QCD, top and LHC
Lecture II: QCD, asymptotic freedom and infrared safety

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Slides available from http://theory.fnal.gov/people/ellis/Talks/

Bibliography

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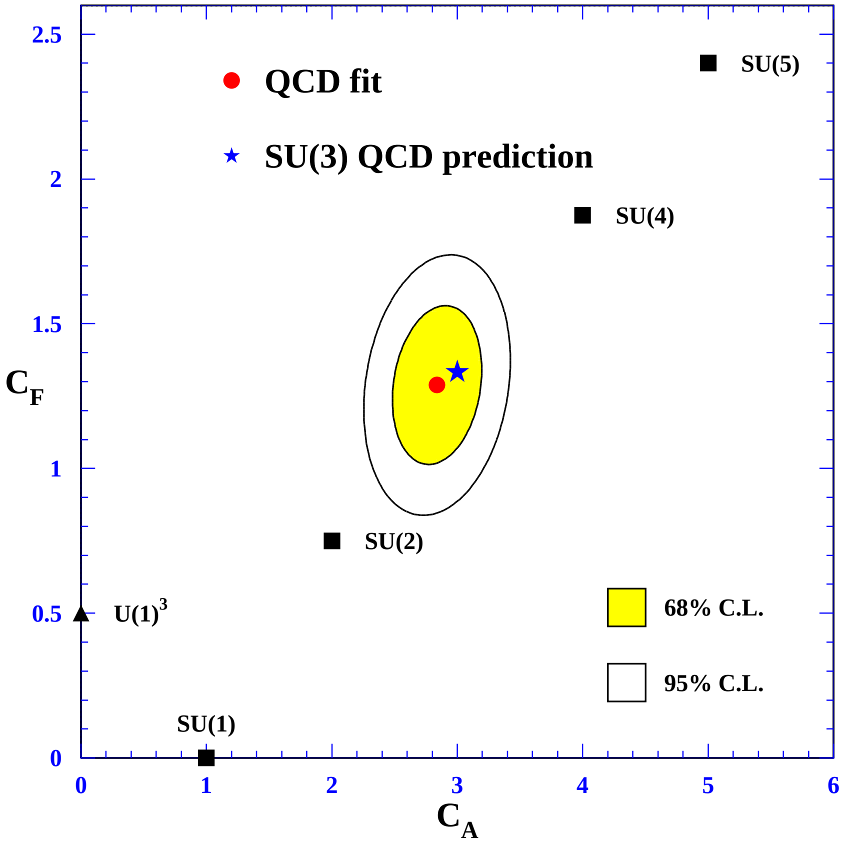
Plan

- Running coupling
  - Running coupling
  - Asymptotic freedom
  - Infrared divergences and infrared finiteness
- e^+e^- annihilation
  - Total cross section
  - Shape distributions
  - Jet fractions
- Jet algorithms in hadronic collisions
  - Inclusive k_T algorithm
  - Cambridge-Aachen algorithm
  - Anti k_T algorithm

Color factor



- hep-ex/0012044v2

Running coupling

- Consider dimensionless physical observable R which depends on a single large energy scale, $Q \gg m$ where m is any mass. Then we can set $m \to 0$ (assuming this limit exists), and dimensional analysis suggests that R should be independent of Q.
- This is not true in quantum field theory. Calculation of R as a perturbation series in the coupling $\alpha_S = g^2/4\pi$ requires renormalization to remove ultraviolet divergences. This introduces a second mass scale μ — point at which subtractions which remove divergences are performed. Then R depends on the ratio Q/μ and is not constant. The renormalized coupling α_S also depends on μ.
- But μ is arbitrary! Therefore, if we hold bare coupling fixed, R cannot depend on μ. Since R is dimensionless, it can only depend on Q^2/μ^2 and the renormalized coupling α_S. Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S}\right] R = 0 \; .$$

- This is the simplest expression of the renormalization group. At fixed bare coupling, physical predictions cannot depend on an arbtitrary choice of renormalization scale.

- Introducing
$$\tau = \ln\left(\frac{Q^2}{\mu^2}\right), \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} ,$$
we have
$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha_S)\frac{\partial}{\partial \alpha_S}\right] R = 0.$$
This renormalization group equation is solved by defining running coupling $\alpha_S(Q)$:
$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)} , \quad \alpha_S(\mu) \equiv \alpha_S .$$
Then
$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)) , \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)} .$$
and hence $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$. Thus all scale dependence in R comes from running of $\alpha_S(Q)$.
- We shall see QCD is asymptotically free: $\alpha_S(Q) \to 0$ as $Q \to \infty$. Thus for large Q we can safely use perturbation theory. Then knowledge of $R(1, \alpha_S)$ to fixed order allows us to predict variation of R with Q.

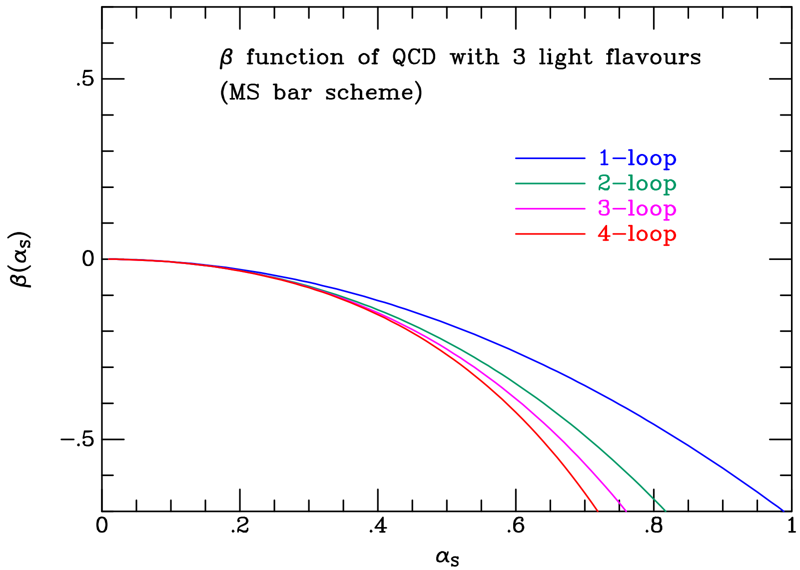
Beta function

- Running of the QCD coupling α_S is determined by the β function, which has the expansion

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S) + \mathcal{O}(\alpha_S^4)$$

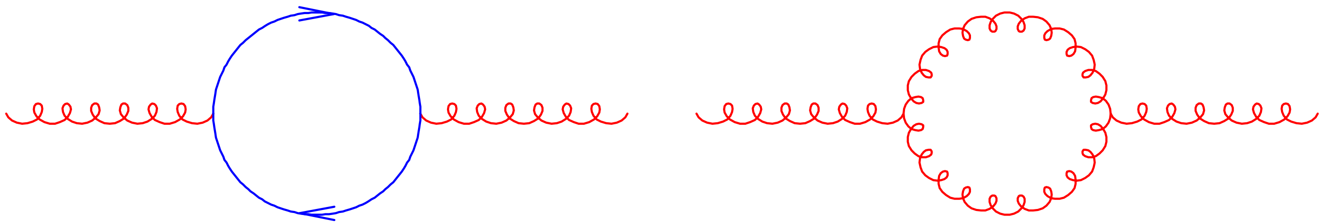
$$b = \frac{(11C_A - 2N_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_A N_f - 3C_F N_f)}{2\pi(11C_A - 2N_f)},$$

where N_f is number of "active" light flavours. Terms up to $\mathcal{O}(\alpha_S^5)$ are known.



