QCD, top and LHC
Lecture III: Parton branching

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Slides available from http://theory.fnal.gov/people/ellis/Talks/

Parton branching

- Solution to the Dirac Equation
- Branching Probabilities
- DGLAP equation
- Parton luminosities
- Shower Monte Carlos
- Sudakov form factor
- Monte Carlo method
- Soft gluon emission
- Angular ordering

Dirac eqn. Massless fermions

- The fermions involved in high energy processes can often be taken to be massless.
- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

- The Weyl representation is more suitable at high energy

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix},$$

- In the Weyl representation upper and lower components have different helicities.

$$\gamma_R = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \gamma_L = \frac{1}{2}(1-\gamma_5) = = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

Weyl representation

- Both representations satisfy the same commutation relations.

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

- in the Weyl representation $\gamma^0\gamma^i = \begin{pmatrix} \sigma^i & \mathbf{0} \\ \mathbf{0} & -\sigma^i \end{pmatrix}$. σ are the Pauli matrices.
- To derive an explicit solution for the massless Dirac equation $\not{p}u(p) = 0$, write out an explicit expression for $\not{p} = \gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3$ in the Weyl representation, NB $p^\pm = p^0 \pm p^3$, and for a massless particle $p^+p^- = (p^1)^2 + (p^2)^2$

$$\not{p} = \begin{pmatrix} 0 & 0 & p^+ & p^1 - ip^2 \\ 0 & 0 & p^1 + ip^2 & p^- \\ p^- & -p^1 + ip^2 & 0 & 0 \\ -p^1 - ip^2 & p^+ & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 0 & p^+ & p^1 - ip^2 \\ 0 & 0 & p^1 + ip^2 & p^- \\ p^- & -p^1 + ip^2 & 0 & 0 \\ -p^1 - ip^2 & p^+ & 0 & 0 \end{pmatrix} \begin{bmatrix} \sqrt{p^+} \\ (p^1 + ip^2)/\sqrt{p^+} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Spinor solutions of Dirac equation

- The massless spinor solns of Dirac eqn are

$$u_+(p) = \begin{bmatrix} \sqrt{p^+} \\ \sqrt{p^-} e^{i\varphi_p} \\ 0 \\ 0 \end{bmatrix}, \quad u_-(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^-} e^{-i\varphi_p} \\ -\sqrt{p^+} \end{bmatrix},$$

where

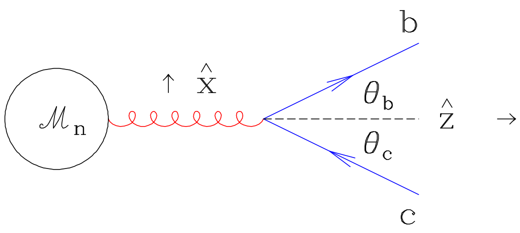
$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+ p^-}}, \qquad p^\pm = p^0 \pm p^3.$$

- Form massless particles charge conjugation is almost trivial; $u_\pm(p) = v_\mp(p)$
- In the Weyl representation the conjugate spinors are

$$u_+^\dagger(p) = \left[\sqrt{p^+}, \sqrt{p^-} e^{-i\varphi_p}, 0, 0\right]$$

$$u_-^\dagger(p) = \left[0, 0, \sqrt{p^-} e^{i\varphi_p}, -\sqrt{p^+}\right]$$

Parton branching - kinematics



$$
\begin{aligned}
p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\
p_b &= (E_b, +E_b \sin\theta_b, 0, +E_b \cos\theta_b) \\
p_c &= (E_c, -E_c \sin\theta_c, 0, +E_c \cos\theta_c)
\end{aligned}
$$

- the kinematics and notation for the branching of parton a into $b + c$. We assume that

$$p_b^2,\, p_c^2 \ll p_a^2 \equiv t$$

- a is an outgoing parton, which is called timelike branching since $t > 0$.
- The opening angle is $\theta = \theta_b + \theta_c$. Defining the energy fraction as

$$z = E_b/E_a = 1 - E_c/E_a \ ,$$

  we have for small angles, $t = 2E_bE_c(1-\cos\theta) = z(1-z)E_a^2\theta^2$
- using transverse momentum conservation,

$$\theta = \frac{1}{E_a}\sqrt{\frac{t}{z(1-z)}} = \frac{\theta_b}{1-z} = \frac{\theta_c}{z} \ .$$

Branching probabilities

- Consider the case where

$$
\begin{aligned}
p_a &= (E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a}) \\
p_b &\sim (E_b, +E_b\theta_b, 0, +E_b) \\
p_c &\sim (E_c, -E_c\theta_c, 0, +E_c)
\end{aligned}
$$

Thus for example

$$u_+^\dagger(p_b) = \sqrt{2E_b}\left[1, \frac{\theta_b}{2}, 0, 0\right]$$

and

$$u_+(p_c) \equiv v_-(p_c) = \sqrt{2E_c}\begin{bmatrix} 1 \\ -\frac{\theta_c}{2} \\ 0 \\ 0 \end{bmatrix}$$

Hence for polarization vectors $\varepsilon_{in} = (0, 1, 0, 0), \varepsilon_{out} = (0, 0, 1, 0)$

$$g\bar{u}_+^b\ \gamma^0\gamma^1\ v_-^c = g\sqrt{4E_bE_c}\left(1, \frac{\theta_b}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{\theta_c}{2} \end{pmatrix} = -g\sqrt{E_bE_c}(\theta_b - \theta_c)$$

$$-g\bar{u}^b_+ \gamma_\mu \varepsilon_a^{p\text{in}\,\mu} v^c_- = g\sqrt{E_b E_c}(\theta_b - \theta_c) = g\sqrt{z(1-z)}(1-2z)E_a\theta = g(1-2z)\sqrt{t}$$

$$-g\bar{u}^b_+ \gamma_\mu \varepsilon_a^{p\text{out}\,\mu} v^c_- = ig\sqrt{E_b E_c}(\theta_b + \theta_c) = ig\sqrt{z(1-z)}E_a\theta = ig\sqrt{t}$$

and the matrix element relation for the branching is

$$|\mathcal{M}_{n+1}|^2 \sim \frac{g^2}{t} T_R F(z; \varepsilon_a, \lambda_b, \lambda_c)|\mathcal{M}_n|^2$$

where the colour factor is now $\text{Tr}(t^A t^A)/8 = T_R = 1/2$. The non-vanishing functions $F(z; \varepsilon_a, \lambda_b, \lambda_c)$ for quark and antiquark helicities λ_b and λ_c are

ε_a	λ_b	λ_c	$F(z; \varepsilon_a, \lambda_b, \lambda_c)$
in	$\pm$	$\mp$	$(1-2z)^2$
out	$\pm$	$\mp$	1

Summing over the polarizations we get

$$2\Big[(1-2z)^2 + 1\Big] = 4(z^2 + (1-z)^2).$$

- Angular momentum argument for vanishing of amplitude in forward direction.

