Landau-Stark states for cold atoms in a parabolic lattice

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Quantum particle in a 2D lattice

The Hamiltonian

$$\widehat{H} = \frac{(\hat{\mathbf{p}} - \mathbf{A})^2}{2M} + V(\mathbf{r}) + \mathbf{F}\cdot\mathbf{r}\,, \quad V(x+a,y) = V(x,y+a) = V(x,y)$$

- Bloch states ($A = 0,\ F = 0$)
- Landau-states ($A \neq 0,\ F = 0$)
- Wannier-Stark states ($A = 0,\ F \neq 0$)
- Landau-Stark states ($A \neq 0,\ F \neq 0$)

Applications of the Landau-Stark states:

- Cold atoms subject to a synthetic magnetic field ($\mathbf{F}\cdot\mathbf{r} \to \omega_{trap} r^2$)
- Alternative approach to Hall physics (for example, integer quantum Hall effect)

Tight-binding approximation ($|\alpha| \ll 1/2$)

Hamiltonian for the symmetric gauge $\mathbf{A} = B(-y/2, x/2, 0)$

$$(\widehat{H}\psi)_{l,m} = -\frac{J_x}{2}\left(\psi_{l+1,m}e^{-i\pi\alpha m} + \psi_{l-1,m}e^{i\pi\alpha m}\right)$$
$$-\frac{J_y}{2}\left(\psi_{l,m+1}e^{i\pi\alpha l} + \psi_{l,m-1}e^{-i\pi\alpha l}\right) + a(F_x l + F_y m)\psi_{l,m}$$

The spectrum:

- $E(\kappa_x, \kappa_y) = -J_x\cos(a\kappa_x) - J_y\cos(a\kappa_y)$ (Bloch states)
- $-\frac{J_x}{2}(b_{l+1} + b_{l-1}) - J_y\cos(2\pi\alpha l + a\kappa)b_l = E_n(\kappa)b_l$ (Landau states)
- $E_{n,k} = aF_x n + aF_y k$ (Wannier-Stark states, $F_x, F_y \neq 0$)
- What is the spectrum of the Landau-Stark states?

The answer: The spectrum is defined by the rationality condition on $\beta = F_x/F_y$.

Localized Landau-Stark states (irrational β)

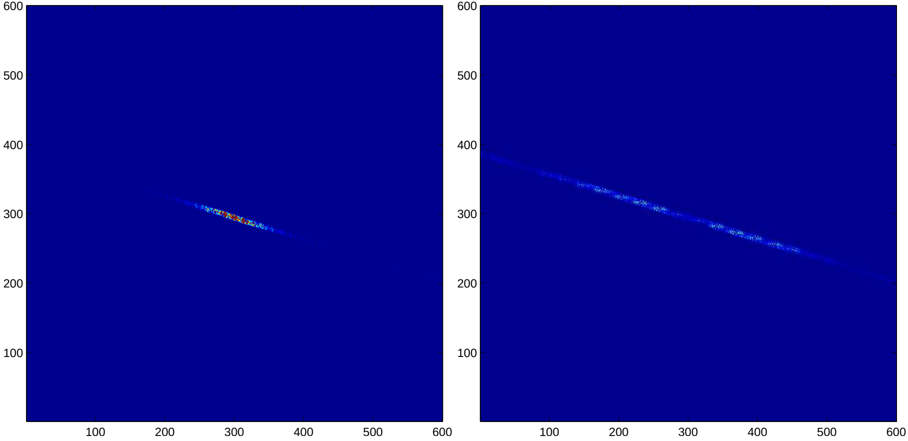


Figure: Examples of the localized Landau-Stark states for $F = 0.5$ (left) and $F = 0.4$ (right). The other parameters are $\beta = (\sqrt{5} - 1)/4 \approx 1/3$ and $\alpha \approx 1/6$. The occupation probabilities $|\psi_{l,m}|^2$ are shown as a color map.

Spectrum of the localized Landau-Stark states

If $\Psi_{l,m}$ is an eigenstate with the energy E, then the state

$$\tilde{\Psi}_{l,m} = \Psi_{l-n,m-k} e^{-i2\pi\alpha nm}$$

is also eigenstate with the energy

$$\tilde{E} = E + a(F_x n + F_y k)$$

Localization lengths

- $\xi_{\|} \sim J/aF$
- $\xi_{\perp} \propto \exp(C|F - F_{cr}|)$, $F < F_{cr} \sim \alpha$

A.R.Kolovsky, and G.Mantica, *Cyclotron-Bloch dynamics of a quantum particle in a two-dimensional lattice*, Phys. Rev. E **83**, 041123 (2011)

A.R.Kolovsky, I.Chesnokov, and G.Mantica, *Cyclotron-Bloch dynamics of a quantum particle in a two-dimensional lattice II: arbitrary field directions*, Phys. Rev. E **86**, 041146 (2012)