- ⋆ if $\frac{d\alpha_S}{d\tau} = -b\alpha_S^2(1 + b'\alpha_S)$ and $\alpha_S \to \bar{\alpha}_S(1 + c\bar{\alpha}_S)$, it follows that $\frac{d\bar{\alpha}_S}{d\tau} = -b\bar{\alpha}_S^2(1 + b'\bar{\alpha}_S) + O(\bar{\alpha}_S^4)$
- ⋆ first two coefficients b, b' are thus invariant under scheme change.

Asymptotic freedom



(a) (b)

- Roughly speaking, quark loop diagram (a) contributes negative N_f term in b, while gluon loop (b) gives positive C_A contribution, which makes β function negative overall. (This statement is literally correct in the background field method).
- QED β function is $\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \ldots$
  Thus b coefficients in QED and QCD have opposite signs.
- From earlier slides,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q)\Big[1 + b'\alpha_S(Q)\Big] + \mathcal{O}(\alpha_S^4).$$

Neglecting b' and higher coefficients gives $\alpha_S(Q) = \frac{\alpha_S(\mu)}{1+\alpha_S(\mu)b\tau}\ , \quad \tau = \ln\left(\frac{Q^2}{\mu^2}\right).$

Asymptotic freedom

- As Q becomes large, $\alpha_S(Q)$ decreases to zero: this is asymptotic freedom. Notice that sign of b is crucial. In QED, $b < 0$ and coupling *increases* at large Q.
- Including next coefficient b' gives implicit equation for $\alpha_S(Q)$:

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b' \ln\left(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) - b' \ln\left(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\right)$$

- What type of terms does the solution of the renormalization group equation take into account in the physical quantity R?
  Assume that R has perturbative expansion

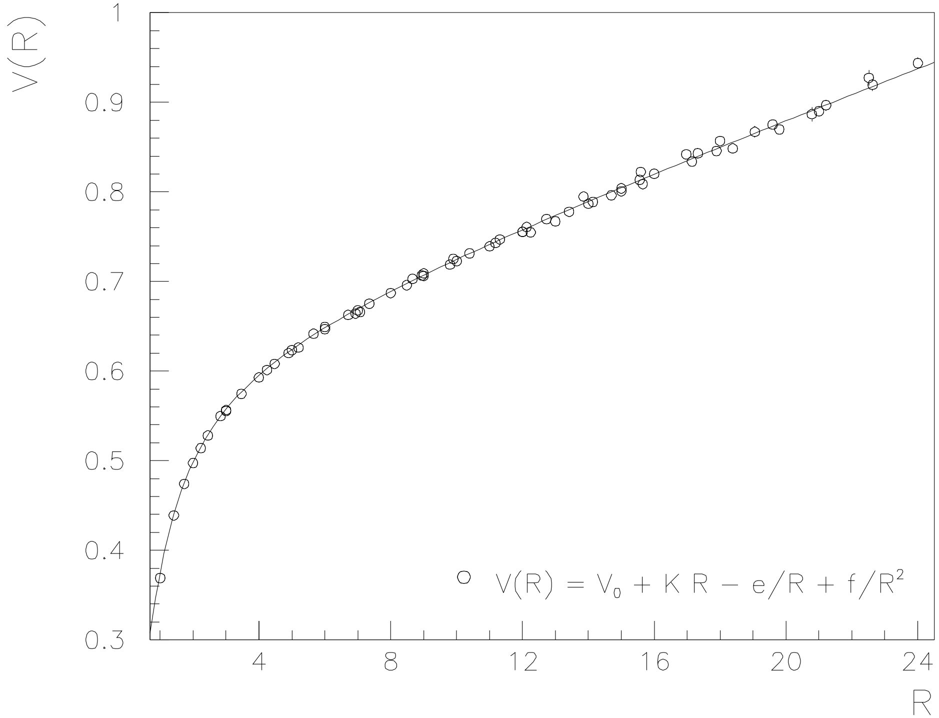
$$R = \alpha_S + \mathcal{O}(\alpha_S^2)$$

Solution $R(1, \alpha_S(Q))$ can be re-expressed in terms of $\alpha_S(\mu)$:

$$\begin{aligned} R(1, \alpha_S(Q)) &= \alpha_S(\mu) \sum_{j=0}^{\infty} (-1)^j (\alpha_S(\mu) b\tau)^j \\ &= \alpha_S(\mu) \Big[1 - \alpha_S(\mu) b\tau + \alpha_S^2(\mu)(b\tau)^2 + \ldots\Big] \end{aligned}$$

Thus there are logarithms of Q^2/μ^2 which are automatically resummed by using the running coupling. Neglected terms have fewer logarithms per power of α_S.

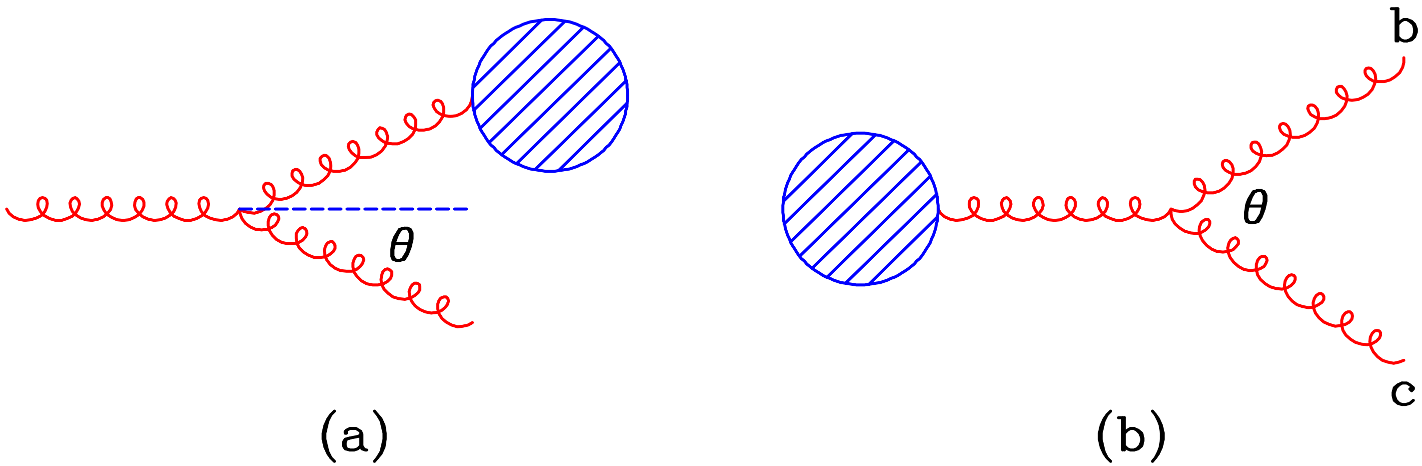
The static force between two quarks



- Lattice calculation of the static $q\bar{q}$ potential, performed in the quenched approximation.
- At small distances $V(R) = V_0 - \frac{\alpha_S(aR)}{R}$, logarithmic modification of Coulomb potential.
- At large distances $V(R) = V_0 + KR$, linearly rising potential. This is infra-red slavery.