Branching probabilities

$$\int \frac{d\phi}{2\pi} CF = \hat{P}_{ba}(z)$$

where $\hat{P}_{ba}(z)$ is the appropriate splitting function

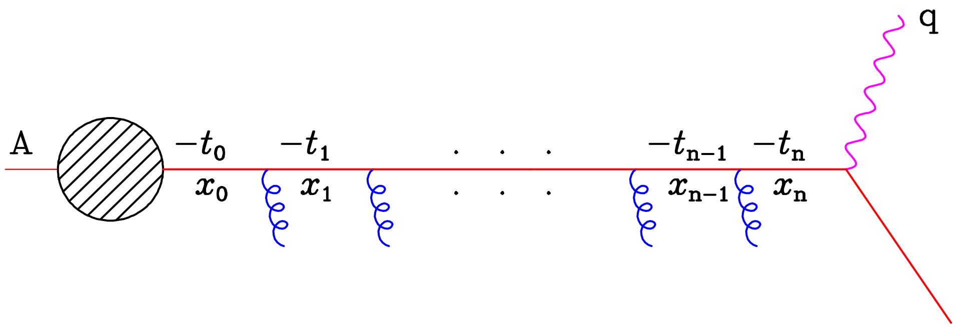
$$d\sigma_{n+1} = d\sigma_n \, \frac{dt}{t} \, dz \, \frac{\alpha_S}{2\pi} \, \hat{P}_{ba}(z) \; .$$

- Including all the color factors we find the results for the unregulated branching probabilities.

$$\begin{aligned}
\hat{P}_{qq}(z) &= C_F \left[\frac{1+z^2}{(1-z)}\right] \; , \\
\hat{P}_{qg}(z) &= T_R \left[z^2 + (1-z)^2\right] , \quad T_R = \frac{1}{2} \; , \\
\hat{P}_{gq}(z) &= C_F \left[\frac{1+(1-z)^2}{z}\right] \; , \\
\hat{P}_{gg}(z) &= C_A \left[\frac{z}{(1-z)} + \frac{1-z}{z} + z\,(1-z)\right]
\end{aligned}$$

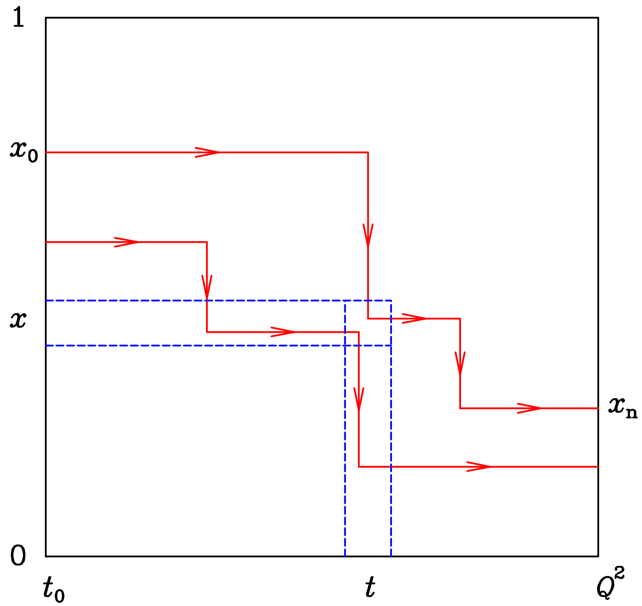
DGLAP equation

- Consider enhancement of higher-order contributions due to multiple small-angle parton emission, for example in deep inelastic scattering (DIS)



- Incoming quark from target hadron, initially with low virtual mass-squared $-t_0$ and carrying a fraction x_0 of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared $q^2 = -Q^2$.
- Cross section will depend on Q^2 and on momentum fraction distribution of partons seen by virtual photon at this scale, $D(x, Q^2)$.

- To derive evolution equation for Q^2-dependence of $D(x, Q^2)$, first introduce pictorial representation of evolution, also useful later for Monte Carlo simulation.



- Represent sequence of branchings by path in (t, x)-space. Each branching is a step downwards in x, at a value of t equal to (minus) the virtual mass-squared after the branching.
- At $t = t_0$, paths have distribution of starting points $D(x_0, t_0)$ characteristic of target hadron at that scale. Then distribution $D(x, t)$ of partons at scale t is just the x-distribution of paths at that scale.

Change in parton distribution

- Consider change in the parton distribution $D(x,t)$ when t is increased to $t+\delta t$. This is number of paths arriving in element $(\delta t, \delta x)$ minus number leaving that element, divided by δx.
- Number arriving is branching probability times parton density integrated over all higher momenta $x' = x/z$,

$$\begin{aligned}\delta D_{\text{in}}(x,t) &= \frac{\delta t}{t}\int_x^1 dx'\, dz \frac{\alpha_S}{2\pi}\hat{P}(z) D(x',t)\,\delta(x - zx') \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z) D(x/z,t)\end{aligned}$$

- For the number leaving element, must integrate over lower momenta $x' = zx$:

$$\begin{aligned}\delta D_{\text{out}}(x,t) &= \frac{\delta t}{t} D(x,t)\int_0^x dx'\, dz \frac{\alpha_S}{2\pi}\hat{P}(z)\,\delta(x' - zx) \\ &= \frac{\delta t}{t} D(x,t)\int_0^1 dz \frac{\alpha_S}{2\pi}\hat{P}(z)\end{aligned}$$

Change in parton distribution

- Change in population of element is

$$\begin{aligned} \delta D(x,t) &= \delta D_{\text{in}} - \delta D_{\text{out}} \\ &= \frac{\delta t}{t} \int_0^1 dz \, \frac{\alpha_S}{2\pi} \hat{P}(z) \left[\frac{1}{z} D(x/z,t) - D(x,t) \right] . \end{aligned}$$

- Introduce plus-prescription with definition

$$\int_0^1 dx \; f(x) \, g(x)_+ \; = \; \int_0^1 dx \; [f(x) - f(1)] \, g(x) \; .$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+ \; ,$$

- Plus-prescription, like the Dirac-delta function, is only defined under integral sign.
- Plus-prescription includes some of the effects of virtual diagrams.