A.R.Kolovsky, and G.Mantica, *Driven Harper model*, Phys. Rev. B **86**, 054306 (2012)

Localization length: numerical results

Inverse participation ratio: $P = \left(\sum_{l,m} |\Psi_{l,m}|^4\right)^{-1}$

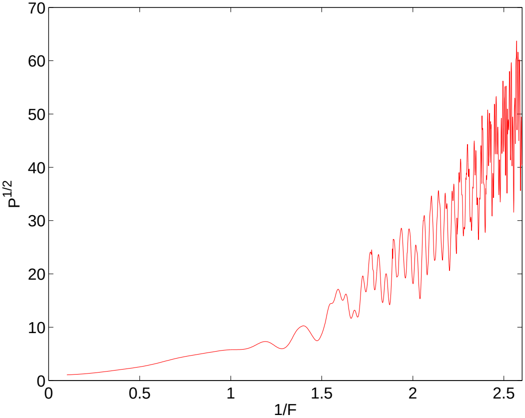


Figure: Localization length $\xi_\perp$ as the function of $1/F$. The other parameters are $\beta = (\sqrt{5}-1)/4 \approx 1/3$, and $\alpha \approx 1/6$.

Semiclassical approach (driven Harper model)

The classical Hamiltonian (quantization rule $[\hat{x},\hat{p}] = i2\pi\alpha$)

$$H_{cl}(t) = -J'_x \cos(p - F_x t) - J'_y \cos(x + F_y t) \ , \quad J'_{x,y} = 2\pi\alpha J_{x,y}$$

This system can be studied:

- on the torus $(-\pi \le p, x < \pi)$
- on the cylinder $(-\pi \le p < \pi, \ -\infty < x < \infty)$
- in the plane $(-\infty < p, x < \infty)$

In the plane: using the canonical substitution $p' = p - F_x t$ and $x' = x + F_y t$ the Hamiltonian takes time-independent form:

$$H'_{cl} = -J'_x \cos(p') - J'_y \cos(x') + F_x x' + F_y p'$$

Phase portrait: transporting islands

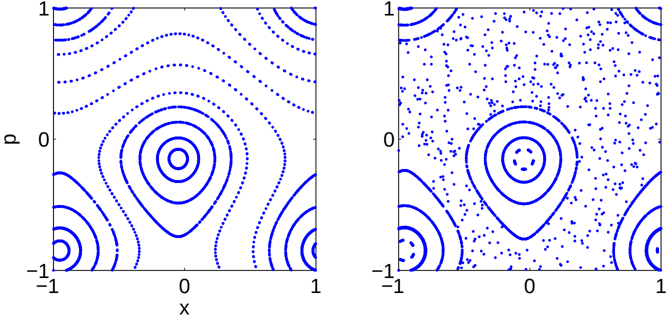


Figure: Stroboscopic map of the classical driven Harper for rational $\beta = 1/3$, left panel, and irrational $\beta = (\sqrt{5} - 1)/4 \approx 1/3$, right panel. Transporting islands disappear if $F > F_{cr}$.

Spectrum of the extended Landau-Stark states (rational β)

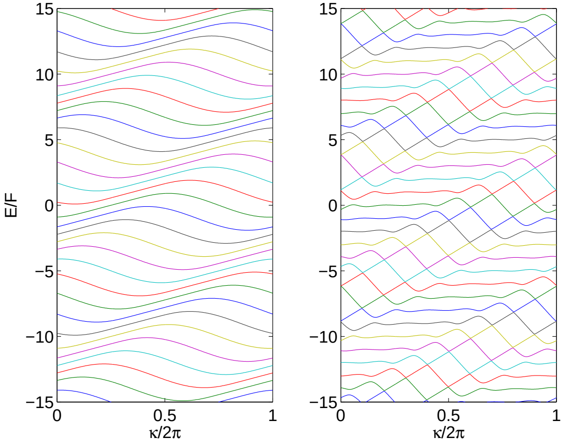


Figure: Examples of the band energy spectrum of extended Landau-Stark states. Parameters are $\alpha = 1/10$, $\beta = 0$, and $F = 1$ (left) and $F = 0.3$ (right).

Intermediate conclusion

- The spectrum of the Landau-Stark states crucially depends on the rationality condition for the parameter $\beta = F_x/F_y$.
- If β is a rational number the spectrum is continuous (band structured) and Landau-Stark states are extended Bloch-like waves in the direction orthogonal to **F**.
- If β is an irrational number the spectrum is discrete, $E = F_x n + F_y k$, and Landau-Stark states are truly localized states, i.e., the localization length $\xi_\perp < \infty$.
- Open problems: (i) Scaling law for the localization length $\xi_\perp$ for arbitrary α; (ii) Other lattice geometries; (iii) Wannier-Stark localization in the presence of a magnetic field beyond the single-band approximation; (iv) ...