Infrared divergences

- Even in high-energy, short-distance regime, long-distance aspects of QCD cannot be ignored. Soft or collinear gluon emission gives infrared divergences in PT. Light quarks ($m_q \ll \Lambda$) also lead to divergences in the limit $m_q \to 0$ (mass singularities).



⋆ Spacelike branching: gluon splitting on incoming line (a)

$$p_b^2 = -2E_aE_c(1-\cos\theta) \le 0\ .$$

Propagator factor $1/p_b^2$ diverges as $E_c \to 0$ (soft singularity) or $\theta \to 0$ (collinear or mass singularity).

If a and b are quarks, inverse propagator factor is

$$p_b^2 - m_q^2 = -2E_a E_c (1 - v_a \cos\theta) \leq 0 \ ,$$

Hence $E_c \to 0$ soft divergence remains; collinear enhancement becomes a divergence as $v_a \to 1$, i.e. when quark mass is negligible. If emitted parton c is a quark, vertex factor cancels $E_c \to 0$ divergence.

- Timelike branching: gluon splitting on outgoing line (b)

$$p_a^2 = 2E_b E_c (1 - \cos\theta) \geq 0 \ .$$

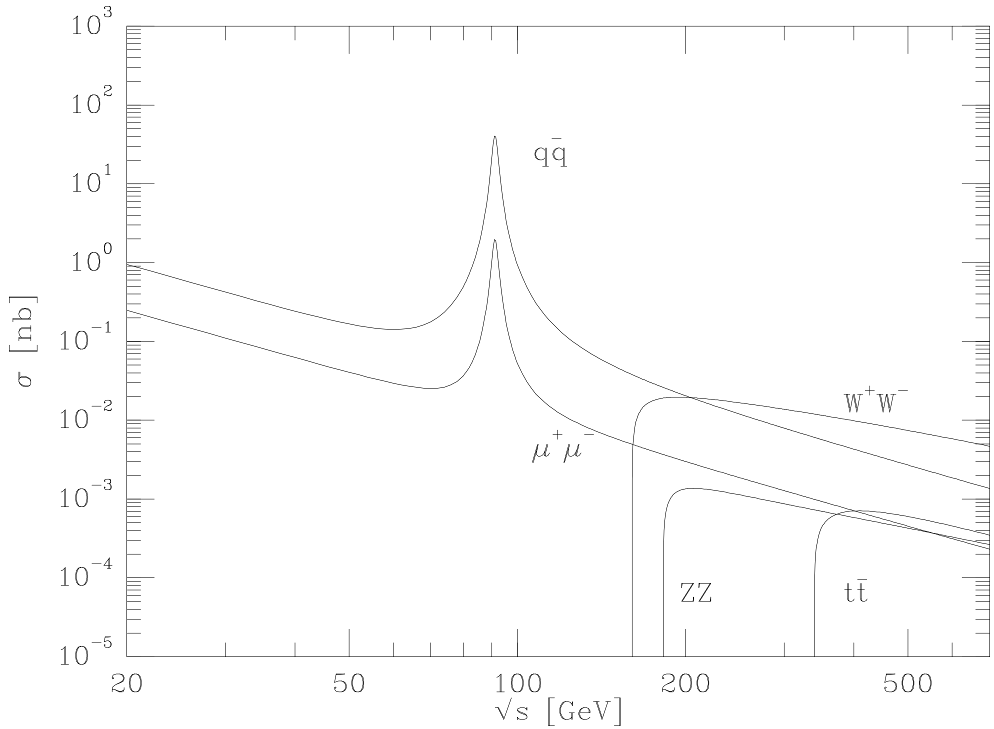
  Diverges when either emitted gluon is soft (E_b or $E_c \to 0$) or when opening angle $\theta \to 0$. If b and/or c are quarks, collinear/mass singularity in $m_q \to 0$ limit. Again, soft quark divergences cancelled by vertex factor.

- Similar infrared divergences in loop diagrams, associated with soft and/or collinear configurations of virtual partons within region of integration of loop momenta.
- Infrared divergences indicate dependence on long-distance aspects of QCD not correctly described by PT. Divergent (or enhanced) propagators imply propagation of partons over long distances. When distance becomes comparable with hadron size ~ 1 fm, quasi-free partons of perturbative calculation are confined/hadronized non-perturbatively, and apparent divergences disappear.

- Can still use PT to perform calculations, provided we limit ourselves to two classes of observables:
  - ⋆ Infrared safe quantities, i.e. those insensitive to soft or collinear branching. Infrared divergences in PT calculation either cancel between real and virtual contributions or are removed by kinematic factors. Such quantities are determined primarily by hard, short-distance physics; long-distance effects give power corrections, suppressed by inverse powers of a large momentum scale.
  - ⋆ Factorizable quantities, i.e. those in which infrared sensitivity can be absorbed into an overall non-perturbative factor, to be determined experimentally.
- In either case, infrared divergences must be *regularized* during PT calculation, even though they cancel or factorize in the end.
  - ⋆ Gluon mass regularization: introduce finite gluon mass, set to zero at end of calculation. However, as we saw, gluon mass breaks gauge invariance.
  - ⋆ Dimensional regularization: analogous to that used for ultraviolet divergences, except we must *increase* dimension of space-time, $\epsilon = 2 - \frac{D}{2} < 0$. Divergences are replaced by powers of $1/\epsilon$.

e^+e^- *annihilation cross section*

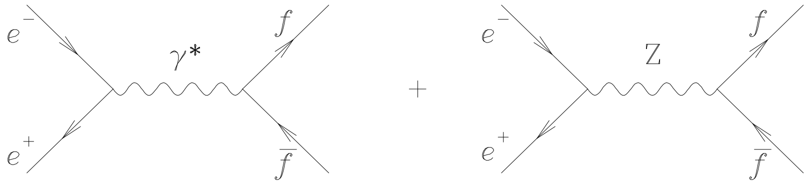
- $e^+e^- \to \mu^+\mu^-$ is a fundamental electroweak processes.
- Same type of process, $e^+e^- \to q\bar{q}$, will produce hadrons. Cross sections are roughly proportional.



- At high energy cross sections fall like $1/s$.

e^+e^- *to hadrons*

- Since formation of hadrons is non-perturbative, how can perturbation theory give hadronic cross section? This can be understood by visualizing event in space-time:
  - ⋆ e^+ and e^- collide to form γ or Z^0 with virtual mass $Q = \sqrt{s}$. This fluctuates into $q\bar{q}$, $q\bar{q}g$,. . . , occupy space-time volume $\sim 1/Q$. At large Q, rate for this short-distance process given by PT.