DGLAP

We obtain the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation:

$$t\frac{\partial}{\partial t}D(x,t) = \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P(z)D(x/z,t)\ .$$

- Here $D(x,t)$ represents parton momentum fraction distribution inside incoming hadron probed at scale t.
- In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

Quarks and gluons

- For several different types of partons, must take into account different processes by which parton of type i can enter or leave the element $(\delta t, \delta x)$. This leads to coupled DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}D_i(x,t) = \sum_j \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}P_{ij}(z)D_j(x/z,t)\ .$$

- Quark ($i = q$) can enter element via either $q \to qg$ or $g \to q\bar{q}$, but can only leave via $q \to qg$. Thus plus-prescription applies only to $q \to qg$ part, giving

$$\begin{aligned}
P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\
P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\left[z^2+(1-z)^2\right] \\
P_{gg}(z) &= 2C_A\left[\left(\frac{z}{1-z}+\frac{1}{2}z(1-z)\right)_+ + \frac{1-z}{z}+\frac{1}{2}z(1-z)\right] - \frac{2}{3}N_f T_R\,\delta(1-z) \\
P_{gq}(z) &= P_{g\bar{q}}(z) = \hat{P}_{qq}(1-z) = C_F\frac{1+(1-z)^2}{z}\ .
\end{aligned}$$

- Using definition of the plus-prescription, P_{qq} and P_{gg} can be written in more common forms

$$
\begin{aligned}
P_{qq}(z) &= C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right] \\
P_{gg}(z) &= 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{1}{6}(11C_A - 4N_f T_R)\,\delta(1-z)\ .
\end{aligned}
$$

Solution by moments

- Given $f_i(x,t)$ at some scale $t = t_0$, factorized structure of DGLAP equation means we can compute its form at any other scale.
- One strategy for doing this is to take moments (Mellin transforms) with respect to x:

$$\tilde{f}_i(N,t) = \int_0^1 dx\, x^{N-1}\, f_i(x,t)\ .$$

Inverse Mellin transform is

$$f_i(x,t) = \frac{1}{2\pi i}\int_C dN\, x^{-N}\, \tilde{f}_i(N,t)\ ,$$

where contour C is parallel to imaginary axis to right of all singularities of integrand.

- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t\frac{\partial}{\partial t}\tilde{f}_i(x,t) = \sum_j \gamma_{ij}(N,\alpha_S)\tilde{f}_j(N,t)$$

- Lowest order approximation for γ, $\gamma_{ij}^{(0)}(N) = \int_0^1 dx\, x^{N-1} P_{ij}(x)$

$$t\frac{\partial}{\partial t}\tilde{f}_i(x,t) = \frac{\alpha_S}{2\pi}\sum_j \gamma_{ij}^{(0)}(N)\tilde{f}_j(N,t)$$

Scaling violation

- Consider combination of parton distributions which is flavour non-singlet, e.g. $D_V = D_{q_i} - D_{\bar{q}_i}$ or $D_{q_i} - D_{q_j}$. Then mixing with the flavour-singlet gluons drops out and solution for fixed α_S is

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{t}{t_0}\right)^{\gamma_{qq}(N,\alpha_S)} ,$$

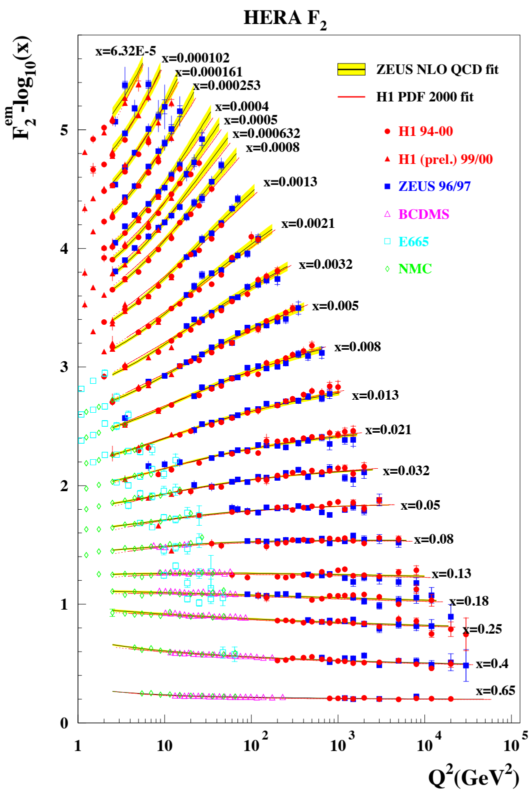
- We see that dimensionless function D_V, instead of being scale-independent function of x as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale t (hence the name anomalous dimensions).
- For running coupling $\alpha_S(t)$, scaling violation is power-behaved in $\ln t$ rather than t. Using leading-order formula $\alpha_S(t) = 1/b\ln(t/\Lambda^2)$, we find

$$\tilde{D}_V(N,t) = \tilde{D}_V(N,t_0)\left(\frac{\alpha_S(t_0)}{\alpha_S(t)}\right)^{d_{qq}(N)}$$

where $d_{qq}(N) = \gamma_{qq}^{(0)}(N)/2\pi b$.

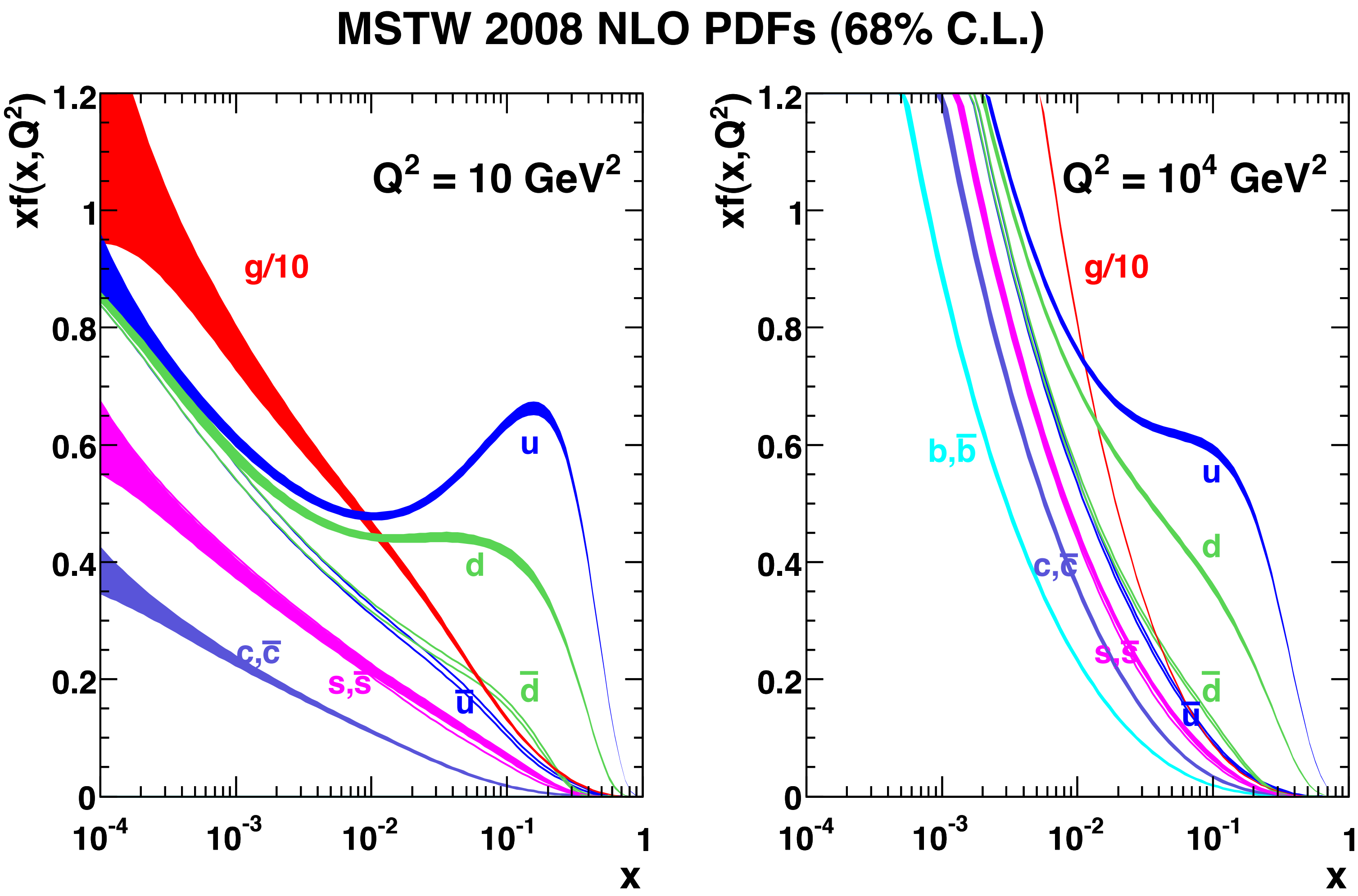
- $$\gamma^0_{qq} = C_F\Big[-\frac{1}{2} + \frac{1}{(N(N+1)} - 2\sum_{k=2}^{N}\frac{1}{k}\Big]$$