Cold atoms in parabolic lattices

The system

$$(\widehat{H}\psi)_{l,m} = -\frac{J}{2}\left(\psi_{l+1,m}e^{-i\pi\alpha m} + \psi_{l-1,m}e^{i\pi\alpha m}\right)$$
$$-\frac{J}{2}\left(\psi_{l,m+1}e^{i\pi\alpha l} + \psi_{l,m-1}e^{-i\pi\alpha l}\right) + \frac{\gamma}{2}(l^2+m^2)\psi_{l,m}$$

where $\gamma \sim \omega_{trap}^2$ is the harmonic confinement.

Locally, at (l, m) near (l_0, m_0), we have

$$\frac{\gamma}{2}(l^2+m^2)\psi_{l,m} \approx F_x l + F_y m$$

where $(F_x, F_y) = -\gamma(l_0, m_0)$ is the gradient force pointing the lattice origin.

Semiclassical approach ($\alpha \ll 1$)

Classical counterpart of the tight-binding Hamiltonian

$$H_{cl} = -J_x \cos(p_x - \pi\alpha y) - J_y \cos(p_y + \pi\alpha x) + \frac{\gamma}{2}(x^2 + y^2)$$

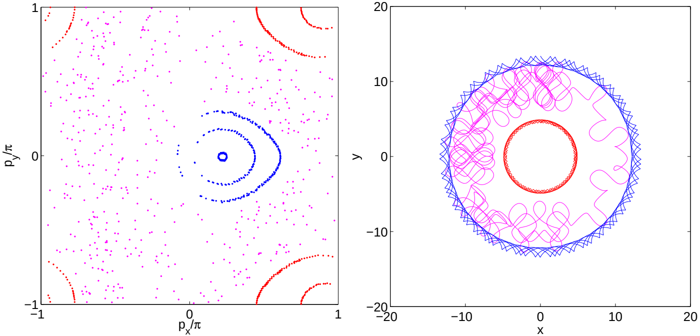


Figure: Poincare cross-section of the energy shell $E = 2.5$ (left panel) and examples of classical trajectories in the coordinate space (right panel). Encircling frequency is $\Omega = \gamma/2\pi\alpha$.

Eigenstates

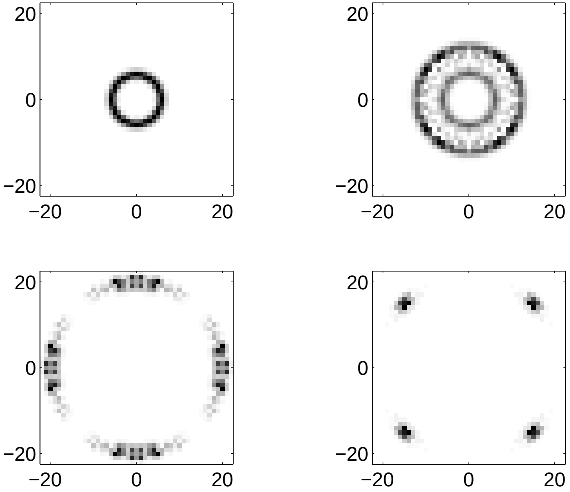


Figure: Examples of the eigenstates for increasing energy ($|\psi_{l,m}|^2$ are shown as the gray-scaled map). Above $E_{cr} = (2\pi\alpha J)^2/\gamma$ the transporting islands disappear and eigenstates become localized states.

Wave packet dynamics

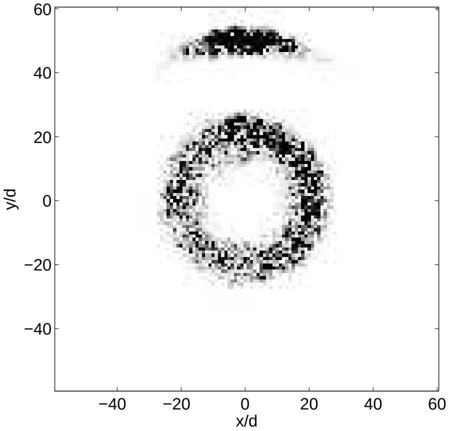
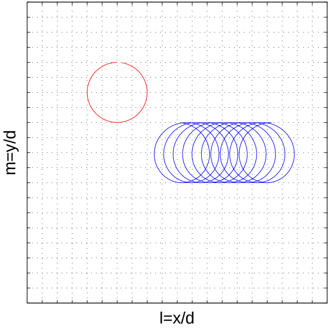


Figure: Long-time ($t \gg 2\pi/\Omega$) wave-packet dynamics for two different initial conditions.