  - ⋆ Subsequently, at much later time $\sim 1/\Lambda$, produced quarks and gluons form hadrons. This modifies outgoing state, but occurs too late to change original probability for event to happen.
- Well below Z^0, process $e^+e^- \to f\bar{f}$ is purely electromagnetic, with lowest-order (Born) cross section (neglecting quark masses)

$$\sigma_0 = \frac{4\pi\alpha^2}{3s}\, Q_f^2$$

The ratio R

- Thus ($3 = N =$ number of possible $q\bar{q}$ colours)

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2\ .$$

- On Z^0 pole, $\sqrt{s} = M_Z$, neglecting γ/Z interference

$$\sigma_0 = \frac{4\pi\alpha^2\kappa^2}{3\Gamma_Z^2}\,(a_e^2 + v_e^2)\,(a_f^2 + v_f^2)$$

where $\kappa = \sqrt{2}G_F M_Z^2/4\pi\alpha = 1/\sin^2(2\theta_W) \simeq 1.5$. Hence

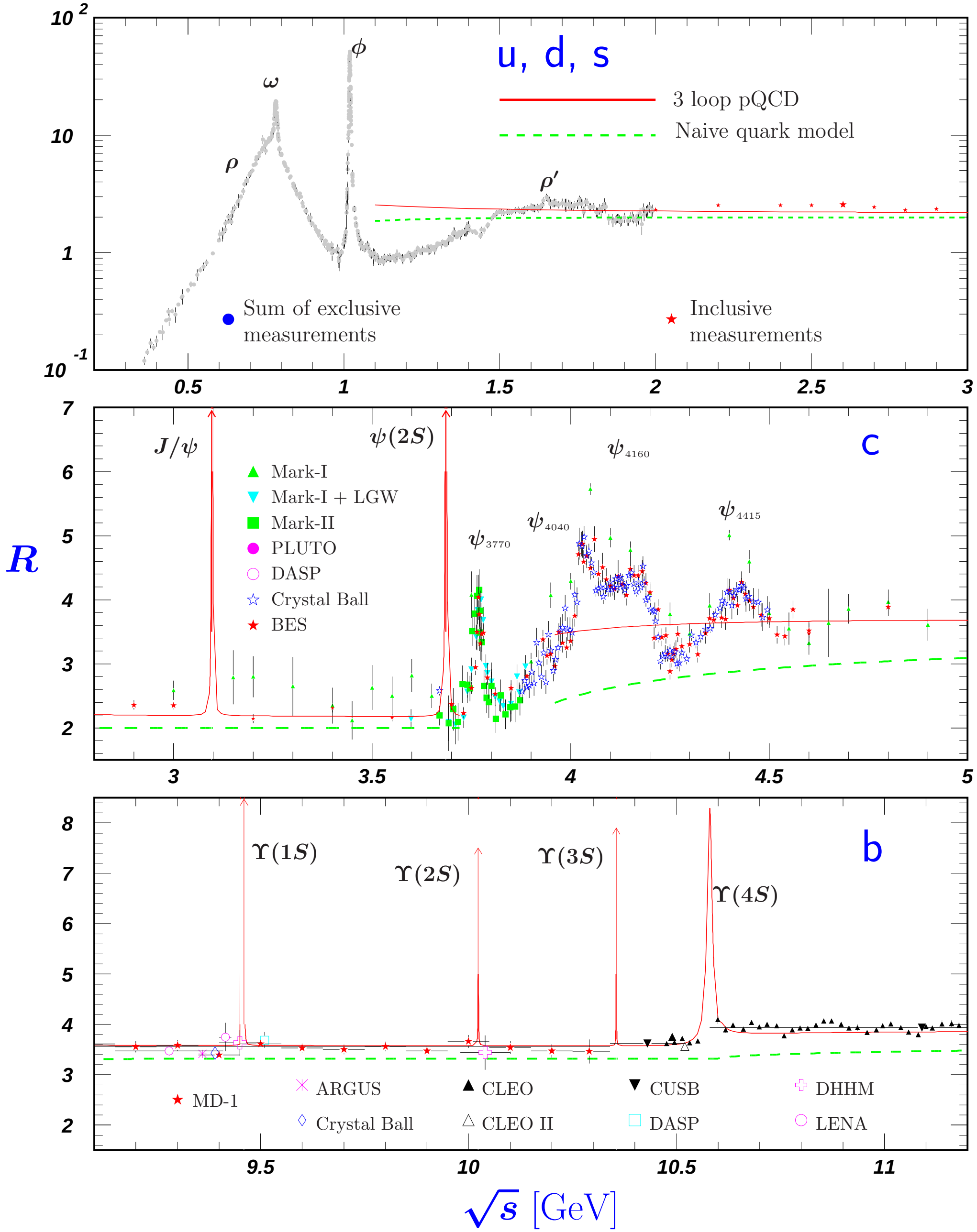
$$R_Z = \frac{\Gamma(Z \to \text{hadrons})}{\Gamma(Z \to \mu^+\mu^-)} = \frac{\sum_q \Gamma(Z \to q\bar{q})}{\Gamma(Z \to \mu^+\mu^-)} = \frac{3\sum_q (a_q^2 + v_q^2)}{a_\mu^2 + v_\mu^2}$$

The couplings to the Z are specified by the $SU(2)_L \times U(1)$ structure

$$v_f = T_f^3 - 2Q_f \sin^2\theta_W\,, \quad a_f = T_f^3$$

where $T_f^3 = \frac{1}{2}$ for $f = \nu, u, \ldots$ and $T_f^3 = -\frac{1}{2}$ for $f = e, d, \ldots$.

Comparison with data



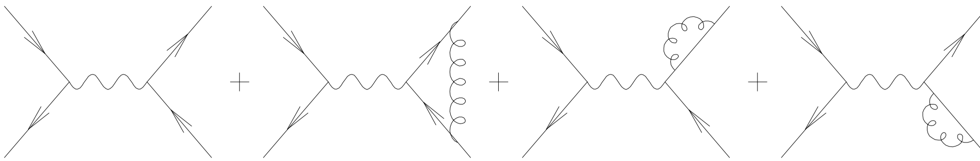
$$R = 3[(\tfrac{2}{3})^2 + 2(\tfrac{1}{3})^2] = 2$$

$$R = 3[2(\tfrac{2}{3})^2 + 2(\tfrac{1}{3})^2] = \frac{10}{3}$$

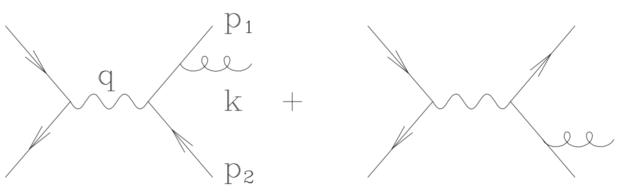
$$R = 3[2(\tfrac{2}{3})^2 + 3(\tfrac{1}{3})^2] = \frac{11}{3}$$

QCD corrections to e^+e^- to hadrons

- Measured cross section is about 5% higher than σ_0, due to QCD corrections. For massless quarks, corrections to R and R_Z are equal. To $\mathcal{O}(\alpha_S)$ we have:



(a)



(b)