Combined data on F2 proton



- Now $d_{qq}(1) = 0$ and $d_{qq}(N) < 0$ for $N \geq 2$. Thus as t increases V *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data:

Parton distributions



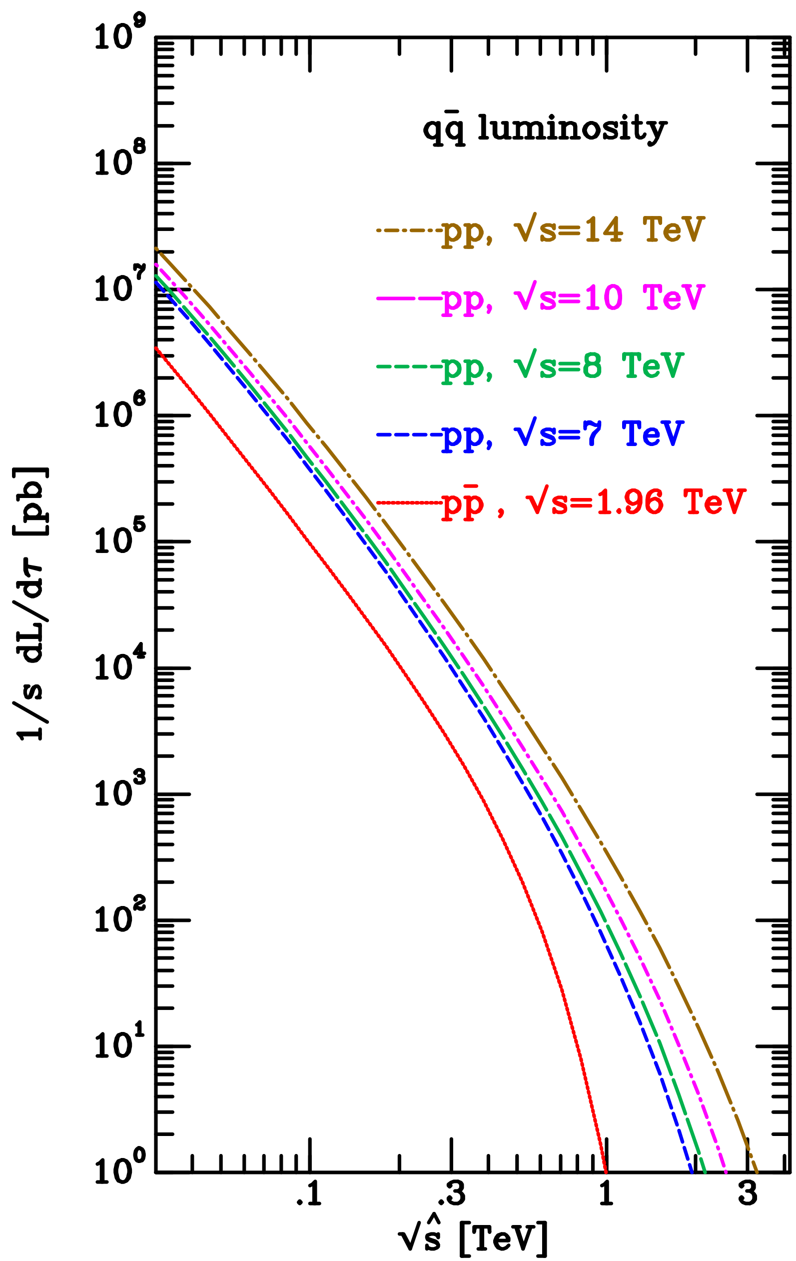
Parton luminosity

- Parton luminosity is determined by the parton distribution functions, $f_i(x_1, \mu^2)$ and $f_j(x_2, \mu^2)$.
- $f_j(x_2, \mu^2)$ need to be determined by data.
- the available centre-of-mass energy-squared of the parton-parton collision, $\hat{s}$, is less than the overall hadron-hadron collision energy, s, by a factor of $x_1 x_2 \equiv \tau$.
- Define differential parton luminosities

