is the only fixed point of the Hénon renormalization R, and $R^n F \to F_*$ exponentially for any infinitely renormalizable Hénon map F.
(2) Is it true that the trace of the unstable manifold $\mathcal{W}^u(F_*)$ by the two-parameter Hénon family $F_{c,b} : (x, y) \mapsto (x^2 + c - by, x)$ is a (real analytic) curve γ on which the Jacobian b assumes all values $0 < b < 1$. If so, does this curve converge to some particular point $(c, 1)$ as $b \to 1$ as computer experiments indicate?

Problem II:

(1) Is the conjugacy $h\colon \mathcal{O}_F \to \mathcal{O}_G$ always Hölder?
(2) Can $\mathcal{O}_F$ have bounded geometry when $b_F \neq 0$? If so, does this property depend only on the average Jacobian b_F?
(3) Does the Hausdorff dimension of $\mathcal{O}_F$ depend only on the average Jacobian b_F? (This question was suggested by A. Avila.)

Problem III:

(1) A *wandering domain* is an open set in the basin of attraction of $\mathcal{O}_F$. Do wandering domains exist?
(2) If a map $F \in \mathcal{I}_\Omega(\overline{\varepsilon})$ does not have wandering domains then the union $\mathcal{F}^s$ of all stable manifolds of periodic points is dense in the domain of F. Does there exist $F \in \mathcal{I}_\Omega(\overline{\varepsilon})$ such that $\mathcal{F}^s$ is not laminar even if there are no heteroclinic tangencies?
(3) For $F \in \mathcal{I}_\Omega(\overline{\varepsilon})$ let $\mathcal{F}^s_\tau$ be the union of stable manifolds of the points in the orbit of the tip. Is $\mathcal{F}^s_\tau$ dense in $\mathrm{Dom}(F)$?

Problem IV:

The unique invariant measure on the Cantor attractor $\mathcal{O}_F$ has characteristic exponents 0 and $\ln b_F < 0$. Can the stable characteristic exponent of the tip τ_F differ from $\ln b_F$?

Problem V:

Can we still speak of rigidity of the Cantor attractor $\mathcal{O}_F$?

(1) Are the Cantor attractors rigid within the topological conjugacy classes of maps restricted to a neighborhood of the Cantor attractor?
(2) Prove or disprove that two Cantor attractors $\mathcal{O}_F$ and $\mathcal{O}_{\tilde{F}}$ are smoothly equivalent if and only if they have the same average Jacobian.

Problem VI:

(1) Can different Morse-Smale components

$$MS_1, MS_2 \subset \bigcup_{n\geq 0} \mathcal{I}^n_\Omega(\overline{\varepsilon})$$

have the same type, that is the maps in MS_1 are conjugate to the maps in MS_2?

(2) As we have shown, the Morse-Smale Hénon maps are dense in the zero entropy region with small Jacobian. Are they dense in the full zero entropy region of dissipative Hénon maps? How about other real analytic families of dissipative two dimensional maps?

(3) The discussion that led to Theorem 7.2 was based on the renormalization structure. However, the non-locally finiteness of the collection of Morse-Smale components might be a more general phenomenon. Study the combinatorics of Morse-Smale components in other real analytic families of dissipative two dimensional maps.

(4) Are the real Morse-Smale Hénon maps from Theorem 7.2 hyperbolic on $\mathbb{C}^2$? To what extent the topology of the real heteroclinic web determines the topology of the corresponding Hénon map on $\mathbb{C}^2$?

Problem VII: Is the convergence of the statistics of the *bad* pieces,

$$\lim_{n\to\infty} \mu(\mathcal{B}_n \setminus \mathcal{S}_n(\epsilon)) = 0,$$

governed by some form of universality? This question is related to Problem V on the regularity of the conjugation $h : \mathcal{O}_G \to \mathcal{O}_G$ when $b_F = b_G$.

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