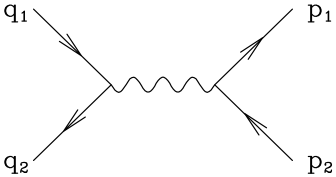
- Real emission diagrams (b):
  - ⋆ Write 3-body phase-space integration as

$$d\Phi_3 = [...] d\alpha\, d\beta\, d\gamma\, dx_1\, dx_2 \ ,$$

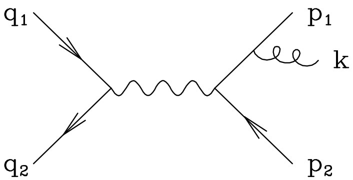
    α, β, γ are Euler angles of 3-parton plane,
    $x_1 = 2p_1 \cdot q/q^2 = 2E_q/\sqrt{s}$,
    $x_2 = 2p_2 \cdot q/q^2 = 2E_{\bar{q}}/\sqrt{s}$.
  - ⋆ Applying Feynman rules and integrating over Euler angles:

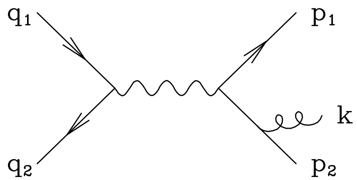
$$\sigma^{q\bar{q}g} = 3\sigma_0 C_F \frac{\alpha_S}{2\pi} \int dx_1\, dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \ .$$

Calculation of matrix element squared



$$\overline{\sum}|\mathcal{M}|^2 = e^4 Q_q^2 N_c \frac{2[(p_1 \cdot q_1)^2 + (p_1 \cdot q_2)^2]}{q_1 \cdot q_2^2}$$





Calculation of matrix element squared II

- Matrix element squared with gluon emission proportional to,

$$
\begin{aligned}
& \mathrm{Tr}\{\not{q}_1\gamma^\mu\not{q}_2\gamma^\nu\}\frac{1}{(2q_1\cdot q_2)^2} \\
\times \quad & [\mathrm{Tr}\{\not{p}_1\not{\epsilon}(\not{p}_1+\not{k})\gamma^\mu\not{p}_2\gamma^\nu(\not{p}_1+\not{k})\not{\epsilon}\}\frac{1}{((p_1+k)^2)^2} \\
+ \quad & \mathrm{Tr}\{\not{p}_1\gamma^\mu(\not{p}_2+\not{k})\not{\epsilon}\not{p}_2\not{\epsilon}(\not{p}_2+\not{k})\gamma^\nu\}\frac{1}{((p_2+k)^2)^2} \\
- \quad & 2\mathrm{Tr}\{\not{p}_1\not{\epsilon}(\not{p}_1+\not{k})\gamma^\mu\not{p}_2\not{\epsilon}(\not{p}_2+\not{k})\gamma^\nu\}\frac{1}{(p_1+k)^2(p_2+k)^2}]
\end{aligned}
$$

$$
\overline{\sum}|\mathcal{M}|^2 = e^4 Q_q^2 N_c 2C_F g_s^2 \frac{(p_1\cdot q_1)^2+(p_1\cdot q_2)^2+(p_2\cdot q_1)^2+(p_2\cdot q_2)^2}{q_1\cdot q_2\; p_1\cdot k\; p_2\cdot k}
$$

- Matrix elements squared summed/averaged over initial polarizations; summed over colours of final quarks.
- Notice cancellation of double pole to single pole.

Calculation of trace with Form

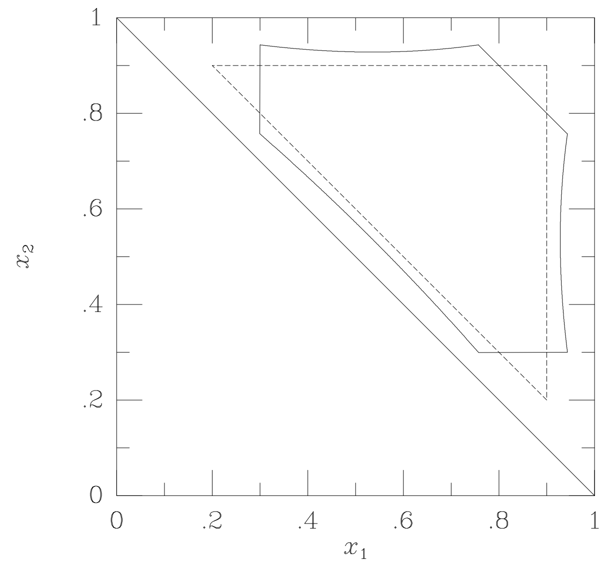
```
FORM 4.0 (Apr 10 2012) 64-bits                    Run: Sun Jun  9 07:57:
    Off Stats;
    * Declare vectors
    V p1,p2,k,q1,q2,p1k,p2k;
    * Declare indices
    I al,mu,nu,j,j1;
    * Declare ans to be a scalar;
    S ans;
    L Msq=-g_(j1,q1,mu,q2,nu)/q1.q2^2*(
      +g_(j,p1,al,p1k,mu,p2,nu,p1k,al)/p1k.p1k^2
      +g_(j,p1,mu,p2k,al,p2,al,p2k,nu)/p2k.p2k^2
      -2*g_(j,p1,al,p1k,mu,p2,al,p2k,nu)/p1k.p1k/p2k.p2k)-ans;
    * Take the 4-dimensional trace over the string with indices j,j1;
    Trace4,j;Trace4,j1;
    * Subtract the answer
    Id,ans=32*(p1.q1^2+p1.q2^2+p2.q1^2+p2.q2^2)/q1.q2/p1.k/p2.k;
    * Apply kinematic substitutions (q1+q2=p1+p2+k);
    Id,p1k.p1k^-1=1/2/p1.k;Id,p2k.p2k^-1=1/2/p2.k;
    Id,p1k=p1+k;Id,p2k=p2+k;
    Id,p1.p2=q1.q2-q1.k-q2.k;
    Id,p1.q1=q1.q2-k.p2-p1.q2;Id,k.q1=p1.q2+p2.k-p2.q1;
    Id,p2.q2=q1.q2-k.p1-p2.q1;Id,k.q2=p2.q1+p1.k-p1.q2;
    Id,p1.p1=0;Id,p2.p2=0;Id,q1.q1=0;Id,q2.q2=0;Id,k.k=0;
    Print +s;
    .end

   Msq = 0;
```

Integration of real radiation

- Integration region: $0 \leq x_1, x_2, x_3 \leq 1$ where $x_3 = 2k \cdot q/q^2 = 2E_g/\sqrt{s} = 2 - x_1 - x_2$.
- Integral divergent at $x_{1,2} = 1$:

$$
\begin{aligned}
1 - x_1 &= \frac{1}{2} x_2 x_3 (1 - \cos\theta_{qg}) \\
1 - x_2 &= \frac{1}{2} x_1 x_3 (1 - \cos\theta_{\bar{q}g})
\end{aligned}
$$



- Divergences:
  - ⋆ collinear when $\theta_{qg} \to 0$ or $\theta_{\bar{q}g} \to 0$;
  - ⋆ soft when $E_g \to 0$, i.e. $x_3 \to 0$.
- Singularities are not physical – simply indicate breakdown of PT when energies and/or invariant masses approach QCD scale Λ.