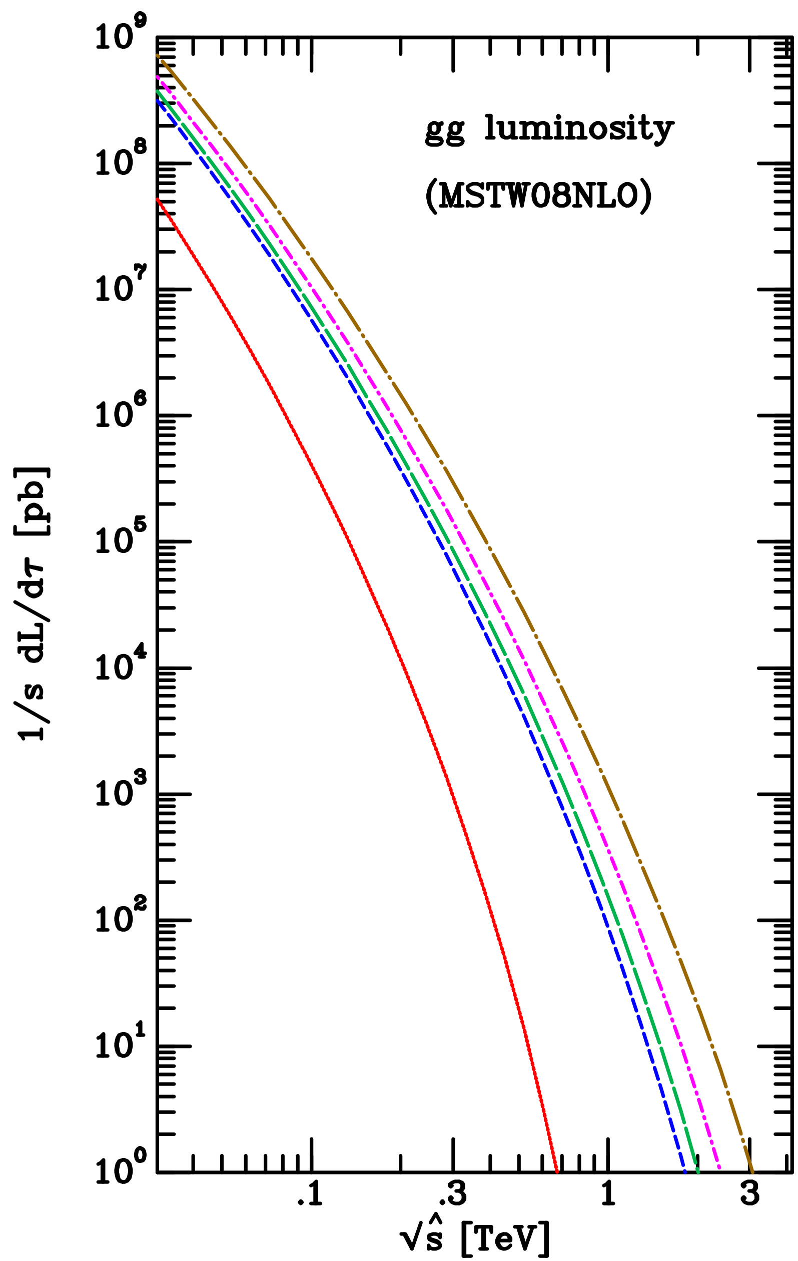
$$\tau \frac{dL_{ij}}{d\tau} = \frac{1}{1+\delta_{ij}} \int_0^1 dx_1 dx_2$$
$$\times \Big[\big(x_1 f_i(x_1, \mu^2)\; x_2 f_j(x_2, \mu^2) \big) + \big(1 \leftrightarrow 2\big) \Big] \delta(\tau - x_1 x_2).$$

- The collider luminosity is quite distinct from the parton luminosity. The former is a property of a machine, whereas the latter is a property of the proton.
- We now assume that $\hat{\sigma}$ depends only on $\hat{s}$.

$$\sigma(s) = \sum_{\{ij\}} \int_{\tau_0}^1 \frac{d\tau}{\tau} \left[\frac{1}{s} \frac{dL_{ij}}{d\tau} \right] \left[\hat{s} \hat{\sigma}_{ij} \right],$$

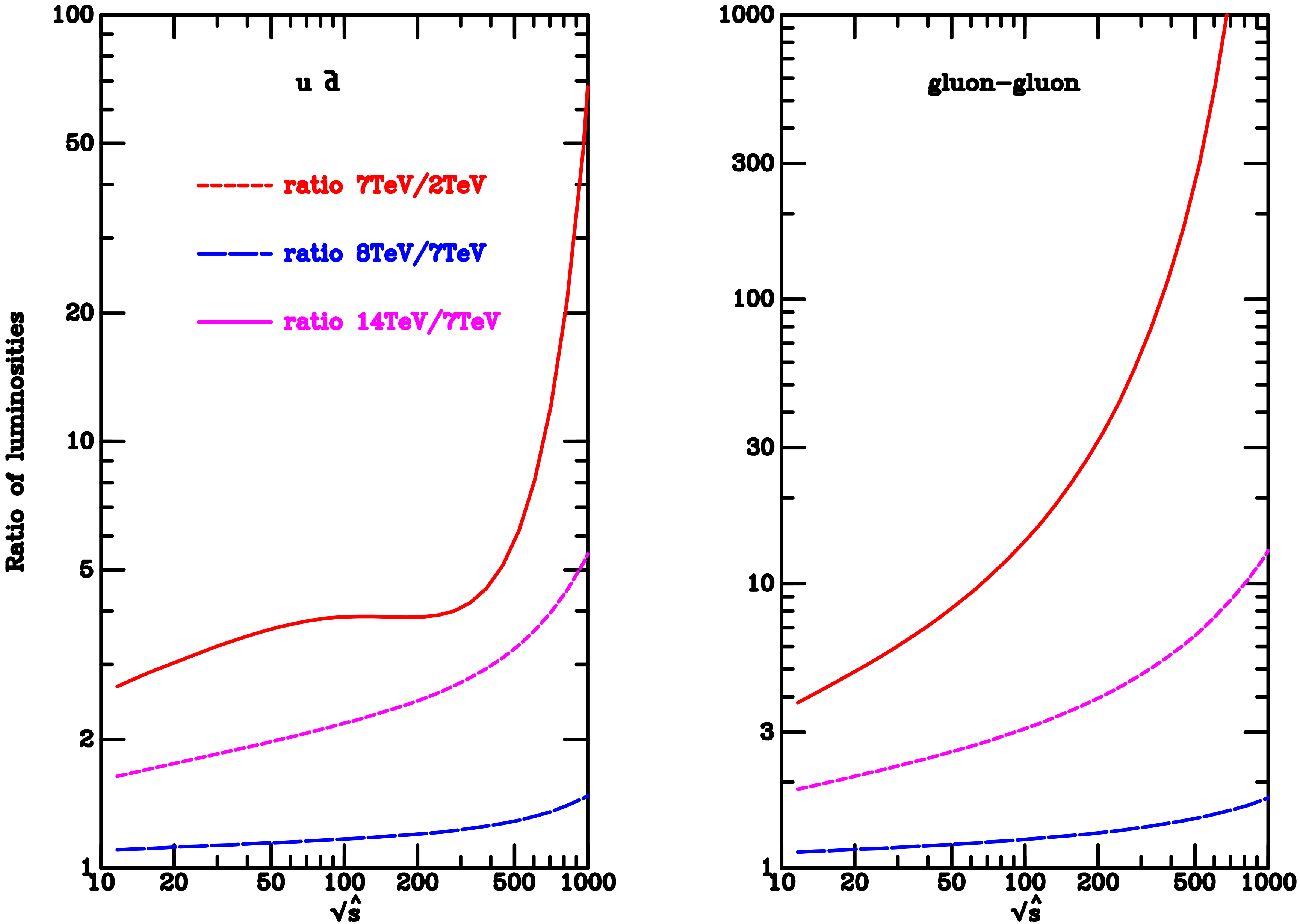


q$\bar{q}$ luminosity
pp, √s=14 TeV
pp, √s=10 TeV
pp, √s=8 TeV
pp, √s=7 TeV
p$\bar{p}$, √s=1.96 TeV
1/s dL/dτ [pb]
√ŝ [TeV]



gg luminosity
(MSTW08NLO)
1/s dL/dτ [pb]
√ŝ [TeV]