Classical Hall's effect (1879)



- cyclotron frequency $\omega_c = eB/M$
- drift velocity $v^* = F/B$

Hall resistance

Linear response regime (small F)

$$v_x = \sigma_{xy} F \ , \quad v_y = \sigma_{yy} F \ , \quad F \ll 1$$

Conductivity tensor

$$\sigma_{xy} \sim \frac{1}{\gamma} \frac{\omega_c/\gamma}{1 + (\omega_c/\gamma)^2} \ , \quad \sigma_{yy} \sim \frac{1}{\gamma} \frac{1}{1 + (\omega_c/\gamma)^2}$$

where γ is the relaxation rate, ω_c the cyclotron frequency and the proportionality coefficient is given by the density of the carriers n_s.

In the limit $\gamma \to 0$ we have

$$R_H = \frac{1}{\sigma_{xy}} = \frac{B}{e n_s}$$

Quantized conductance

The Nakato-Kubo equation

$$\sigma_{xy} = \frac{e^2\hbar}{i} \sum_{E_\alpha < E_F < E_\beta} \frac{(v_y)_{\alpha\beta}(v_x)_{\beta\alpha} - (v_x)_{\alpha\beta}(v_y)_{\beta\alpha}}{(E_\alpha - E_\beta)^2}$$

$$\sigma_{xy} = \frac{e^2}{h} \sum_\alpha n_\alpha$$

Topological invariants

$$n_\alpha = \frac{1}{2\pi i} \int [\nabla_\kappa \times \mathbf{K}(\kappa)] \mathrm{d}^2\kappa \, , \quad \mathbf{K}(\kappa) = \langle u_{\alpha,\kappa} | \nabla_\kappa | u_{\alpha,\kappa} \rangle$$

[*] R.B.Laughlin, D.J.Thouless, M.Kohmoto, ...

Alternative approach

Master equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\widehat{H}_0 + eFy, \hat{\rho}] - \gamma(\hat{\rho} - \hat{\rho}_0) \ , \quad \hat{\rho}_0 = \sum_{i=1}^{E_F} |\psi_i\rangle\langle\psi_i|$$

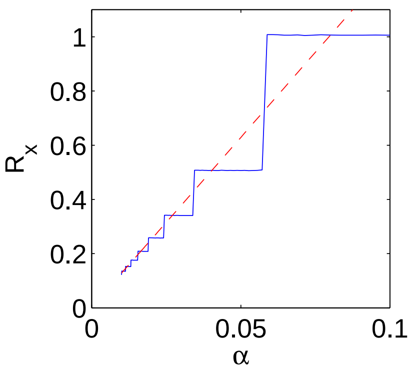
We find the stationary current

$$v_{x,y} = \mathrm{Tr}[\hat{v}_{x,y}\hat{\rho}(t=\infty)]$$

by using the semi-analytical method of Ref. [3] and then consider the limits $F \to 0$ and $\gamma \to 0$.

[3] A.R.Kolovsky, *Hall conductivity beyond the linear response regime*, Europhys. Lett. **96**, 50002 (2011).

Hall resistance



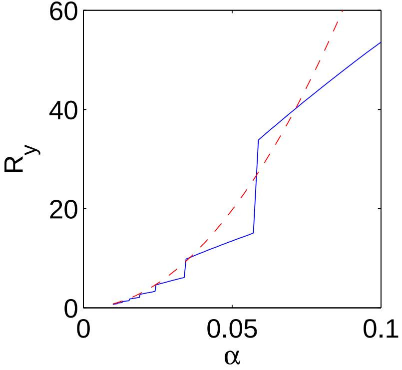


Figure: The Hall (left) and Ohm (right) resistence as the function of the Peierls phase α. The other parameters are $J_x = J_y = 1$, $F = 0.01$ and $\gamma = 0.01$. The dashed lines are the classical Hall effect.

Conclusions

- Cold atoms in optical lattices in the presence of harmonic confinement and a gauge field can be conviniently (exhaustively) described in terms of the Landau-Stark states.
- We considered the case of a uniform 'magnetic' field with the Peierls phase $\alpha \ll 1$.
- In this case the single-particle eigenstates are extended states for $E < E_{cr} = (2\pi\alpha J)^2/\gamma$ and localized states for $E > E_{cr}$.
- The extended states are further sorted into regular transporting states with the equidistant spectrum $E_n = \pm J + \hbar\Omega$ ($\Omega = \gamma/2\pi\alpha$ is the encircling frequency) and chaotic counter-transporting states.
- Open problems: (i) Large values of the Peierls phase α; (ii) Staggered magnetic fields; (iii) ...
- Landau-Stark states is a powerful tool for analyzing other fundamental problems in physics like, for example, the integer quantum Hall effect.