Integration with dimensional regularization

- Collinear and/or soft regions do not in fact make important contribution to R. To see this, make integrals finite using dimensional regularization, $D = 4 + 2\epsilon$ with $\epsilon < 0$. Then

$$
\begin{aligned}
\sigma^{q\bar{q}g} &= 2\sigma_0 \frac{\alpha_S}{\pi} H(\epsilon) \\
\times \quad & \int \frac{dx_1 dx_2}{P(x_1, x_2)} \left[\frac{(1-\epsilon)(x_1^2 + x_2^2) + 2\epsilon(1-x_3)}{[(1-x_1)(1-x_2)]} - 2\epsilon \right]
\end{aligned}
$$

where $\quad H(\epsilon) = \dfrac{3(1-\epsilon)(4\pi)^{2\epsilon}}{(3-2\epsilon)\Gamma(2-2\epsilon)} = 1 + \mathcal{O}(\epsilon)\ .$

and $\quad P(x_1, x_2) = \left[(1-x_1)(1-x_2)(1-x_3)\right]^{\epsilon}$

Hence

$$
\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_S}{\pi} H(\epsilon) \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right] .
$$

- Soft and collinear singularities are regulated, appearing instead as poles at $D = 4$.

Completion of total cross section

- Virtual gluon contributions (a): using dimensional regularization again

$$\sigma^{q\bar{q}} \;=\; 3\sigma_0 \left\{1 + \frac{2\alpha_S}{3\pi}\, H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right]\right\} .$$

- Adding real and virtual contributions, poles cancel and result is finite as $\epsilon \to 0$:

$$R \;=\; 3\sum_q Q_q^2 \, \left\{1 + \frac{\alpha_S}{\pi} + \mathcal{O}(\alpha_S^2)\right\} .$$

  Thus R is an infrared safe quantity.

- Coupling α_S evaluated at renormalization scale μ. UV divergences in R cancel to $\mathcal{O}(\alpha_S)$, so coefficient of α_S independent of μ. At $\mathcal{O}(\alpha_S^2)$ and higher, UV divergences make coefficients renormalization scheme dependent:

$$\begin{aligned} R &= 3\, K_{QCD} \sum_q Q_q^2 \, , \\ K_{QCD} &= 1 + \frac{\alpha_S(\mu^2)}{\pi} + \sum_{n\geq 2} C_n \left(\frac{s}{\mu^2}\right) \left(\frac{\alpha_S(\mu^2)}{\pi}\right)^n \end{aligned}$$

Higher order coefficients

- In $\overline{\text{MS}}$ scheme with scale $\mu = \sqrt{s}$,

$$\begin{aligned} C_2(1) &= \frac{365}{24} - 11\zeta(3) - [11 - 8\zeta(3)]\frac{N_f}{12}, \quad \zeta(3) = 1.2020569\ldots \\ &\simeq 1.986 - 0.115 N_f \end{aligned}$$

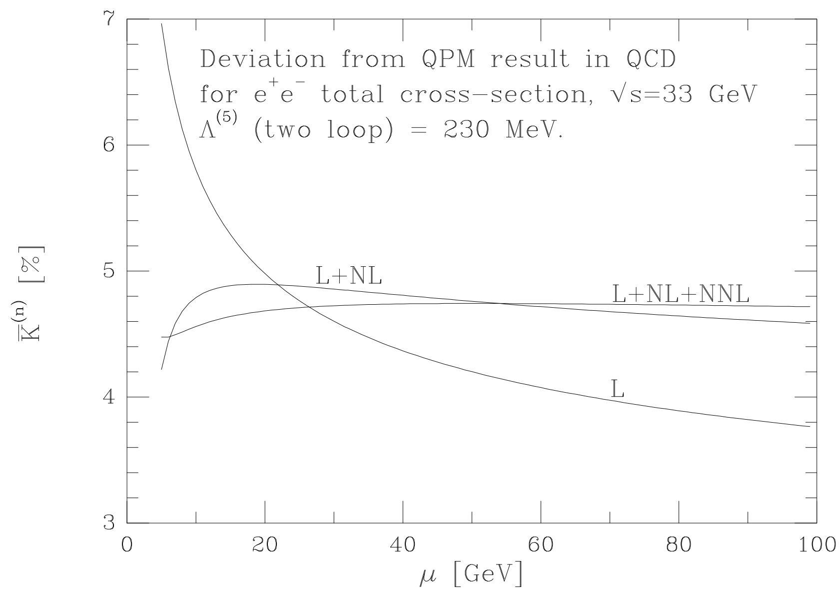
- Coefficient C_3 is also known.
- Scale dependence of $C_2,\ C_3\ \ldots$ fixed by requirement that, order-by-order, series should be independent of μ. For example

$$C_2\left(\frac{s}{\mu^2}\right) = C_2(1) - \frac{\beta_0}{4}\log\frac{s}{\mu^2}$$

where $\beta_0 = 4\pi b = 11 - 2N_f/3$.

Scheme and scale dependence

- Scale and scheme dependence only cancels completely when series is computed to all orders. Scale change at $\mathcal{O}(\alpha_S^n)$ induces changes at $\mathcal{O}(\alpha_S^{n+1})$. The more terms are added, the more stable is prediction with respect to changes in μ.



- Residual scale dependence is an important source of uncertainty in QCD predictions. One can vary scale over some 'physically reasonable' range, e.g. $\sqrt{s}/2 < \mu < 2\sqrt{s}$, to try to quantify this uncertainty. But there is no real substitute for a full higher-order calculation.

Shape distributions

- Shape variables measure some aspect of shape of hadronic final state, e.g. whether it is pencil-like, planar, spherical etc.
- For $d\sigma/dX$ to be calculable in PT, shape variable X should be infrared safe, i.e. insensitive to emission of soft or collinear particles. In particular, X must be invariant under $\boldsymbol{p}_i \to \boldsymbol{p}_j + \boldsymbol{p}_k$ whenever $\boldsymbol{p}_j$ and $\boldsymbol{p}_k$ are parallel or one of them goes to zero.
- Examples are Thrust and C-parameter:

$$
\begin{aligned}
T &= \max \frac{\sum_i |\boldsymbol{p}_i \cdot \boldsymbol{n}|}{\sum_i |\boldsymbol{p}_i|} \\
C &= \frac{3}{2} \frac{\sum_{i,j} |\boldsymbol{p}_i|\,|\boldsymbol{p}_j| \sin^2\theta_{ij}}{(\sum_i |\boldsymbol{p}_i|)^2}
\end{aligned}
$$

After maximization, unit vector $\boldsymbol{n}$ defines *thrust axis*.

- In Born approximation final state is $q\bar{q}$ and $1-T=C=0$. Non-zero contribution at $\mathcal{O}(\alpha_S)$ comes from $e^+e^- \to q\bar{q}g$. Recall distribution of $x_i = 2E_i/\sqrt{s}$:

$$
\frac{1}{\sigma}\frac{d^2\sigma}{dx_1 dx_2} = C_F \frac{\alpha_S}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \,.
$$

Distribution of shape variable X is obtained by integrating over x_1 and x_2 with constraint $\delta(X - f_X(x_1, x_2, x_3 = 2 - x_1 - x_2))$, i.e. along contour of constant X in (x_1, x_2)-plane.