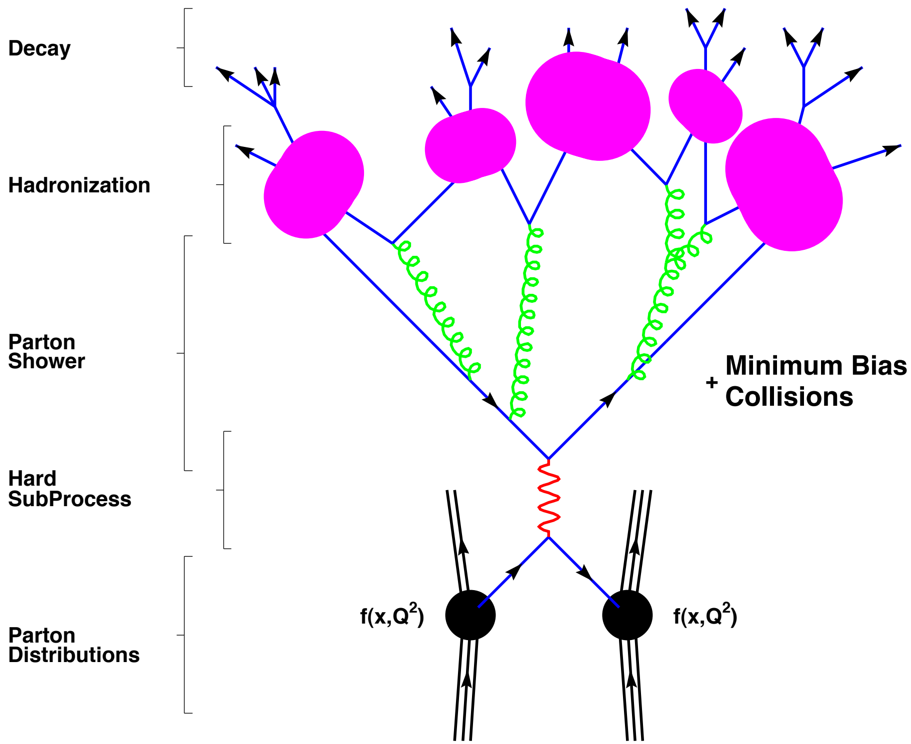
Ratios of luminosities



Shower Monte Carlos

Attempt to give a complete description of a hadron scattering event. Analysis of events at the Tevatron and the LHC will be performed using these programs.

- Large number of standard model hard scattering processes
- Inclusion of some BSM processes.
- Inclusion of real (and virtual) radiation using QCD-based parton shower approximation
- Fragmentation of partons into the observed hadrons.
- Model for resonance decay included



Leading general purpose programs

- Pythia, T. Sjöstrand, L. Lönnblad, S. Mrenna and P.Z. Skands,
  http://www.thep.lu.se/$\sim$torbjorn/Pythia.html
- Pythia 8, T. Sjöstrand C++
  http://www.thep.lu.se/$\sim$torbjorn/Pythia.html
- HERWIG, G. Corcella, I.G. Knowles, G. Marchesini, S. Moretti, K. Odagiri, P. Richardson, M.H. Seymour, B.R. Webber,
  http://hepwww.rl.ac.uk/theory/seymour/herwig/
- Herwig++
  S. Gieseke, A. Ribon, P. Richardson, M.H. Seymour, P. Stephens, B.R. Webber
  http://projects.hepforge.org/herwig/
- SHERPA,
  Tanju Gleisberg, Frank Krauss, Andreas Schälicke, Steffen Schumann, Jan Winter
  http://www.sherpa-mc.de/
- ISAJET,
  F. Paige, S. Protopopescu, H. Baer and X. Tata
  http://www.phy.bnl.gov/$\sim$isajet

Sudakov form factor

- DGLAP equations are convenient for evolution of parton distributions. Expressed in terms of the unregulated branching probability we have,

$$t\frac{\partial}{\partial t}D(x,t) \quad = \quad \int_x^1 \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t) - \int_0^1 dz\frac{\alpha_S}{2\pi}\hat{P}(z)D(x,t)$$

- To study structure of final states, slightly different form is useful. Consider again simplified treatment with only one type of branching. Introduce Sudakov form factor:

$$\Delta(t) \equiv \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int dz\,\frac{\alpha_S}{2\pi}\hat{P}(z)\right] \ ,$$

$$t\frac{\partial}{\partial t}D(x,t) \quad = \quad \int \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t) + \frac{D(x,t)}{\Delta(t)}t\frac{\partial}{\partial t}\Delta(t) \ ,$$

$$t\frac{\partial}{\partial t}\left(\frac{D}{\Delta}\right) \quad = \quad \frac{1}{\Delta}\int \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}(z)D(x/z,t) \ .$$

Sudakov form factor

- This is similar to DGLAP, except D replaced by D/Δ and regularized splitting function P replaced by unregularized $\hat{P}$. Integrating,

$$D(x,t) \quad = \quad \Delta(t)D(x,t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z,t')\ .$$

- This has simple interpretation. First term is contribution from paths that do not branch between scales t_0 and t. Thus Sudakov form factor $\Delta(t)$ is probability of evolving from t_0 to t without branching. Second term is contribution from paths which have their last branching at scale t'. Factor of $\Delta(t)/\Delta(t')$ is probability of evolving from t' to t without branching.

Sudakov form factor

- Generalization to several species of partons straightforward. Species i has Sudakov form factor

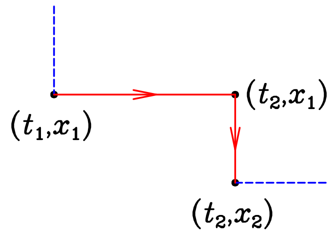
$$\Delta_i(t) \equiv \exp\left[-\sum_j \int_{t_0}^{t} \frac{dt'}{t'} \int dz\, \frac{\alpha_S}{2\pi} \hat{P}_{ji}(z)\right] ,$$

which is probability of it evolving from t_0 to t without branching. Then

$$t\frac{\partial}{\partial t}\left(\frac{D_i}{\Delta_i}\right) = \frac{1}{\Delta_i}\sum_j \int \frac{dz}{z}\frac{\alpha_S}{2\pi}\hat{P}_{ij}(z) D_j(x/z,t) \; .$$

Monte Carlo method

- Monte Carlo branching algorithm operates as follows: given virtual mass scale and momentum fraction (t_1, x_1) after some step of the evolution, or as initial conditions, it generates values (t_2, x_2) after the next step.