- For thrust, $f_T = \max\{x_1, x_2, x_3\}$ and we find

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = C_F \frac{\alpha_S}{2\pi}\left[\frac{2(3T^2 - 3T + 2)}{T(1-T)}\log\left(\frac{2T-1}{1-T}\right) - \frac{3(3T-2)(2-T)}{(1-T)}\right].$$

  This diverges as $T \to 1$, due to soft and collinear gluon singularities. Virtual gluon contribution is negative and proportional to $\delta(1-T)$, such that correct total cross section is obtained after integrating over $\frac{2}{3} \leq T \leq 1$, the physical region for two- and three-parton final states.

- $\mathcal{O}(\alpha_S^2)$ corrections also known. Comparisons with data provide test of QCD matrix elements, through shape of distribution, and measurement of α_S, from overall rate. Care must be taken near $T = 1$ where (a) hadronization effects become large, and (b) large higher-order terms of the form $\alpha_S^n \log^{2n-1}(1-T)/(1-T)$ appear in $\mathcal{O}(\alpha_S^n)$.

Jet fractions

- To define fraction f_n of n-jet final states ($n = 2, 3, \ldots$), must specify jet algorithm.
- Development of a jet a series of sequential branchings. Majority of QCD branching is soft and/or collinear, with following divergences:

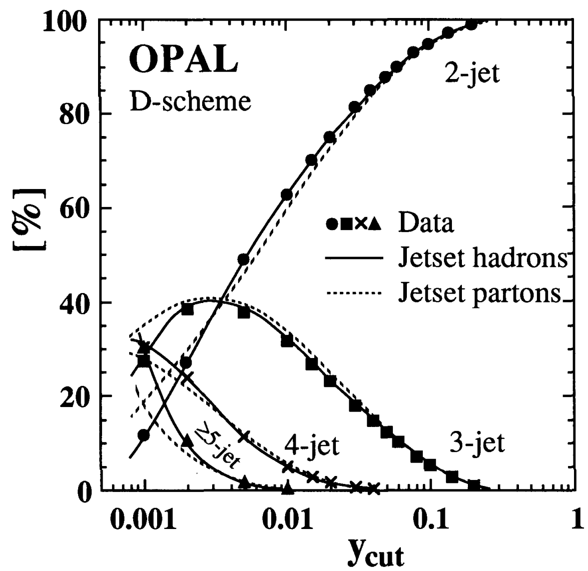
$$[dk_j]\,|M_{g\to g_i g_j}|^2 = \frac{2\alpha_s C_A}{\pi}\frac{dE_i}{E_i E_j}\frac{d\theta_{ij}}{\theta_{ij}} \quad E_j \ll E_i, \theta_{ij} \ll 1$$

- To invert branching process, take pair which are closest in a metric defined by the divergence structure of the theory

- This is the philosophy of k_T or Durham algorithm:
  - ⋆ Define jet resolution y_{cut} (dimensionless).
  - ⋆ For each pair of final-state momenta p_i, p_j define

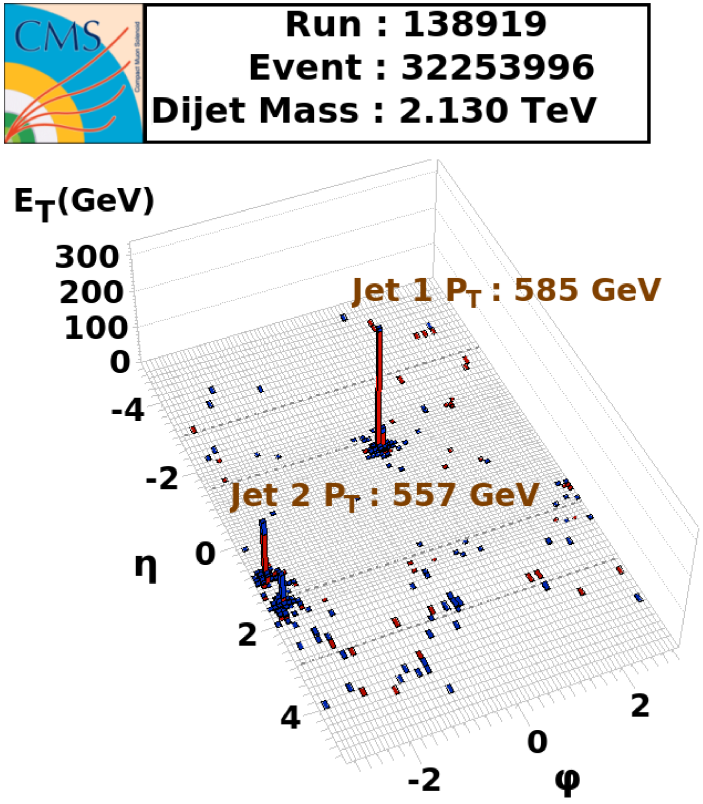
$$y_{ij} = 2\min\{E_i^2, E_j^2\}(1 - \cos\theta_{ij})/s$$

  - ⋆ If $y_{IJ} = \min\{y_{ij}\} < y_{\text{cut}}$, combine I, J into one object K with $p_K = p_I + p_J$.
  - ⋆ Repeat until $y_{IJ} > y_{\text{cut}}$. Then remaining objects are jets.



Jet algorithms

- At a hadron collider jets are clearly visible by eye.
- There are many possible mathematical procedures for defining a jet. A jet algorithm has to specify
  - ⋆ Which particles/partons are grouped together in a jet.
  - ⋆ How the momenta of the chosen particles are combined to form a pseudo-particle
- A proper jet algorithm should be insensitve to the emission of soft and collinear radiation
  - ⋆ From a theoretical point of view, this is a requirement for a finite result, which will be calculable in QCD perturbation theory
  - ⋆ From a experimental point of view, the detector will not be able to resolve collinear and/or soft hadrons.



- "Lego" plot in terms of azimuthal angle and rapidity $y = \frac{1}{2}\ln(\frac{E+p_z}{E-p_z})$. For a massless particle y is identical to the pseudorapidity, $\eta = -\ln\tan(\theta/2)$.
- Rapidity is additive under longitudinal boosts.

Sequential recombination jet algorithms

- Development of a jet a series of sequential branchings. Majority of QCD branching is soft and/or collinear, with following divergences:

$$[dk_j]\,|M_{g\to g_i g_j}|^2 = \frac{2\alpha_s C_A}{\pi}\frac{dE_i}{E_i E_j}\frac{d\theta_{ij}}{\theta_{ij}} \quad E_j \ll E_i, \theta_{ij} \ll 1$$

- To invert branching process, take pair which are closest in a metric defined by the divergence structure of the theory
- Definition of the k_T/Durham algorithm for hadron collisions.
  1. Calculate (or update) distances between all (pseudo-)particles i and j, (related to the relative k_T between the particles)

$$y_{ij} = 2min(E_i^2, E_j^2)(1 - \cos\theta_{ij})$$

  2. Find smallest of y_{ij}
     - ⋆ If $y > y_{cut}$ stop clustering
     - ⋆ else recombine i and j and repeat from step 1

Inclusive k_T algorithm

- The inclusive k_T algorithm for hadron-hadron collisions is a generalization of the e^+e^- variant.
- It belongs to the class of sequential recombination jet algorithms, which define both a jet and a clustering history.
- Introduces the new concept of a particle beam distance and the angular radius R

$$\begin{aligned} d_{ij} &= min(p_{Ti}^2, p_{Tj}^2)\frac{\Delta R_{ij}^2}{R^2}, \quad \Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2 \\ d_{iB} &= p_{Ti}^2 \end{aligned}$$

1. Find smallest of d_{ij} , d_{iB}
2. if it is ij , combine $i + j$ and return to step 1
3. if it is iB, call i a jet, remove it from list of particles, and return to step 1
4. stop when no particles are left.

- S.D. Ellis and Soper, (hep-ph/9305266);
- Jets all separated by at least R on the lego plot.
- NB: number of jets not IR safe (soft jets near beam); number of jets above p_t cut is IR safe.
- depends on two parameters, R and p_T^{cut}

Cambridge-Aachen for hadronic collisions

- We can classify the a family sequential recombination algorithms as follows

$$
\begin{aligned}
d_{ij} &= min(k_{Ti}^{2p}, k_{Tj}^{2p}) \frac{\Delta R_{ij}^2}{R^2} \\
d_{iB} &= k_{Ti}^{2p}
\end{aligned}
$$

- The Cambridge-Aachen is the simplest jet algorithm and corresponds to $p = 0$, Wobisch and Wengler, hep-ph/990728
- Recombine the pair of objects closest in R_{ij}
- Repeat until all $R_{ij} > R$.
- The remaining objects are jets.
- Because of clustering hierachy in angle, C/A has been shown to provide the best performance when it comes to resolving jet substructure
  - ⋆ undoing the pair-wise clustering of a jet step-by-step yields its subjets
  - ⋆ Promising strategies to find e.g. high-pT top quarks and Higgs bosons are based on subjets using the C/A algorithm
- Leads to ragged edge jets

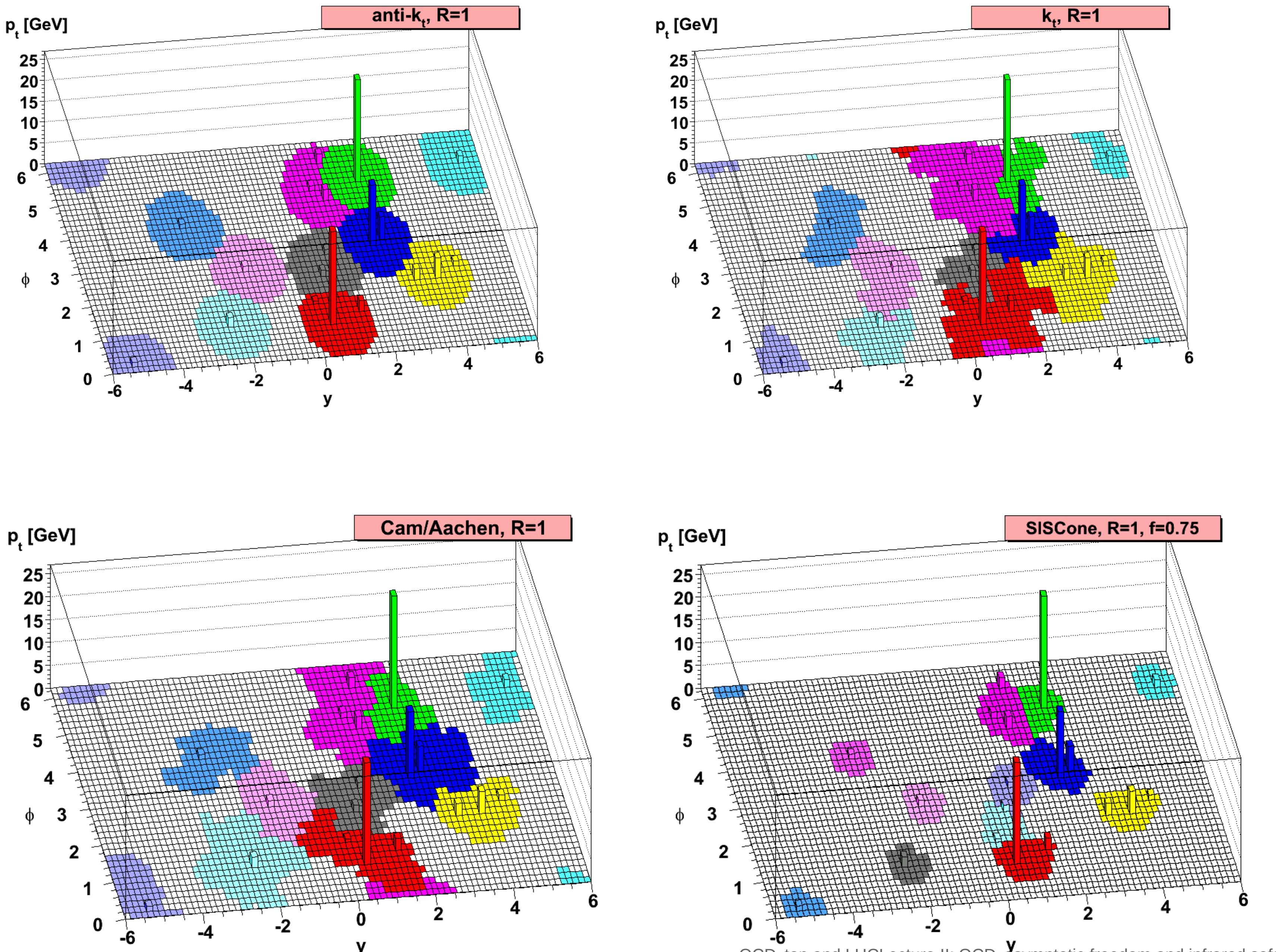
Anti-k_t

Formulated similarly to k_t (Cacciari, Salam & Soyez 0802.1189), but with

$$dij = min(1/k_{Ti}^2, 1/k_{Tj}^2)\frac{\Delta R_{ij}^2}{R^2}$$

- Anti-kt privileges the collinear divergence of QCD, favours clusterings that involve hard particles, and disfavours clustering between pairs of soft particles.
- Most pairwise clusterings involve at least one hard particle
- The algorithm involves two parameters, R the angular reach for the jets, and p_T threshold for the final jets to be taken into account.
- However since the algorithm still involves a combination of energy and angle in its distance measure, this is a collinear-safe growth, (a collinear branching gets clustered first).
- Anti-k_t leads to circular jets, which are experimentally favoured for acceptance corrections.
- Clustering sequence is not usefully related to QCD branching.

Comparison of jet algorithms, arXiv:0802.1189



Recap

- Perturbative QCD has infrared singularities due to collinear or soft parton emission. We can calculate infra-red safe or factorizable quantities in perturbation theory because of the property of asymptotic freedom.
- Total e^+e^- cross section is an example of an infra-red finite quantity.
- IR singularities are normally regularized by dimensional regulaization.
- Higher order corrections can be calculated for IR safe or factorizable quantities; because α_S is not so small, they are necessary to find agreement with the data.
- Jets are specified by a jet algorithm, depending on one or two parameters. Different algorithms (and parameters) will best capture different features of the data.