  - ⋆ Since probability of evolving from t_1 to t_2 without branching is $\Delta(t_2)/\Delta(t_1)$, t_2 can be generated with the correct distribution by solving

$$\frac{\Delta(t_2)}{\Delta(t_1)} = \mathcal{R}$$

    where $\mathcal{R}$ is random number (uniform on $[0,1]$).
  - ⋆ If t_2 is higher than hard process scale Q^2, this means branching has finished.
  - ⋆ Otherwise, generate $z = x_2/x_1$ with distribution proportional to $(\alpha_S/2\pi)P(z)$, where $P(z)$ is appropriate splitting function, by solving

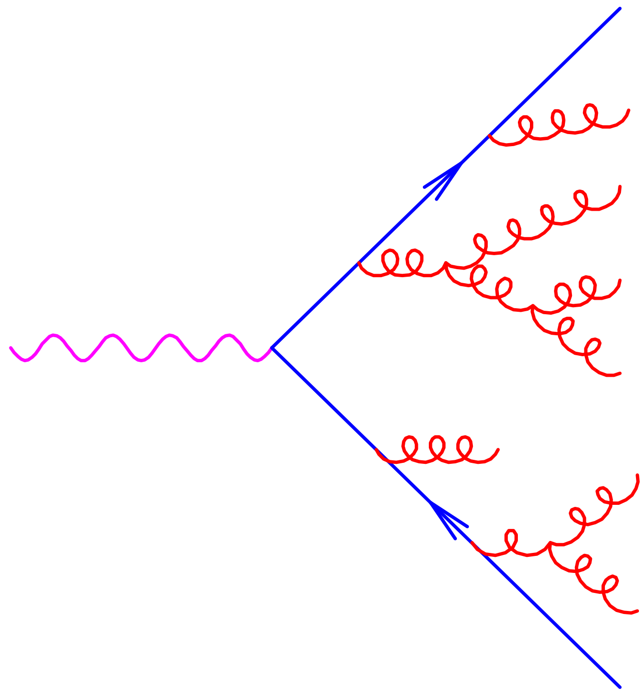
$$\int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_S}{2\pi} P(z) = \mathcal{R}' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_S}{2\pi} P(z)$$

- In DIS, (t_i, x_i) values generated define virtual masses and momentum fractions of exchanged quark, from which momenta of emitted gluons can be computed. Azimuthal emission angles are then generated uniformly in the range $[0, 2\pi]$. More generally, e.g. when exchanged parton is a gluon, azimuths must be generated with polarization angular correlations.
- Each emitted (timelike) parton can itself branch. In that case t evolves downwards towards cutoff value t_0, rather than upwards towards hard process scale Q^2. Probability of evolving downwards without branching between t_1 and t_2 is now given by

$$\frac{\Delta(t_1)}{\Delta(t_2)} = \mathcal{R} \; .$$

Thus branching stops when $\mathcal{R} < \Delta(t_1)$.

Parton Cascade



- Due to successive branching, parton cascade or shower develops. Each outgoing line is source of new cascade, until all outgoing lines have stopped branching. At this stage, which depends on cutoff scale t_0, outgoing partons have to be converted into hadrons via a hadronization model.

Soft gluon emission

- Parton branching formalism discussed so far takes account of collinear enhancements to all orders in PT. There are also soft enhancements: When external line with momentum p and mass m (not necessarily small) emits gluon with momentum q, propagator factor is

$$\frac{1}{(p \pm q)^2 - m^2} = \frac{\pm 1}{2p \cdot q} = \frac{\pm 1}{2\omega E(1 - v\cos\theta)}$$

  where ω is emitted gluon energy, E and v are energy and velocity of parton emitting it, and θ is angle of emission. This diverges as $\omega \to 0$, for any velocity and emission angle.

- Including numerator, soft gluon emission gives a colour factor times universal, spin-independent factor in amplitude

$$F_{\text{soft}} = \frac{p \cdot \epsilon}{p \cdot q}$$

  where ϵ is polarization of emitted gluon.

- For example, emission from quark gives numerator factor $N \cdot \epsilon$, where

$$
\begin{aligned}
N^\mu &= (\not{p} + \not{q} + m)\gamma^\mu u(p) \underset{\omega \to 0}{\longrightarrow} (\gamma^\nu \gamma^\mu p_\nu + \gamma^\mu m) u(p) \\
&= (2p^\mu - \gamma^\mu \not{p} + \gamma^\mu m) u(p) \; = \; 2p^\mu u(p) \; .
\end{aligned}
$$

(using Dirac equation for on-mass-shell spinor $u(p)$).

- Universal factor F_{soft} coincides with classical eikonal formula for radiation from current p^μ, valid in long-wavelength limit.
- No soft enhancement of radiation from off-mass-shell internal lines, since associated denominator factor $(p+q)^2 - m^2 \to p^2 - m^2 \neq 0$ as $\omega \to 0$.

- Enhancement factor in amplitude for each external line implies cross section enhancement is sum over all pairs of external lines $\{i,j\}$:

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega}\frac{d\Omega}{2\pi}\frac{\alpha_S}{2\pi}\sum_{i,j} C_{ij}W_{ij}$$

where $d\Omega$ is element of solid angle for emitted gluon, C_{ij} is a colour factor, and radiation function W_{ij} is given by

$$W_{ij} = \frac{\omega^2 p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q} = \frac{1-v_iv_j\cos\theta_{ij}}{(1-v_i\cos\theta_{iq})(1-v_j\cos\theta_{jq})}\ .$$

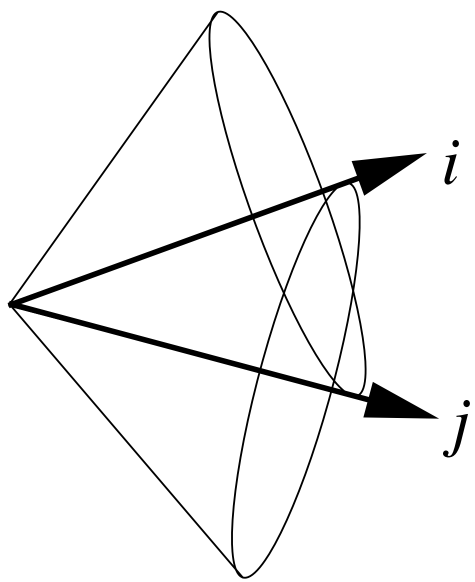
Colour-weighted sum of radiation functions $C_{ij}W_{ij}$ is antenna pattern of hard process.

- Radiation function can be separated into two parts containing collinear singularities along lines i and j. Consider for simplicity massless particles, $v_{i,j}=1$. Then $W_{ij} = W_{ij}^i + W_{ij}^j$ where

$$W_{ij}^i = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{jq}}\right) \equiv \frac{1}{2(1-\cos\theta_{iq})}\left(1+\frac{\cos\theta_{iq}-\cos\theta_{ij}}{1-\cos\theta_{jq}}\right)$$

- This function has remarkable property of angular ordering. Write angular integration in polar coordinates w.r.t. direction of i, $d\Omega = d\cos\theta_{iq}\, d\phi_{iq}$. Performing azimuthal integration, we find

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^i = \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}\text{, otherwise 0.}$$



Thus, after azimuthal averaging, contribution from W_{ij}^i is confined to cone, centred on direction of i, extending in angle to direction of j. Similarly, W_{ij}^j, averaged over ϕ_{jq}, is confined to cone centred on line j extending to direction of i.

Angular ordering

- To prove angular ordering property, write $n_i = (0,0,1), n_j = (0, \sin\theta_{ij}, \cos\theta_{ij}), n_q = (\sin\theta_{iq}\sin\phi_{iq}, -\sin\theta_{iq}\cos\phi_{iq}, \cos\theta_{iq})$, so that

$$1 - \cos\theta_{jq} = a - b\cos\phi_{iq}$$

where $a = 1 - \cos\theta_{ij}\cos\theta_{iq}$, $b = \sin\theta_{ij}\sin\theta_{iq}$. Defining $z = \exp(i\phi_{iq})$, we have

$$I^i_{ij} \equiv \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1-\cos\theta_{jq}} = \frac{1}{i\pi b} \oint \frac{dz}{(z_+ - z)(z - z_-)}$$

where z-integration contour is the unit circle and

$$z_\pm = \frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}\,.$$

Now only pole at $z = z_-$ can lie inside unit circle, so

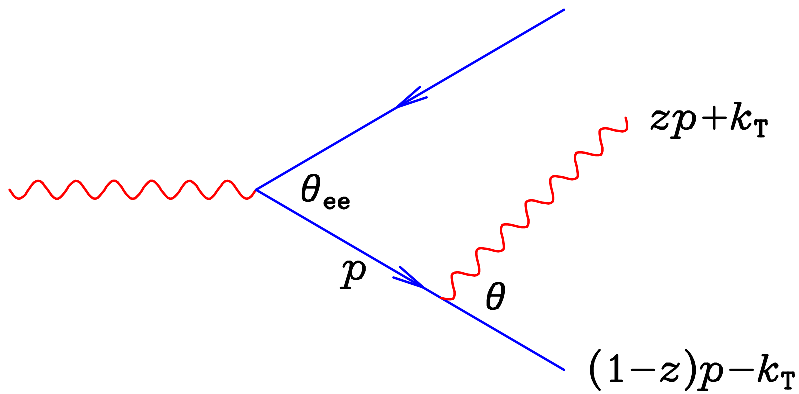
$$I^i_{ij} = \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{a - b\cos\phi_{iq}} = \sqrt{\frac{1}{a^2 - b^2}} \equiv \frac{1}{|\cos\theta_{iq} - \cos\theta_{ij}|}\,.$$

Angular ordering (cont)

$$
\begin{aligned}
\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi}\, W_{ij}^i &= \frac{1}{2(1-\cos\theta_{iq})}[1+(\cos\theta_{iq}-\cos\theta_{ij})I_{ij}^i] \\
&= \frac{1}{1-\cos\theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}\text{, otherwise 0.}
\end{aligned}
$$

Chudakov effect

- Angular ordering is coherence effect common to all gauge theories. In QED it causes Chudakov effect – suppression of soft bremsstrahlung from e^+e^- pairs, which has simple explanation in old-fashioned (time-ordered) perturbation theory.



- Consider emission of soft photon at angle θ from electron in pair with opening angle $\theta_{ee} < \theta$. For simplicity assume $\theta_{ee}, \theta \ll 1$.
- Transverse momentum of photon is $k_T \sim zp\theta$ and energy imbalance at $e \to e\gamma$ vertex is

$$\Delta E \sim k_T^2/zp \sim zp\theta^2 \ .$$

- Time available for emission is $\Delta t \sim 1/\Delta E$. In this time transverse separation of pair will be $\Delta b \sim \theta_{ee}\Delta t$.

Chudakov effect

- For non-negligible probability of emission, photon must resolve this transverse separation of pair, so

$$\Delta b > \lambda/\theta \sim (zp\theta)^{-1}$$

  where λ is photon wavelength.
- This implies that

$$\theta_{ee}(zp\theta^2)^{-1} > (zp\theta)^{-1} \ ,$$

  and hence $\theta_{ee} > \theta$. Thus soft photon emission is suppressed at angles larger than opening angle of pair, which is angular ordering.
- Photons at larger angles cannot resolve electron and positron charges separately – they see only total charge of pair, which is zero, implying no emission.
- More generally, if i and j come from branching of parton k, with (colour) charge $\boldsymbol{Q}_k = \boldsymbol{Q}_i + \boldsymbol{Q}_k$, then radiation outside angular-ordered cones is emitted coherently by i and j and can be treated as coming directly from (colour) charge of k.

Coherent branching

- Angular ordering provides basis for coherent parton branching formalism, which includes leading soft gluon enhancements to all orders.
- In place of virtual mass-squared variable t in earlier treatment, use angular variable

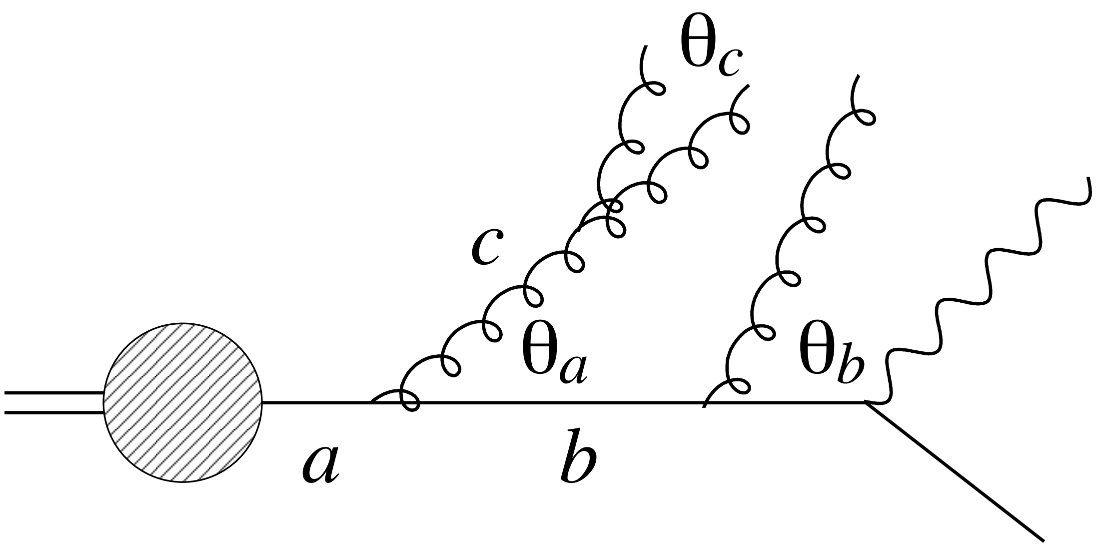
$$\zeta = \frac{p_b \cdot p_c}{E_b\, E_c} \simeq 1 - \cos\theta$$

as evolution variable for branching $a \to bc$, and impose angular ordering $\zeta' < \zeta$ for successive branchings. Iterative formula for n-parton emission becomes

$$d\sigma_{n+1} = d\sigma_n \frac{d\zeta}{\zeta} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z)\ .$$

Coherent branching

- Note that for spacelike branching $a \to bc$ (a incoming, b spacelike), angular ordering condition is



$$\theta_b > \theta_a > \theta_c \ ,$$

Recap

- Parton evolution can be represented as a branching process from higher values of x
- DGLAP equation predicts growth at small x and shrinkage at large x with increasing Q^2.
- The Sudakov form factor $\Delta(t)$ is the probability of evolving from t_0 to t without branching.
- branching from (t_1, x_1) to (t_2, x_2) with the right probability can be performed with by choosing three random numbers, (t, x, ϕ)
- Branching is subject to an angular ordering constraint. Large angle emission is dynamically suppressed.